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q-Deformed Quantum Coherent States and Some of Their Applications

A Thesis

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By

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

ٱقْرَأْ بِٱسْمِ رَبِّكَ ٱلَّذِى خَلَقَ ﴿١﴾ خَلَقَ ٱلْإِنسَٰنَ مِنْ
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To My Parents

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Summary

The concept of q-deformation has found many important applications in a variety of fields in physics, such as quantum optics, atomic physics, solid state physics, nuclear physics and cosmology. This has motivated its extension to many well-established other concepts such as coherent states well-known in quantum optics. On the other hand, the interpretation of the physical meaning of the q-deformation remains an outstanding problem.

The present work is an attempt to apply the concept of q-deformed coherent states to solve this interpretation problem. The q-deformed 1-D quantum harmonic oscillator is used as a model for the application of the methodology of using q-deformed coherent states to solve this problem. This is achieved by first deriving the classical Liouville equation for the q-deformed 1-D classical harmonic oscillator in the undeformed and deformed oscillator phase spaces. Then, this equation is solved by using the method of characteristics which gives the classical probability distribution function for this oscillator in phase space. The behavior of this function is then investigated by using a computer visualization method based on a computer program constructed in Mathematica® language.

On the quantum level, the Heisenberg equation of motion for the density operator corresponding to this 1-D quantum harmonic oscillator is expressed in the present work in terms of the standard quasiprobability distribution functions, again in the deformed and undeformed phase spaces. This helps to derive the quantum Liouville equations for this q-deformed oscillator in these phase spaces. The classical limits of these resulting Liouville equations are then approached by extending a standard procedure based on the non-deformed coherent states to the q-deformed case. In addition to the application of the standard q-deformed coherent states, a novel approach based on q-deformed coherent states due to Arik and Coon is also employed in this investigation.

Summary

The results of detailed mathematical derivations in this process of approaching the classical limit reveal that this limit is statistical in nature. This is similar to the case of the ordinary undeformed oscillator which has been proved previously. They also reveal, together with the complementary computer visualizations, more information about the physical meaning of the q-deformation. This includes the observations that the q-deformed 1-D oscillator can be interpreted as a nonlinear oscillator where the nonlinearity parameter depends on $\hbar$. Also, the behavior of the classical limits of the quantum Liouville equations for this oscillator is observed to show whorl shapes that can be contrasted with their classical analogs. This whorl shape behavior can be considered as a phenomenon connected with q-deformation in general; the anharmonic oscillator being a special case.

Some connection with phase space having a non-commutative geometry, resulting from q-deformation, also finds evidence in some of the results presented in this thesis.

List of Symbols

$\hbar$: Planck's constant divided by 2π (Dirac's constant)

υ: speed of light in matter

c: speed of light in empty space

m: mass of a particle

ω: natural angular frequency of the 1-D classical harmonic oscillator

t: time

$\mathscr{q}$: position coordinate in phase space

$\mathscr{p}$: momentum coordinate in phase space

q: real deformation parameter

λ: nonlinearity parameter

$[x]_q$: q-number

D_x: dilatation (shift) operator

$\mathcal{D}_x^q$: Jackson derivative operator with respect to x

$\mathcal{A}$ and $\mathcal{B}$: two dynamical variables

α, α^*: two independent complex variables representing coordinates in complex phase space

$H(\alpha, \alpha^*)$: Hamiltonian of the 1-D classical harmonic oscillator

α_q, α_q^*: two independent q-deformed complex variables representing the coordinates in q-deformed complex phase space

$\{\mathcal{A}, \mathcal{B}\}$: Poisson bracket of $\mathcal{A}$ and $\mathcal{B}$ with respect to canonical variables $\mathscr{q}$ and $\mathscr{p}$ respectively

ω_q : q-deformed oscillator frequency

List of Symbols

$\mathbb{H}_q(\alpha,\alpha^*)$: Hamiltonian of the q-deformed 1-D classical harmonic oscillator in α-representation

$\mathcal{H}_q(\alpha_q,\alpha_q^*)$: Hamiltonian of the q-deformed 1-D classical harmonic oscillator in α_q-representation

$\chi_q(\alpha,\alpha^*)$: Poisson bracket $\{\alpha_q,\alpha_q^*\}_{\alpha,\alpha^*}$ multiplied by $i\hbar$ for q-deformed 1-D classical harmonic oscillator in α-representation

$\eta_q(\alpha,\alpha^*)$: Poisson bracket $\{\alpha,\mathbb{H}_q(\alpha,\alpha^*)\}_{\alpha,\alpha^*}$ divided by $\hbar\omega\alpha$ for q-deformed 1-D classical harmonic oscillator in α-representation

$P_{CL}(\alpha,\alpha^*;t)$: classical probability distribution function for the undeformed classical harmonic oscillator in α-representation

$\mathcal{P}_{CL}^q(\alpha,\alpha^*;t)$: classical probability distribution function for the q-deformed classical harmonic oscillator in α-representation

$P_{CL}^q(\alpha_q,\alpha_q^*;t)$: classical probability distribution function for the q-deformed classical harmonic oscillator in α_q-representation

$\hat{a},\hat{a}^\dagger$: annihilation and creation boson operators

$\hat{N}$: number operator

$\bar{n}$: expectation value of the number operator

$\hat{H}$: Hamiltonian operator of the undeformed 1-D quantum harmonic oscillator

$\hat{\mathbb{H}}_q$: Hamiltonian operator of the q-deformed 1-D quantum harmonic oscillator in α-representation

$\hat{\mathcal{H}}_q$: Hamiltonian operator of the q-deformed 1-D quantum harmonic oscillator in α_q-representation

$|\alpha\rangle$: coherent state (Gluaber-Sudarshan state)

$|\alpha\rangle\langle\alpha|$: projection operator for coherent states

List of Symbols

$\langle \hat{A} \rangle$: expectation value of operator $\hat{A}$

$[\hat{A}, \hat{B}]$: commutator of operators $\hat{A}$ and $\hat{B}$

Δq : uncertainty in position

Δp : uncertainty in momentum

$\hat{D}$: displacement operator

$\hat{\rho}$: density operator

s : ordering parameter with $s = 1, 0, -1$

$\varphi^{(s)}(\alpha, \alpha^*; t)$: quasiprobability distribution function

$Q(\alpha, \alpha^*; t)$: Husimi Q-function. This function represents the quasiprobability distribution function $\varphi^{(s)}(\alpha, \alpha^*; t)$ in the α-representation with $s = 1$

$W(\alpha, \alpha^*; t)$: Wigner W- function. This function represents the quasiprobability distribution function $\varphi^{(s)}(\alpha, \alpha^*; t)$ in the α-representation with $s = 0$

$P(\alpha, \alpha^*; t)$: Glauber-Sudarshan P- function. This function represents the quasiprobability distribution function $\varphi^{(s)}(\alpha, \alpha^*; t)$ in the α-representation with $s = -1$

$\xi(\alpha, \alpha^*; t)$: probability distribution function for the undeformed 1-D quantum harmonic oscillator in α-representation

$\varphi_q^{(s)}(\alpha, \alpha^*; t)$: q-analog of the quasiprobability distribution function $\varphi^{(s)}(\alpha, \alpha^*; t)$ in α-representation

$Q_q(\alpha, \alpha^*; t)$: q-analog of the Husimi function $Q(\alpha, \alpha^*; t)$ in α-representation

$P_q(\alpha, \alpha^*; t)$: q-analog of the Glauber-Sudarshan P-function $P(\alpha, \alpha^*; t)$ in α-representation

$\xi_q(\alpha, \alpha^*; t)$: q-analog of the $\xi(\alpha, \alpha^*; t)$ in α-representation

List of Symbols

$|n\rangle$: number state

$|0\rangle$: ground state

$\mathcal{N}\left(|\alpha|^2\right)$: normalization constant in α -representation

$\mathcal{P}(n)$: Poissonian distribution with n excitations in a coherent state $|\alpha\rangle$

$\mathbb{P}_q\left(\alpha_q, \alpha_q^*; t\right)$: q-analog of the Glauber-Sudarshan P-function $P\left(\alpha, \alpha^*; t\right)$ in α_q -representation

$\left[\hat{N}\right]_q$: q-deformed number operator

$\mathcal{N}\left(\left|\alpha_q\right|^2\right)$: normalization constant in α_q -representation

$|n\rangle_q$: q-deformed number state

$\hat{a}_q, \hat{a}_q^\dagger$: q-deformed annihilation and creation boson operators

$\frac{D}{D\alpha_q}$: q-differential operator which is also called Jackson derivative (i.e., $\mathcal{D}_x^q$)

$\bar{n}_q$: expectation value of q-deformed number operator $\left[\hat{N}\right]_q$

$\left[\hat{A}, \hat{B}\right]_q$: q-commutator of operators $\hat{A}$ and $\hat{B}$

$\left\|\alpha_q\right\rangle$: unnormalized q-deformed coherent state

$\|n\rangle_q$: unnormalized q-deformed number state

$\|0\rangle_q$: unnormalized q-deformed ground state

$\mathbb{S}$: basic integral

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Chapter One

Introduction

Introduction

1.1 The q-Deformation

The notion of deformation is inherent in physics, where quantum mechanics can be considered as a deformation of Newtonian mechanics with deformation parameter $\hbar$ and, hence, in the limit $\hbar \to 0$, quantum mechanics reduces to classical mechanics. Similarly, special relativity is the deformation of Newtonian mechanics with deformation parameter υ/c and such that, in the limit $\upsilon/c \to 0$ reduces to Newtonian mechanics.

From these physical examples, one can give the following mathematical definition for q-deformation which is a q-analog theorem [1-3], where the identity expression is a generalization involving a new parameter, that returns the original theorem, in the limit as $q \to 1$[1-3]. The history of q-calculus and q-hypergeometric functions dates back to the 18th century, where Euler (1707-1783), was the first to introduce the q-deformation parameter in his Introduction [1]. The formal power series was introduced by Gudermann (1798-1852) and Weierstrass (1815-1897) [1]. The basic hypergeometric series was introduced in the 19th century, specifically in 1846 by the German mathematician Heine [1, 2]. In the years 1909 and 1910, Jackson [4, 5] introduced the first explicit attempt to join the q-deformation with differential equations to obtain what is called the q-difference equations. Jackson is also known for the invention of the q-derivative (Jackson Derivative JD) [1, 3, 5, 6] and Jackson Integral [1, 3].

The applications of q-deformation in physics were first strongly connected with the development of the subject of quantum groups and then expanded to cover many fields in physics. In the period 1945-1950, Tamm and Dancoff [7, 8] introduced what is called Tamm-Dancoff deformed algebra. Deformed algebras are applied in quantum field theory [8]. Many years later, in 1976, Arik and Coon [9] introduced a deformed oscillator algebra in the Hilbert space H_q, where q is a real parameter. The deformed algebra with new generators was introduced by Feinsilver in 1987 [10, 11]. In 1989, the q-deformed oscillator algebra was introduced independently by Biedenharn and Macfarlane [12,13]; this q-deformed oscillator algebra was derived by adopting new definitions for the q-deformed creation and annihilation operators. Later on, specifically in the years 2002 and 2003, Quesne's oscillator algebra [14, 15] was introduced, where a new family of q-deformed coherent states were constructed. In this context, one can refer the reader to refs. [16-18].

The concept of q-deformation has found its way into real physics applications, such as the q-deformed fermions and q-deformation in thermostatics and statistical physics [19-22] and phase-diffusion of the q-deformed oscillator [23]. Other applications are the q-deformation of the Heisenberg algebra, Heisenberg equation of motion, uncertainty relation and Coulomb problem for q-Hydrogen atom [24-27]. Among these applications of the q-deformation in physics, one singles out an important application that arises as a result of attempts by many researchers [28-37] to apply the q-deformation and its generalization (f-deformation) to the well-known concept of coherent states [14,15,38].

1.2 Coherent States

In 1926, Schrödinger [39] discovered the "non-spreading wavepackets" of the harmonic oscillator. The original definition introduced by Schrödinger for these packets is that they have minimum-uncertainty product and correspond to the classical trajectory in phase space. Many years later, and specifically in 1963, Glauber [40] called the Schrödinger [39] non-spreading wavepackets the "Coherent States" for the first time, and defined them as the eigenstates of the boson annihilation operator. The P-representation of the coherent states was introduced in 1963 by Glauber [40] and Sudarshan [41] independently. Other works by Glauber appeared in 1963 [42, 43]. The coherent states were always considered as the most classical ones among the quantum states. This notion was introduced in 1968 by Carruthers and Nieto [44]. Also, Glauber introduced more detailed clarifications of these states in 1969 [45, 46]. Thus, after the work of Glauber [40, 42] and Sudarshan [41], the coherent states became widely known and intensively used by many physicists where these states find many applications in the fields of physics and mathematical physics from solid state physics to cosmology and they represent the core of quantum optics. The work of Dodonov [47] represents an excellent review about coherent states, their types and applications. It is worth mentioning here that another technique to utilize the Glauber P-representation [40] was invented by Fan [48]. This technique is called integration within ordered product of operators (IWOP).

1.3 The f-Deformed and q-Deformed Coherent States and Some of Their Applications

A dominant direction in mathematical physics in the last decades of the 20th century was related to various deformations of the harmonic oscillator canonical commutation relation, $\left[\hat{a},\hat{a}^{\dagger}\right]=1$, where $\hat{a}$ and $\hat{a}^{\dagger}$ are the well-known annihilation and creation operators respectively [49, 50, 51]. The corresponding deformed bosonic operators $\hat{a}_q$ and $\hat{a}_q^{\dagger}$ were introduced, where the subscript "q" refers to the "q-deformation". However, the first study to obtain the eigenstates of the operator $\hat{a}_q^{\dagger}\hat{a}_q$ was performed by Iwata [52] in 1951. Many years later, specifically in 1976, Arik and Coon [9], and Kuryskin [53] considered a generalization of the work of Iwata [52] to involve the case of several dimensions in order to obtain the q-deformed coherent states. A realization of the q-commutation relation, $\left[\hat{a}_q, \hat{a}_q^{\dagger}\right]_q = 1$, was achieved by Jannussis et al [54] in terms of the usual bosonic operators $\hat{a}$ and $\hat{a}^{\dagger}$ by means of the nonlinear transformation $\hat{a}_q = \hat{a}\, f\left(\hat{N}\right)$, where $\hat{N} = \hat{a}^{\dagger}\hat{a}$, and $f\left(\hat{N}\right)$ represent a real operator function of $\hat{N}$. Biedenharn and Macfarlane [12, 13] in 1989 introduced independently a different type of q-deformed coherent states. The squeezing properties of these states were studied in 1990 and 1991 by Solomon and Katriel [55-56]. An excellent review of the f-deformed and q-deformed coherent states can be found in refs. [38, 47].

Thus, q-deformation has many applications in physics, where the q-deformed coherent states found their way in the field of quantum optics after the famous work of Biedenharn and Macfarlane [12, 13]. Also, the non-classical states are applied in different areas of physics, such as high energy physics, cosmology, condensed matter physics, molecular physics, and Bose-Einstein condensation [47]. The recent literature on the subject of the q-deformed coherent states and their applications in physics includes q-analogs of squeezed states, some of their non-classical properties [57], and q-deformed entangled states introduced on the basis of the IWOP-technique [58]. Moreover, the q-deformation can even be useful in actual real life situations where in 2013, Capolupo et al. [59] studied the benefit of using q-deformed coherent states to study filtered water with fractal self-similar properties.

It is worth mentioning that q-deformed coherent states are a special type of more generalized coherent states called Nonlinear Coherent States (NLCSs). These states, defined in the years 1996 and 1997 by Filho et al., and Man’ko et al.

[60,61] as the eigenstate of the f-deformed boson annihilation operator, represent the generalization of the q-deformed boson annihilation operator $\hat{a}_q$ [60,61].

The NLCSs are sometimes called the f-deformed coherent states in the work of Man'ko et al. [61] on the f-oscillator. However, the nonlinear coherent states have attracted much attention in recent years [62-74], mostly because they exhibit non-classical properties of the radiation field, such as squeezing, sub-Poissonian photon statistics and photon anti-bunching [62-74].

1.4 Interpretation of the Classical Limit of Quantum Systems

Since the formulation of the quantum theory by Schrödinger in 1926 [49, 50], many attempts were performed to interpret the classical limit of the quantum systems. One of these attempts was implemented by Schrödinger to produce what was called later on the coherent states [47] for the harmonic oscillator. So, this attempt can be considered as the first attempt to approach the classical limit where these coherent states have a minimum uncertainty product. Another attempt was introduced by Dirac [75] in 1927, when he considered classical mechanics as the limiting case of quantum mechanics when $\hbar \to 0$. This limit implies that the time dependent Schrödinger equation [49, 50, 51] for a single particle in an external field reduces to the well-known Newton's equation of motion. But this limit is still a controversial problem and represents one of the problems still facing the interpretation of quantum mechanics.

However, Ghosh et al. [76] proved in 1977 that the classical limit of the quantum harmonic oscillator is statistical in nature, where the fluid dynamical equations belonging to what is called the single particle Schrödinger fluid have been obtained. These fluid equations reveal much of the physics involved in the classical limit of quantum systems and shed light on the outstanding problem of the interpretation of quantum mechanics. Another example of the interpretation of quantum systems by using the classical limit was introduced in 1986 by Milburn [77] when he interpreted the quantum anharmonic oscillator by using the time–evolution equation for the Husimi function (quasiprobability distribution function) [77] in terms of undeformed phase space [77]. A simulation using the solution of this equation was obtained by him to reveal the nature of the quantum anharmonic oscillator in phase space. The behavior of the Husimi function exhibited a whorl structure which becomes increasingly more convoluted on a finer and finer scale as $t \to \infty$. Also, Habeeb in 1987 [78] interpreted the quantum damped oscillator system in the light of the work of Ghosh [76]. In this work the interpretation of such system was deduced from the conservative form of the obtained fluid-dynamical equations, where it was found that the classical limit for this quantum oscillator cannot be considered as the quantum analog of the classical damped oscillator. In 2009, Jafarpour and Tahamtan [79] obtained the classical limit for the octic anharmonic oscillator from the expectation value of the eigenenergy and eigenstate for the Rayleigh-Schrödinger perturbation theory. The classical limit for this oscillator revealed that there is a frequency shift proportional to the sixth power of the amplitude of this system.

Finally, it should be stated that the statistical description of microscopic systems is usually obtained by employing the quasiprobabilty distribution functions [45, 76, 80-86] in phase space. The first attempts in this direction were introduced by Wigner in 1932 [45, 80-86,107] to study the quantum corrections to classical statistical mechanics. His particular type of distribution function has become to be known as the Wigner distribution function [45, 80-86,107]. This function is normalized but can have a negative value [45, 80-86,107]. It has found many applications, primarily in statistical mechanics, and also in areas such as quantum optics. Another type of quasiprobability distribution function is the well-known P-representation of Glauber and Sudarshan [40, 41], which, like the Wigner function, can be well defined or singular [45], and has also found extensive use.

Also, the Q-function (Husimi function) was introduced in 1940 by Husimi [45, 80-86,107], which is a normalized function and has always positive values, and represents a third type of quasiprobability distribution functions.

1.5 Physical Interpretation of q-Deformation

There have been many attempts to reveal the physical meaning and interpretation of q-deformation. In this context, the q-deformed quantum harmonic oscillator has been used as a good example of this situation. In 1991, Buzek [87] evaluated the time-evolution of the mean values of the q (position) and the p (momentum) for the q-oscillator in order to obtain the periodic classical behavior, where the non- periodic behavior of this oscillator was interpreted as the interaction of the quantum oscillator with another system. In 1992, Shabanov [88] studied also the physical meaning and interpretation of the same oscillator used by Buzek but in a different manner. Shabanov obtained the q-deformed variables via the standard Heisenberg commutation relations, and defined the q-deformation parameter, q, to be a function of $\hbar$ and some dimensional parameter, ℓ_q where $q = e^{-\hbar / \ell_q^2 \omega}$. To interpret this oscillator, he applied the classical limit $\hbar \to 0$, $q \to 1$ for the canonical variables to arrive at the classical theory. The second attempt by Shabanov [89] was more rigorous than the first attempt, where he introduced in 1993 the path integral in his approach. Hence, the classical theory was obtained by applying the semi classical approximation. It turns out that the q-oscillator can be interpreted as a particle with a friction force acting on the particle that is proportional to its velocity. In the same year, Chaichian and Demichev [90] constructed a q-deformed path integral and applied the quasi-classical limit with some specific conditions to obtain the classical equation of motion. Also, Man’ko et al. [91] studied both the quantum and classical q-oscillator via the Dirac dequantization method to construct the classical q-oscillator from the corresponding quantum q-oscillator and interpret the q-oscillator as a classical oscillator with a special type of nonlinearity, where the frequency of the oscillator is a function of the energy which is a constant of the motion. Man'ko also published further works in the same context in 1997 and 1998 [61, 92] in which he introduced the concept of the f-oscillator. Gruver [93] studied the dynamical properties of the q-deformed oscillator and found that the q-oscillator is an anharmonic oscillator with a q-deformation parameter which can be interpreted as a measure of anharmonicity. Many years later, and specifically in 2007, Jafarov et al. [94] introduced a different technique to understand the q-deformation for the quantum harmonic oscillator by studying the behavior of the density plot for both Wigner and Husimi quasiprobability distribution functions for this oscillator. Another attempt to interpret q-deformation was made in the year 2014 where Batouli and El Baz [95] studied the q-deformation for the quantum harmonic oscillator in a way similar to the work of Buzek [87] but with some modifications. These modifications led to a different interpretation of the q-deformation of this oscillator, where it was found that the q-deformed quantum harmonic oscillator is the quantum version of the classical forced oscillator with a modified q-dependent frequency, such that in the limit $q \to 1$ the driving force disappears.

From another point of view, the q-deformation can be interpreted in terms of non-commutative quantum mechanics, introduced by Lavagno et al. in 2006 [96], where the meaning of q-deformation was investigated by applying non-commutative q-calculus. Then, they obtained the generalized q-classical theory which is defined by means of the q-deformed Poisson bracket. Also, Eftekharzadeh et al. and Benatti et al. [97-99] studied in the period 2005-2014 the interpretation of the non-commutative quantum mechanics by applying a special classical limiting to the non-commutativity.

Despite all the attempts that were made to interpret q-deformation, there is still a problem in understanding the physics behind this kind of deformation. This has motivated the present work which is an attempt to investigate the physical nature of q-deformation in the quantum oscillator case via the q-deformed coherent states on the basis of Glauber-Sudarshan P-representation to obtain the Heisenberg equation of motion (quantum Liouville equation) then approach the classical limit to recover the classical Liouville equation of the q-deformed oscillator.

1.6 Aims of the Thesis

The major aims of this thesis are:

1- The derivation of the 1-D classical Liouville equation in undeformed and deformed phase spaces for q-deformed classical harmonic oscillator.

2- Investigation into the possibility of finding a well-behaved analytical solution for this equation by using the well-known analytical solution methods to solve partial differential equations. This solution will produce the probability distribution function belonging to the q-deformed 1-D classical harmonic oscillator in phase space.

3- Studying the time-evolution of the probability distribution function to investigate its behavior in phase space. This investigation is performed by writing a computer simulation program in Mathematica®.

4- Generalization of the Glauber-Sudarshan P-representation for the q-deformed 1-D quantum harmonic oscillator in such a way as to handle the q-deformation problem by overcoming the problem of operators disentanglement in order to derive the quantum Liouville equation in the undeformed phase space.

5- Employing Zaslavskii's method [129] to derive the Q-representation for the quantum Liouville equation of this oscillator in the undeformed phase space for comparison purposes.

6- Investigation into the possibility of deriving the q-analog of Glauber-Sudarshan P-representation in the deformed phase space. This permits the derivation of the P-representation for the quantum Liouville equation of the q-deformed 1-D quantum harmonic oscillator in this deformed phase space.

7- Investigation into the possibility to obtain the classical limit for the obtained quantum Liouville equation in order to obtain the corresponding q-deformed 1-D classical Liouville equation.

8- Attempting to solve these equations to obtain the quasiprobability distribution functions for the q-deformed harmonic oscillator.

9- Finally, using a technique similar to that used in the classical treatment, studying the time-evolution of these probability distribution functions in order to investigate their behavior in phase space.

1.7 Thesis Layout

To achieve the aims stated previously, the rest of the thesis is organized as follows. **Chapter Two** is devoted to the introduction of the mathematical concepts and relations that are relevant to coherent states and the classical limit of the ordinary harmonic oscillator. Then, in **Chapter Three**, the concepts of quantum calculus, including q-numbers, q-deformed elementary functions, q-derivative as well as the equations governing the q-deformed classical and quantum harmonic oscillators are given. The q-deformed coherent states and some of their properties are also introduced in this chapter with some mathematical details together with the definition of the q-deformed density operator. **Chapter Four** is devoted to the construction of the q-deformed oscillator and its f–deformed generalized version and the derivation of the classical Liouville equations for these oscillators. A solution for these Liouville equations and a simulation method is also introduced in this chapter and a computer visualizing method is presented to investigate these oscillators in phase space together with a discussion of the obtained results. In **Chapter Five**, the q-deformed quantum oscillator is constructed and then its quantum Liouville equation is derived in terms of the Glauber–Sudarshan quasiprobability distribution function. Also, the classical limit is investigated for the obtained quantum Liouville equation along the same lines used in Chapter Four to investigate the physical meaning of q-deformation. Finally, **Chapter Six** is dedicated to the main conclusions and suggestions for future work.

In addition, (8) Appendices are devoted to give the full mathematical derivations for some relations and mathematical expressions that were used in this thesis.

Chapter Two

Coherent States and the Classical Limit of the Quantum Harmonic Oscillator

Coherent States and the Classical Limit of the Quantum Harmonic Oscillator

The undeformed simple harmonic oscillator (ordinary simple harmonic oscillator) (SHO) represents the standard system that is used in physics to simulate many systems in nature. In this chapter a short review is given about the undeformed classical and quantum SHO to serve the aim of deriving the time-evolution equations of motion for both classical and quantum oscillators.

2.1 The Undeformed 1-D Classical Harmonic Oscillator

The Hamiltonian of the 1-D-SHO with mass m and angular frequency ω is defined as [100]

$$H = \frac{\mathcal{p}^2}{2m} + \frac{m\omega^2 \mathcal{q}^2}{2} \tag{2.1}$$

where $\mathcal{q}$ and $\mathcal{p}$ represent the classical position and momentum respectively. The Poisson bracket for any two dynamical variables $\mathcal{A}$ and $\mathcal{B}$ with respect to the classical coordinates $(\mathcal{q}, \mathcal{p})$ in classical phase-space is defined as [100]:

$$\{\mathcal{A}, \mathcal{B}\}_{\mathcal{q},\mathcal{p}} = \left(\frac{\partial \mathcal{A}}{\partial \mathcal{q}}\right)_{\mathcal{p}} \left(\frac{\partial \mathcal{B}}{\partial \mathcal{p}}\right)_{\mathcal{q}} - \left(\frac{\partial \mathcal{A}}{\partial \mathcal{p}}\right)_{\mathcal{q}} \left(\frac{\partial \mathcal{B}}{\partial \mathcal{q}}\right)_{\mathcal{p}} \tag{2.2}$$

Therefore, the Poisson bracket $\{\mathcal{q}, \mathcal{p}\}$ with respect to canonical variables $(\mathcal{q}, \mathcal{p})$ is (the subscript $\mathcal{q}, \mathcal{p}$ will be dropped from now on):

$$\{\mathcal{q}, \mathcal{p}\} = 1 \tag{2.3}$$

since q and p are considered to be independent variables. The independent variables q and p that have Poisson bracket defined in eqn. (2.3) are called canonical variables [100]. One can also define the Hamiltonian in eqn. (2.1) in terms of non-canonical complex independent variables α and α^*, where [77,91,101]:

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}\, q + \frac{i p}{\sqrt{2\hbar m\omega}} \tag{2.4}$$

$$\alpha^* = \sqrt{\frac{m\omega}{2\hbar}}\, q - \frac{i p}{\sqrt{2\hbar m\omega}} \tag{2.5}$$

The appearance of $\hbar$ in eqns. (2.4) and (2.5) is to provide a convenient scaling for various physical quantities even though one is dealing with a classical system here [77].

The Poisson bracket $\{\alpha, \alpha^*\}$ with respect to canonical variables q and p is given as [101]:

$$\{\alpha, \alpha^*\} = -\left(\frac{i}{\hbar}\right) \tag{2.6}$$

But, since from eqns. (2.4) and (2.5):

$$q = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha + \alpha^*\right) \tag{2.7}$$

$$p = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\alpha - \alpha^*\right) \tag{2.8}$$

then, substituting eqns. (2.7) and (2.8) into eqn. (2.1) one obtains [91]:

$$H\left(\alpha, \alpha^*\right) = \hbar\omega\alpha\alpha^* \tag{2.9}$$

2.1.1 The Classical Equation of Motion

The equation of motion for the system defined by the Hamiltonian of eqn. (2.9) is obtained from Hamilton's equations as [91]:

$$\dot{\alpha} = \left\{\alpha, H\left(\alpha, \alpha^*\right)\right\} \tag{2.10}$$

where [91,101]:

$$\left\{\alpha, H\left(\alpha, \alpha^*\right)\right\} = \left\{\alpha, \alpha^*\right\} \cdot \left\{\alpha, H\left(\alpha, \alpha^*\right)\right\}_{\alpha, \alpha^*} \tag{2.11}$$

But since,

$$\left\{\alpha, H\left(\alpha, \alpha^*\right)\right\}_{\alpha, \alpha^*} = \left(\frac{\partial \alpha}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial H\left(\alpha, \alpha^*\right)}{\partial \alpha^*}\right)_{\alpha} - \left(\frac{\partial \alpha}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial H\left(\alpha, \alpha^*\right)}{\partial \alpha}\right)_{\alpha^*} \tag{2.12}$$

and,

$$\left.\begin{aligned} \left(\frac{\partial \alpha}{\partial \alpha}\right)_{\alpha^*} &= \left(\frac{\partial \alpha^*}{\partial \alpha^*}\right)_{\alpha} = 1 \\ \left(\frac{\partial \alpha}{\partial \alpha^*}\right)_{\alpha} &= \left(\frac{\partial \alpha^*}{\partial \alpha}\right)_{\alpha^*} = 0 \end{aligned}\right\} \tag{2.13}$$

then, substituting eqns. (2.13) into eqn. (2.12), one obtains:

$$\left\{\alpha, H\left(\alpha, \alpha^*\right)\right\}_{\alpha, \alpha^*} = \left(\frac{\partial H\left(\alpha, \alpha^*\right)}{\partial \alpha^*}\right)_{\alpha} \tag{2.14}$$

Substituting eqn. (2.9) into eqn. (2.14) and using the result together with eqn. (2.6) in eqn. (2.11), one obtains the Poisson bracket $\left\{\alpha, H\left(\alpha, \alpha^*\right)\right\}$. Finally, substituting the result into eqn. (2.10), one obtains:

$$\dot{\alpha} = -i\omega\alpha \tag{2.15}$$

The solution of eqn. (2.15) is

$$\alpha(t) = \alpha(0)\, e^{-i\omega t} \tag{2.16}$$

where $\alpha(0)$ is the amplitude or the value of $\alpha(t)$ at $t = 0$. Similarly, the complex conjugates of eqns. (2.15) and (2.16) are given as:

$$\dot{\alpha}^* = i\omega\alpha^* \tag{2.17}$$

and,

$$\alpha^*(t) = \alpha^*(0)\, e^{i\omega t} \tag{2.18}$$

2.1.2 The Classical Liouville Equation

The classical Liouville equation (or time-evolution equation) in phase space is given in terms of the Poisson bracket as [100]

$$\frac{\partial \mathcal{A}}{\partial t} = \left\{ H(\alpha, \alpha^*), \mathcal{A} \right\} \tag{2.19}$$

Letting, $\mathcal{A} = P_{CL}(\alpha, \alpha^*; t)$ in eqn. (2.19), this equation becomes:

$$\frac{\partial P_{CL}(\alpha, \alpha^*; t)}{\partial t} = \left\{ H(\alpha, \alpha^*), P_{CL}(\alpha, \alpha^*; t) \right\} \tag{2.20}$$

where, $P_{CL}(\alpha, \alpha^*; t)$ represents the classical probability distribution function for the 1-D SHO in phase-space. But since,

$$\left\{ H(\alpha, \alpha^*), P_{CL}(\alpha, \alpha^*; t) \right\} = \left\{ \alpha, \alpha^* \right\} \cdot \left\{ H(\alpha, \alpha^*), P_{CL}(\alpha, \alpha^*; t) \right\}_{\alpha, \alpha^*} \tag{2.21}$$

and,

$$\left\{H(\alpha,\alpha^*),P_{CL}(\alpha,\alpha^*;t)\right\}_{\alpha,\alpha^*} = \left(\frac{\partial H(\alpha,\alpha^*)}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial P_{CL}(\alpha,\alpha^*;t)}{\partial\alpha^*}\right)-\left(\frac{\partial H(\alpha,\alpha^*)}{\partial\alpha^*}\right)_{\alpha}\left(\frac{\partial P_{CL}(\alpha,\alpha^*;t)}{\partial\alpha}\right) \tag{2.22}$$

then, using the Hamiltonian of eqn. (2.9), one obtains:

$$\left.\begin{aligned}\left(\frac{\partial H(\alpha,\alpha^*)}{\partial\alpha}\right)_{\alpha^*} &= \hbar\omega\alpha^* \\ \left(\frac{\partial H(\alpha,\alpha^*)}{\partial\alpha^*}\right)_{\alpha} &= \hbar\omega\alpha\end{aligned}\right\} \tag{2.23}$$

Substituting eqn. (2.23) in eqn. (2.22), and using the result in eqn. (2.21) together with the eqn. (2.6), one obtains the expression for the Poisson bracket $\left\{H(\alpha,\alpha^*),P_{CL}(\alpha,\alpha^*;t)\right\}$. Using this result, eqn. (2.20) becomes:

$$\frac{\partial P_{CL}(\alpha,\alpha^*;t)}{\partial t} = -i\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)P_{CL}(\alpha,\alpha^*;t) \tag{2.24}$$

Equation (2.24) represents the classical Liouville equation for the classical undeformed 1-D SHO in phase space.

2.2 The Uneformed 1-D Quantum Harmonic Oscillator

The undeformed boson operators $\hat{a}$ and $\hat{a}^\dagger$ are defined in terms of position and momentum operators $\hat{q}$ and $\hat{p}$ respectively, as [49-51]:

$$\hat{a}=\frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q}+i\hat{p}\right) \qquad \text{(annihilation operator)} \tag{2.25}$$

$$\hat{a}^\dagger=\frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q}-i\hat{p}\right) \qquad \text{(creation operator)} \tag{2.26}$$

The commutation relations for $\left(\hat{a},\hat{a}^{\dagger}\right)$ and $(\hat{q},\hat{p})$ are given as [49-51, 91]:

$$\left[\hat{q},\hat{p}\right]=i\hbar \tag{2.27}$$

$$\left[\hat{a},\hat{a}^{\dagger}\right]=1 \tag{2.28}$$

where,

$$\hat{q}=q \tag{2.29}$$

$$\hat{p}=-i\hbar\frac{d}{dq} \tag{2.30}$$

Therefore, the Hamiltonian operator of the 1-D quantum harmonic oscillator can be defined in terms of the boson operators as [49-51]:

$$\hat{H}=\left(\frac{\hbar\omega}{2}\right)\left(\hat{a}\hat{a}^{\dagger}+\hat{a}^{\dagger}\hat{a}\right)=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)=\hbar\omega\left(\hat{a}\hat{a}^{\dagger}-\frac{1}{2}\right) \tag{2.31}$$

Also, one usually defines the number operator, $\hat{N}$, as

$$\hat{N}=\hat{a}^{\dagger}\hat{a} \tag{2.32}$$

Hence,

$$\hat{N}+1=\hat{a}\hat{a}^{\dagger} \tag{2.33}$$

and in terms of eqns. (2.32) and (2.33), the Hamiltonian operator becomes:

$$\hat{H}=\hbar\omega\left(\hat{N}+\frac{1}{2}\right) \tag{2.34}$$

Then, the expectation value of the Hamiltonian operator can be calculated as [49-51]:

$$\left\langle\hat{H}\right\rangle=\left\langle n\middle|\hat{H}\middle|n\right\rangle=\hbar\omega\left(n+\frac{1}{2}\right) \tag{2.35}$$

where the state $|n\rangle$ is n^{th} excited number state defined as [49-51]:

$$|n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}\,|0\rangle \qquad ;\ n = 0,1,2,\cdots \tag{2.36}$$

It satisfies the relations [49-51]

$$\left.\begin{aligned} &\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \\ &\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ \text{and,}\quad & \\ &\hat{a}|0\rangle = 0 \end{aligned}\right\} \tag{2.37}$$

where the state $|0\rangle$ is the vacuum (or ground) state with occupation number $n=0$.

2.3 Coherent States of the 1-D Quantum Harmonic Oscillator

In quantum systems, one cannot measure the position of a particle and its momentum precisely at the same time because of the Heisenberg uncertainty principle. Therefore, the best way to talk about a quantum state that is analogues to classical motion is a localized state. Such a state is the “Coherent State” [39, 40, 42, 43].

2.3.1 Standard Definitions of Coherent States:

A coherent state $|\alpha\rangle$ can be defined in three different ways as follows:

a) As eigenstate of the boson annihilation operator, or [40,102-108]:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.38}$$

where α is the eigenvalue of the annihilation operator $\hat{a}$ when acting on $|\alpha\rangle$, which is a complex number since $\hat{a}$ is a non-Hermitian operator.

b) As minimum-uncertainty state (MUS):

According to this definition, it is required that the coherent state $|\alpha\rangle$ satisfies the minimum-uncertainty relation as [42, 43, 83]:

$$(\Delta q)^2 \cdot (\Delta p)^2 = \frac{\hbar^2}{4} \tag{2.39}$$

where $(\Delta q)^2$ and $(\Delta p)^2$ are defined as [103],

$$(\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2 = \left(\frac{\hbar}{2m\omega}\right) \tag{2.40}$$

and,

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \left(\frac{\hbar m\omega}{2}\right) \tag{2.41}$$

and the expectation values are taken with respect to the coherent state $|\alpha\rangle$.

c) As the state generated from the ground state $|0\rangle$ by acting with the displacement operator $\hat{D}(\alpha)$, or [102,104,109]:

$$|\alpha\rangle = \hat{D}\,|0\rangle \tag{2.42}$$

where,

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \tag{2.43}$$

It is known that all three definitions of the coherent states $|\alpha\rangle$ can be shown to be equivalent for the SHO [103].

Also, it can be shown that the coherent state $|\alpha\rangle$ as defined above, can be expanded in Fock space as [40, 42, 43]:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} |n\rangle \tag{2.44}$$

where

$$|\alpha|^2 = \alpha\alpha^* \tag{2.45}$$

2.3.2 Some Properties of Coherent States:

Coherent states of the SHO have a number of important properties. In this section, some of these properties that are of relevance to the work in this thesis are reviewed. For further mathematical details about these properties and other properties one is referred to refs. [83, 102-110].

(a) The set of coherent states $\{|\alpha\rangle\}$ is normalizable with normalization constant $\mathcal{N}(|\alpha|^2)$ given as [97,99]:

$$\mathcal{N}(|\alpha|^2) = e^{-\frac{|\alpha|^2}{2}} \tag{2.46}$$

(b) These coherent states are non-orthogonal states in the sense [102-110]:

$$\langle\alpha|\beta\rangle = e^{-\left(\frac{1}{2}\right)\left(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta\right)} \tag{2.47}$$

or,

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{2.48}$$

Hence, for $\alpha \neq \beta$ these states become approximately orthogonal since

$|\alpha-\beta|^2 \to \infty$ as $\langle\alpha|\beta\rangle \to 0$ [102-110].

(c) The set of coherent states $\{|\alpha\rangle\}$ resolves the identity:

This means that [83, 102-110]

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = 1 \tag{2.49}$$

where,

$$d^2\alpha = d\alpha\, d\alpha^* \tag{2.50}$$

In view of eqn. (2.47), the set of coherent states $\{|\alpha\rangle\}$ forms an overcomplete set of states, where $\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$ represents the completeness relation for number states in Hilbert space.

(d) The expectation value of the Hamiltonian operator $\hat{H}$ in the coherent state $|\alpha\rangle$ is given as [108-110]:

$$\langle\hat{H}\rangle = \langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega\left\{\langle\alpha|\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)|\alpha\rangle\right\} = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) \tag{2.51}$$

(e) The expectation value $\bar{n}$ of the number operator $\hat{N}$ in the coherent state $|\alpha\rangle$ is given as [102,108-110]:

$$\bar{n} = \langle\hat{N}\rangle = \langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle = |\alpha|^2 \tag{2.52}$$

(f) The probability $\mathcal{P}(n)$ of measuring n excitations in a coherent state $|\alpha\rangle$ is Poisson distributed since,

$$\mathcal{P}(n) = |\langle n|\alpha\rangle|^2 = \left(\frac{|\alpha|^{2n}}{n!}\right) e^{-|\alpha|^2} \tag{2.53}$$

Substituting $\bar{n}$ from eqn. (2.52) into eqn. (2.53) one gets [102,108-110]:

$$\mathcal{P}(n)=\left|\langle n|\alpha\rangle\right|^2=\left(\frac{(\bar{n})^n}{n!}\right)e^{-\bar{n}} \tag{2.54}$$

It is clear that eqn. (2.53) represents a Poissonian distribution [102,108-110].

2.4 The Density Operator

The density operator $\hat{\rho}$ is an operator associated with the probability of finding any quantum system under consideration in a certain state [49, 50]. In the coherent state representation, this operator can be defined in terms of the weight function $\varphi^{(s)}(\alpha,\alpha^*)$ with ordering parameter, s, as [41, 42, 45, 46]:

$$\hat{\rho}=\int d^2\alpha\, \varphi^{(s)}(\alpha,\alpha^*)\ |\alpha\rangle\langle\alpha| \tag{2.55}$$

The weight function $\varphi^{(s)}(\alpha,\alpha^*)$ represents a quasiprobabilty distribution function [45, 46]. A quasiprobability distribution is just like a true probability distribution from which one can calculate the average values. However, it differs from a true probability in that it can have negative as well as positive values besides other properties [45, 46, 80, 81, 84].

The values of the ordering parameter, s, and their associated quasiprobability distribution functions are illustrated in Table (2.1)

Table (2.1)

Values of the ordering parameter s, their associated types of ordered boson operators and quasiprobability distribution functions.

s	$\varphi^{(s)}(\alpha,\alpha^*)$	type of ordered bosons operators		type of quasiprobability distribution functions	
1	$\varphi^{(1)}(\alpha,\alpha^*)$	normal ordered	$\hat{a}^{\dagger}\hat{a}$	Q-function (Husimi)	$Q(\alpha,\alpha^*)$
0	$\varphi^{(0)}(\alpha,\alpha^*)$	symmetrical ordered	$\left(\frac{1}{2}\right)\left(\hat{a}\hat{a}^{\dagger}+\hat{a}^{\dagger}\hat{a}\right)$	W-function (Wigner)	$W(\alpha,\alpha^*)$
-1	$\varphi^{(-1)}(\alpha,\alpha^*)$	anti-normal ordered	$\hat{a}\hat{a}^{\dagger}$	P-function (Glauber, Sudarshan)	$P(\alpha,\alpha^*)$

2.5 The Quantum Liouville Equation

The equation of motion for any operator $\hat{\mathcal{A}}$ in the Heisenberg picture is given by the Heisenberg equation of motion as [49, 50]:

$$\frac{\partial\hat{\mathcal{A}}}{\partial t}=\left(\frac{i}{\hbar}\right)\left[\hat{\mathcal{A}},\hat{H}\right] \tag{2.56}$$

For the density operator, $\hat{\rho}$, this equation becomes [49, 50]:

$$\frac{\partial\hat{\rho}}{\partial t}=\left(\frac{i}{\hbar}\right)\left[\hat{\rho},\hat{H}\right] \tag{2.57}$$

Eqn. (2.57) is the analog of the classical Liouville equation (2.20) which can be considered as quantum Liouville equation.

2.6 Classical Limit of Quantum Systems

The expectation value of the Heisenberg equation of motion for any operator $\hat{\mathcal{A}}$ can be obtained from eqn. (2.56). In general, it is written as [49, 50]

$$\left\langle \frac{\partial \hat{\mathcal{A}}}{\partial t} \right\rangle = \left(\frac{i}{\hbar} \right) \left\langle \left[\hat{\mathcal{A}}, \hat{H} \right] \right\rangle \tag{2.58}$$

Eqn. (2.58) is the quantum analog of the equation of motion for a classical dynamical variable $\mathcal{A}$ in a Hamiltonian system as given in eqn. (2.19).

It is well known that quantum mechanics should go over to classical mechanics whenever the commutators divided by "$i\hbar$" go over into the corresponding Poisson brackets in the limit $\hbar \to 0$ [49, 50, 75]. Thus, eqn. (2.58) goes over into eqn. (2.19) in the classical limit. This is Ehrenfest's theorem in its general form [49, 50, 75]. Therefore, one can investigate if any quantum system has a classical counterpart by applying this theorem. This application is very important as it sheds light on the outstanding problem of the interpretation of quantum mechanics [76-78].

A good example of applying the classical limit for quantum systems is the 1-D quantum harmonic oscillator, where the standard method to approach the classical limit for this oscillator is achieved via coherent states. This method can be approached by using the density operator, $\hat{\rho}$, in the Heisenberg equation of motion. The best example in this respect is that of Ghosh et al. [76] for the SHO which can be summarized as follows.

It is well known that the density operator, $\hat{\rho}$, can be written in the Glauber-Sudarshan P-representation as [40, 42, 45, 46] (see also Sec. (2.4)):

$$\hat{\rho}=\int d^2\alpha\ P(\alpha,\alpha^*)\ |\alpha\rangle\langle\alpha| \tag{2.59}$$

Substituting the Hamiltonian of eqn. (2.31) in eqn. (2.57), one gets [76]:

$$\frac{\partial\hat{\rho}}{\partial t}=i\omega\left\{\hat{\rho}\ \hat{a}^\dagger\hat{a}-\hat{a}^\dagger\hat{a}\,\hat{\rho}\right\} \tag{2.60}$$

Eqn. (2.60) can be simplified by re-arranging terms such that all annihilation operators, $\hat{a}$, are to the left and all creation operators, $\hat{a}^\dagger$, are to the right for all operator products (anti-normal ordering) [42 45,46]. This process can be achieved by using integration by parts and the following relations [76,108,109,111,112]:

$$\hat{a}\,\hat{\rho}=\int d^2\alpha|\alpha\rangle\langle\alpha|\ \alpha P(\alpha,\alpha^*) \tag{2.61}$$

$$\hat{\rho}\ \hat{a}^\dagger=\int d^2\alpha|\alpha\rangle\langle\alpha|\ \alpha^* P(\alpha,\alpha^*) \tag{2.62}$$

$$\hat{a}^\dagger\hat{\rho}=\int d^2\alpha|\alpha\rangle\langle\alpha|\ \left(\alpha^*-\frac{\partial}{\partial\alpha}\right)P(\alpha,\alpha^*) \tag{2.63}$$

$$\hat{\rho}\ \hat{a}=\int d^2\alpha|\alpha\rangle\langle\alpha|\ \left(\alpha-\frac{\partial}{\partial\alpha^*}\right)P(\alpha,\alpha^*) \tag{2.64}$$

Eqns. (2.61) – (2.64) can be written in the form of one-to-one correspondence as [76,108,109,111,112]:

$$\hat{a}\,\hat{\rho}\rightarrow\alpha P(\alpha,\alpha^*) \tag{2.65}$$

$$\hat{\rho}\ \hat{a}^\dagger\rightarrow\alpha^* P(\alpha,\alpha^*) \tag{2.66}$$

$$\hat{a}^\dagger\hat{\rho}\rightarrow\left(\alpha^*-\frac{\partial}{\partial\alpha}\right)P(\alpha,\alpha^*) \tag{2.67}$$

$$\hat{\rho}\,\hat{a} \rightarrow \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P\left(\alpha, \alpha^*\right) \tag{2.68}$$

Therefore, the Heisenberg equation of motion, eqn. (2.60), is translated into the P-representation as [76]:

$$\frac{\partial P\left(\alpha,\alpha^*;t\right)}{\partial t} = -i\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right) P\left(\alpha,\alpha^*;t\right) \tag{2.69}$$

One can consider eqn. (2.69) as representing the classical Liouville equation by interchanging $P\left(\alpha,\alpha^*;t\right)$ by $P_{CL}\left(\alpha,\alpha^*;t\right)$(i.e.; classical probability distribution function) because eqn. (2.69) does not contain any quantum term (i.e., $\hbar$ does not appear in this equation). Also, defining the mass density and the local hydrodynamic velocity, one can consider another interpretation of the classical limit in terms of the continuity and Euler equations of the incompressible fluid dynamics as can be found in ref. [76].

The importance of this approach to the classical limit of a quantum system will become clearer in the next chapters, specifically when the deformed harmonic oscillator and the approach to its classical limit are introduced.

Chapter Three

The q-Deformed Harmonic Oscillator and q-Deformed Coherent States

The q-Deformed Harmonic Oscillator and q-Deformed Coherent States

3.1 Quantum Calculus

Quantum calculus is a branch of calculus that arises naturally from studying the subject of quantum groups. It is different from ordinary calculus, but they approach each other in the limit $q \to 1$ and coincide when q is equal to unity. In the next sections, some mathematical details about quantum calculus that are needed in the next chapters will be introduced.

3.1.1 q–Numbers

In general, q-numbers are classified into two types according to the q-deformation parameter under consideration; if they are real q-numbers or complex ones. In this thesis, the interest will be in real q-deformation parameter in the range of values $0 < q < 1$. The real q-deformation parameter can be defined in terms of 'x' which could be ordinary number or operator as [16, 17, 113, 114]:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3.1}$$

which is invariant (symmetrical) under the substitution $q \to q^{-1}$. One can also define

$$q = e^{\lambda} \tag{3.2}$$

where λ represents the nonlinearity parameter in the range of values $-\infty < \lambda < 0$, then,

$$\lambda = \ln q \tag{3.3}$$

Using eqns. (3.2) and (3.3) in eqn. (3.1), one can write:

$$[x]_q = \frac{\sinh(\lambda x)}{\sinh(\lambda)} \tag{3.4}$$

There is another definition for the q-number $[x]_q$ as [116,117]:

$$[x]_q = \frac{q^x - 1}{q - 1} \tag{3.5}$$

which is non-symmetrical under the substitution $q \to q^{-1}$.

In a similar manner as before, using eqns. (3.2) and (3.3) in eqn. (3.5), the q-deformed number $[x]_q$ becomes:

$$[x]_q = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1} \tag{3.6}$$

It can be noted that both definitions of the q–number $[x]_q$ coincide with the ordinary number x in the limit $q \to 1$, i.e.,

$$\lim_{q \to 1} [x]_q = x \tag{3.7}$$

3.1.2 q–Deformed Elementary Functions

In addition to q-deformed numbers, some q-deformed elementary functions can be introduced, where according to Euler's 1st and 2nd identities [3]:

$$(1+x)_q^{\infty} = \sum_{j=0}^{\infty} q^{j(j-1)/2} \left(\frac{x^j}{(1-q)(1-q^2)\cdots(1-q^j)} \right) \quad \text{(Euler's 1}^{\text{st}}\text{ identity)} \tag{3.8}$$

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{j=0}^{\infty} \left(\frac{x^j}{(1-q)(1-q^2)\cdots(1-q^j)} \right) \quad \text{(Euler's 2}^{\text{nd}}\text{ identity)} \tag{3.9}$$

Then, according to eqns. (3.8) and (3.9), there are two types of q-exponential functions which are defined as [3]:

$$e_q^x = \sum_{j=0}^{\infty} \left(\frac{x^j}{[j]_q!} \right) \tag{3.10}$$

and,

$$E_q(x) = \sum_{j=0}^{\infty} q^{j(j-1)/2} \left(\frac{x^j}{[j]_q!} \right) \tag{3.11}$$

where $[j]_q!$ represents the q-analog of the factorial, defined as [1,3]

$$[j]_q! = [j]_q\,[j-1]_q\,[j-2]_q \cdots [1]_q \tag{3.12}$$

$$[0]_q! = 1 \tag{3.13}$$

Also, q-trigonometric functions are defined as [1, 3,115]

$$\sin_q(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{[2j+1]_q!} \tag{3.14}$$

$$\cos_q(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{[2j]_q!} \tag{3.15}$$

3.1.3 q–Derivative (Jackson's Derivative)

The q-analog of the ordinary derivative was first introduced by Jackson in 1909 [4]. The Jackson derivative operator with respect to x, denoted as $\mathcal{D}_x^q$, is defined in terms of the dilatation (shift) operator D_x [3,113-116] as:

$$\mathcal{D}_x^q = \frac{D_x - (D_x)^{-1}}{\left(q - q^{-1}\right) x} \tag{3.16}$$

where

$$D_x = q^{x \frac{\partial}{\partial x}} \tag{3.17}$$

Also, the dilatation operator can be defined by using ordinary differential calculus (i.e., $D_x = q^{x \frac{d}{dx}}$) as given in refs. [3,114].

Other possible definitions of $\mathcal{D}_x^q$ are also found in the literature. For example, the definition

$$\mathcal{D}_x^q = \frac{D_x - 1}{(q-1)\, x} \tag{3.18}$$

is also used [114,116].

The Jackson derivative operator, defined in eqns. (3.16) and (3.18), when acting on an arbitrary function $f(x)$, gives [115]:

$$\mathcal{D}_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{\left(q - q^{-1}\right) x} \tag{3.19}$$

for the definition of $\mathcal{D}_x^q$ given in eqn. (3.16), and [116]:

$$\mathcal{D}_x^q f(x) = \frac{f(qx) - f(x)}{(q-1)\,x} \tag{3.20}$$

for the definition of $\mathcal{D}_x^q$ given in eqn. (3.18).

These results follow from the relation [3,112-116]

$$q^{\pm x \frac{\partial}{\partial x}} f(x) = f\left(q^{\pm 1} x\right) \tag{3.21}$$

Also, one notices that both definitions of the Jackson derivative given above reduce to the standard derivative in the limit $q \to 1$ as expected, or:

$$\lim_{q \to 1} \mathcal{D}_x^q f(x) = \frac{d}{dx} f(x) \tag{3.22}$$

It is also useful to note some examples of the action of the q-derivative on the q-deformed elementary functions, such as [3]:

$$\left.\begin{aligned} &\mathcal{D}_x^q e_q^x = e_q^x \\ &\mathcal{D}_x^q E_q(x) = E_q(qx) \\ &\mathcal{D}_x^q \sin_q(x) = \cos_q(x) \\ &\mathcal{D}_x^q \cos_q(x) = -\sin_q(x) \end{aligned}\right\} \tag{3.23}$$

More details about such actions of the q-derivative on other functions can be found in ref. [3].

3.1.4 Analog of Leibniz Rules for q-Differentiation

The analog of Leibniz rules for q-differentiation of two arbitrary functions

$F(x)$ and $G(x)$ are given as [3]:

(i) Sum Rule

$$\mathcal{D}_x^q \left\{F(x)+G(x)\right\}=\mathcal{D}_x^q F(x)+\mathcal{D}_x^q G(x) \tag{3.24}$$

(ii) Product Rule

$$\mathcal{D}_x^q \left\{F(x)\cdot G(x)\right\}=F(qx)\mathcal{D}_x^q G(x)+G(x)\mathcal{D}_x^q F(x)$$

which can also be written as (3.25)

$$\mathcal{D}_x^q \left\{F(x)\cdot G(x)\right\}=G(qx)\mathcal{D}_x^q F(x)+F(x)\mathcal{D}_x^q G(x)$$

(iii) Quotient Rule

$$\mathcal{D}_x^q \left\{\frac{F(x)}{G(x)}\right\}=\frac{G(qx)\mathcal{D}_x^q F(x)-F(qx)\mathcal{D}_x^q G(x)}{G(x)G(qx)}$$

which can also be written as (3.26)

$$\mathcal{D}_x^q \left\{\frac{F(x)}{G(x)}\right\}=\frac{G(x)\mathcal{D}_x^q F(x)-F(x)\mathcal{D}_x^q G(x)}{G(x)G(qx)}$$

where $G(x)$ and $G(qx)\neq 0$.

For more details about q-integration, q-polynomials and many other q-relations and identities, one is referred to refs. [1- 3].

3.2 The q-Deformed Harmonic Oscillator

The q-deformed harmonic oscillator was introduced firstly in connection with studying quantum groups [16], where one can consider the q-deformed quantum harmonic oscillator as a deformation of the standard quantum harmonic oscillator. There are different versions of the q-deformed harmonic oscillator that can be obtained by defining the q-deformed boson operators [9, 12-15]. In the next two sections, q-deformations of the harmonic oscillator in both its classical and quantum versions, are introduced.

3.2.1 The q-Deformed Classical Harmonic Oscillator

There are different approaches to introduce q-deformation for the classical harmonic oscillator. One approach is q-deformation of the Poisson bracket via the Jackson derivative [117]. Another approach is q-deformation of the Lagrangian of the harmonic oscillator [118]. Also, there is the possibility of q-deforming the action integral to obtain the q-deformed equation of motion [119]. In this context, it should be emphasized that the problem of the q-deformed classical oscillator and its interpretation are still open problems. Finally, it should also be stated that a q-deformed classical harmonic oscillator reduces to the standard classical harmonic oscillator in the limit $q \to 1$.

3.2.2 The q-Deformed Quantum Harmonic Oscillator

In general, there are different versions of the q-deformed quantum harmonic oscillator according to the q–commutator that is adopted for each version as well as the definitions of the bosonic operators that satisfy these q-commutators [9, 12-16, 32, 38]. An example of the q-deformed quantum oscillator is given in ref. [12] in which Biedenharn introduced the following q–commutator:

$$\left[\hat{a}_q, \hat{a}_q^\dagger\right]_q = \hat{a}_q \hat{a}_q^\dagger - q^{\mp 1} \hat{a}_q^\dagger \hat{a}_q = q^{\pm \hat{N}} \tag{3.27}$$

In general, and according to Man’ko [91], the q-deformed oscillator may represent a special type of nonlinearity where the frequency of the oscillator depends on the energy of the oscillator (i.e., $|\alpha|^2$). In this context, an f-deformed oscillator, which is a generalization of the q-oscillator, was introduced by Man'ko [61]. The realization of the f-deformed boson operators $\hat{a}_f$ and $\hat{a}_f^{\dagger}$ in terms of the undeformed boson operators $\hat{a}$ and $\hat{a}^{\dagger}$ was achieved via the transformation [61, 92, 96, 120]:

$$\left.\begin{aligned} \hat{a}_f &= \hat{a}\, f\left(\hat{N}\right) = f\left(\hat{N}+1\right)\hat{a} \\ \hat{a}_f^{\dagger} &= f\left(\hat{N}\right)\hat{a}^{\dagger} = \hat{a}^{\dagger} f\left(\hat{N}+1\right) \end{aligned}\right\} \tag{3.28}$$

where $f\left(\hat{N}\right)$ represents a non-negative real operator-valued function of the number operator. It should be noted that the subscript "f" used here refers to the "f-deformation" case. Also, whenever a q-deformation process is used instead of the f-deformation process, then the subscript "f" is interchanged by "q" and vice versa.

The transformation from the f-deformed oscillator to the q-deformed oscillator or to the undeformed oscillator involves substituting specific values for the function $f\left(\hat{N}\right)$ in the transformation of eqn. (3.28) in the form [61, 64, 65, 91]:

$$f\left(\hat{N}\right)=\begin{cases} 1 & \text{for undeformed oscillator} \\ \sqrt{\dfrac{[\hat{N}]_q}{\hat{N}}} & \text{for q-deformed oscillator} \\ \text{otherwise} & \text{for f-deformed oscillator} \end{cases} \tag{3.29}$$

Furthermore, the Hamiltonian operators of the q-deformed and f-deformed quantum harmonic oscillators are defined as [61, 91, 92, 120, 121]:

$$\hat{\mathcal{H}}_q = \left(\frac{\hbar\omega}{2}\right)\left(\hat{a}_q\hat{a}_q^{\dagger} + \hat{a}_q^{\dagger}\hat{a}_q\right) \tag{3.30a}$$

and,

$$\hat{\mathcal{H}}_f = \left(\frac{\hbar\omega}{2}\right)\left(\hat{a}_f\,\hat{a}_f^\dagger + \hat{a}_f^\dagger\,\hat{a}_f\right) \tag{3.30b}$$

respectively.

Substituting the f-deformed boson operators $\hat{a}_f$ and $\hat{a}_f^\dagger$ from eqn. (3.28) into eqn. (3.30b), results in:

$$\hat{\mathbb{H}}_f = \left(\frac{\hbar\omega}{2}\right)\left\{\hat{N}\,f^2\left(\hat{N}\right) + \left(\hat{N}+1\right)\,f^2\left(\hat{N}+1\right)\right\} \tag{3.31}$$

Eqn. (3.31) represents the Hamiltonian operator for the f-deformed quantum harmonic oscillator in terms of the undeformed number operator $\hat{N}$.

The q-deformed number operator, $\hat{N}_q$, in terms of q-deformed boson operators is defined as [28, 32, 33, 87]:

$$\hat{N}_q = \left[\hat{N}\right]_q = \hat{a}_q^\dagger\,\hat{a}_q \tag{3.32}$$

and hence,

$$\left[\hat{N}+1\right]_q = \hat{a}_q\,\hat{a}_q^\dagger \tag{3.33}$$

Substituting eqns. (3.32) and. (3.33) in eqn. (3.30a), yields [12]:

$$\hat{\mathbb{H}}_q = \left(\frac{\hbar\omega}{2}\right)\left\{\left[\hat{N}\right]_q + \left[\hat{N}+1\right]_q\right\} \tag{3.34}$$

Eqn. (3.34) represents the Hamiltonian operator for the q-deformed quantum harmonic oscillator in terms of the undeformed number operator $\hat{N}$.

Also, the q-deformed number operator, $\hat{N}_q$, can be defined in terms of the q-number by using eqns. (3.4) and (3.6) together with $x = \hat{N}$, then eqn. (3.32) becomes [120,121]:

$$\hat{N}_q = \left[\hat{N}\right]_q = \frac{\sinh\left(\lambda \hat{N}\right)}{\sinh\left(\lambda\right)} \tag{3.35a}$$

and,

$$\hat{N}_q = \left[\hat{N}\right]_q = \frac{e^{\lambda \hat{N}} - 1}{e^{\lambda} - 1} \tag{3.35b}$$

The q-deformed boson operators act on the q-deformed number state $|n\rangle_q$ in the q-deformed Hilbert space as [12, 28, 32, 87]

$$\left.\begin{aligned} \hat{a}_q |n\rangle_q &= \sqrt{[n]_q}\ |n-1\rangle_q \\ \hat{a}_q^\dagger |n\rangle_q &= \sqrt{[n+1]_q}\ |n+1\rangle_q \end{aligned}\right\} [n]_q = [1]_q, [2]_q, \cdots \tag{3.36}$$

where,

$$|n\rangle_q = \frac{\left(a_q^\dagger\right)^n}{\sqrt{[n]_q!}}\ |0\rangle_q \tag{3.37}$$

and,

$$|0\rangle_q = |0\rangle \tag{3.38}$$

Also, the completeness relation in q-Hilbert space for $|n\rangle_q$ is given as:

$$\sum_{n=0}^{\infty} |n\rangle_{qq}\langle n| = 1 \tag{3.39}$$

Another formulation of the q-deformed quantum harmonic oscillator, which was introduced by Arik and Coon [9], is based on the definition of the Jackson derivative. The basic idea in this work depends on the basic integral [123]. For other formulations of the q-deformed quantum harmonic oscillator, one is referred to refs. [32, 38, 89, 96].

It should also be stated that, as in the q-deformed classical oscillator case, the q-deformed quantum oscillator reduces to the standard quantum oscillator in the limit $q \to 1$.

3.3 The q-Deformed Coherent States

The importance of the standard (Glauber–Sudarshan) coherent states motivated many researchers to study the q-deformation of these states [26-28, 43, 68]. In what follows, is an attempt made to give an introduction to this subject with some mathematical details that are relevant to the work in the present thesis.

3.3.1 Definition of q-Deformed Coherent States and their Generalizations

In general, the q-deformed coherent state, $|\alpha_q\rangle$ is defined as the eigenstate of the q-deformed annihilation operator [14, 15, 31, 38, 61-65, 87], or:

$$\hat{a}_q |\alpha_q\rangle = \alpha_q |\alpha_q\rangle \tag{3.40}$$

This definition produces a normalizable q-deformed coherent state.

The Hilbert space adjoint of eqn. (3.40) becomes

$$\langle\alpha_q|\hat{a}_q^{\dagger} = \langle\alpha_q|\alpha_q^{*} \tag{3.41}$$

where α_q and α_q^* represent independent arbitrary complex numbers. Also, the q-deformed coherent state reduces to the Glauber–Sudarshan state in the limit $q \to 1$. A generalization of the q-deformed coherent state was introduced by Man'ko in [61]. This generalization leads to the concept of nonlinear coherent state (NLCs). The NLCs involves the notion of the f-deformed oscillator (see eqn. (3.29)) [47, 62, 64, 68] where for each choice of the nonlinearity function $f(\hat{N})$ one gets a different nonlinear coherent state.

The q-deformed coherent state (q–CS) can be expanded in the q-Fock space as [28, 47, 64, 87]

$$\left|\alpha_q\right\rangle = \mathcal{N}_q\left(\left|\alpha_q\right|^2\right) \sum_{n=0}^{\infty} \frac{\alpha_q^n}{\sqrt{[n]_q!}} \left|n\right\rangle_q \tag{3.42}$$

where $\mathcal{N}_q\left(\left|\alpha_q\right|^2\right)$ represents the q-analog of the normalization constant defined as [28, 47, 64, 64, 87]:

$$\mathcal{N}_q\left(\left|\alpha_q\right|^2\right) = e_q^{-\frac{\left|\alpha_q\right|^2}{2}} \tag{3.43}$$

Also, the expectation value, $\bar{n}_q$, of the q-deformed number operator, $\hat{N}_q$, is given as [124]:

$$\bar{n}_q = \left\langle \alpha_q \middle| \hat{N}_q \middle| \alpha_q \right\rangle = \left\langle \alpha_q \middle| \hat{a}_q^\dagger \hat{a}_q \middle| \alpha_q \right\rangle = \alpha_q \alpha_q^* = \left|\alpha_q\right|^2 \tag{3.44}$$

3.3.2 The q-Deformed Density Operator

As it has already been illustrated in Sec. (1.2), the conventional coherent states (Glauber–Sudarshan states) and the P-representation of the density operator [40, 41, 45, 46] play a crucial role in investigating the classical limit of the undeformed quantum harmonic oscillator via the Heisenberg equation of motion [76, 78]. To generalize this notion to the q-deformed density operator, one should overcome the problem of the existence of resolution of unity for q-deformed coherent states, as not all definitions of q-CSs satisfy the resolution of unity [38,122]. This resolution of unity can be obtained by using the standard method to produce an explicit formula for the q-deformed density operator. This method is based on solving the moment problem in order to obtain the q-deformed weight function [38,122]. This weight function represents the q-deformed quasiprobability distribution function [14, 15, 38, 122], where in the limit $q \to 1$, it reduces to the conventional (undeformed) quasiprobability distribution function $\varphi^{(s)}(\alpha,\alpha^*)$ for the standard (Glauber–Sudarshan) coherent states [40, 41, 45, 46].

Chapter Four

Classical Treatment of the q-Deformed Harmonic Oscillator

Classical Treatment of the q-Deformed Harmonic Oscillator

In this chapter, the q-deformed classical harmonic oscillator is defined in more details and treated along the lines of Man'ko [91]. The treatment begins with a generalization that involves the general case of the f-oscillator where the q-oscillator is considered as a special case. Then, the classical equation of motions and the Liouville equations are obtained for both types of oscillator. Finally, solutions for the classical Liouville equations for the q-deformed case are obtained and a computer visualization method is used to investigate the phase space time-evolution of this oscillator for comparison with the anharmonic oscillator treated by Milburn [77].

4.1 The f-Deformed Coordinate Transformation

As mentioned in Sec. (2.1), α and α^* represent two independent complex variables that can be considered as coordinates in a complex phase space. When considering f-deformation [61], these undeformed coordinates can be transformed to f-deformed coordinates α_f and α_f^* by a non-linear transformation as [61, 91]:

$$\alpha_f = f\left(\alpha,\alpha^*\right)\alpha \tag{4.1}$$

$$\alpha_f^* = f^*\left(\alpha,\alpha^*\right)\alpha^* \tag{4.2}$$

where α_f and α_f^* are considered as two independent complex variables and $f = f(\alpha,\alpha^*)$ is a non-negative function of the two independent complex variables α and α^*. The subscript " f " refers to "f-deformation". Generally, the function $f(\alpha,\alpha^*)$ has the following three forms corresponding to the deformation types shown in Table (4.1):

Table (4.1)

Forms of the function $f(\alpha,\alpha^*)$ and their associated types of deformation.

$f(\alpha,\alpha^*)$	type of deformation
1	undeformed case
$\sqrt{\dfrac{\left[\alpha\alpha^*\right]_q}{\alpha\alpha^*}}$	q–deformed case
otherwise	general f–deformed case

In the case of q-deformation, $\sqrt{\dfrac{\left[\alpha\alpha^*\right]_q}{\alpha\alpha^*}}$ can also be written in terms of the q–numbers as [9, 61, 91]:

$$\sqrt{\frac{\left[\alpha\alpha^*\right]_q}{\alpha\alpha^*}} = \sqrt{\frac{\sinh(\lambda\alpha\alpha^*)}{\alpha\alpha^*\sinh(\lambda)}} \quad \text{for} \quad \left[\alpha\alpha^*\right]_q = \frac{q^{\alpha\alpha^*} - q^{-\alpha\alpha^*}}{q - q^{-1}} \tag{4.3a}$$

and,

$$\sqrt{\frac{\left[\alpha\alpha^*\right]_q}{\alpha\alpha^*}} = \sqrt{\frac{e^{\lambda\alpha\alpha^*} - 1}{\alpha\alpha^*(e^{\lambda} - 1)}} \quad \text{for} \quad \left[\alpha\alpha^*\right]_q = \frac{q^{\alpha\alpha^*} - 1}{q - 1} \tag{4.3b}$$

4.2 The Classical Hamiltonian

The classical Hamiltonian of the f-deformed oscillator can be introduced in terms of the two sets of coordinates of the non-linear transformation given in eqns. (4.1) and (4.2) as:

(A) The Classical Hamiltonian in α–Representation

The Hamiltonian of the f-deformed oscillator defined in terms of the (α,α^*) coordinates as given by Man’ko [61, 91] is,

$$\mathbb{H}_f(\alpha,\alpha^*) = \hbar\omega f\ f^*\alpha\alpha^* \tag{4.4}$$

For the case of the q-deformed classical oscillator, the function f is given by eqns. (4.3). Then, eqn. (4.4) in α-representation becomes

$$\mathbb{H}_q(\alpha,\alpha^*) = \begin{cases} \hbar\omega \left\{ \dfrac{\sinh(\lambda\alpha\alpha^*)}{\sinh(\lambda)} \right\} & \text{for } \left[\alpha\alpha^*\right]_q = \dfrac{q^{\alpha\alpha^*} - q^{-\alpha\alpha^*}}{q - q^{-1}} \quad \text{(4.5a)} \\ \hbar\omega \left\{ \dfrac{e^{\lambda\alpha\alpha^*} - 1}{\left(e^{\lambda} - 1\right)} \right\} & \text{for } \left[\alpha\alpha^*\right]_q = \dfrac{q^{\alpha\alpha^*} - 1}{q - 1} \quad \text{(4.5b)} \end{cases}$$

(B) The Classical Hamiltonian in α_f –Representation

The classical Hamiltonian of the f-deformed oscillator, defined in terms of the $\left(\alpha_f,\alpha_f^*\right)$ coordinates as given by Man’ko [61, 91], is

$$\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right) = \hbar\omega\alpha_f\alpha_f^* \tag{4.6}$$

Hence, for the special case of q-deformation, this Hamiltonian can be written as:

$$\mathcal{H}_q\left(\alpha_q,\alpha_q^*\right) = \hbar\omega\alpha_q\alpha_q^* \tag{4.7}$$

4.3 The Poisson Bracket $\{\alpha_f, \alpha_f^*\}$

The Poisson bracket $\{\alpha_f, \alpha_f^*\}$ can be written as (see Appendix–A, eqn. (A.11)):

$$\{\alpha_f, \alpha_f^*\} = \{\alpha, \alpha^*\} \cdot \{\alpha_f, \alpha_f^*\}_{\alpha,\alpha^*} \tag{4.8}$$

where,

$$\{\alpha_f, \alpha_f^*\}_{\alpha,\alpha^*} = \left(\frac{\partial \alpha_f}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha_f^*}{\partial \alpha^*}\right)_{\alpha} - \left(\frac{\partial \alpha_f}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha_f^*}{\partial \alpha}\right)_{\alpha^*} \tag{4.9}$$

Using eqns. (4.1) and (4.2), one obtains:

$$\left.\begin{aligned}
\left(\frac{\partial \alpha_f}{\partial \alpha}\right)_{\alpha^*} &= f + \alpha f_\alpha \\
\left(\frac{\partial \alpha_f}{\partial \alpha^*}\right)_{\alpha} &= \alpha f_{\alpha^*} \\
\left(\frac{\partial \alpha_f^*}{\partial \alpha}\right)_{\alpha^*} &= \alpha^* f_\alpha^* \\
\left(\frac{\partial \alpha_f^*}{\partial \alpha^*}\right)_{\alpha} &= f^* + \alpha^* f_{\alpha^*}^*
\end{aligned}\right\} \tag{4.10}$$

where the following notations have been employed,

$$\left.\begin{aligned}
f_\alpha &= \left(\frac{\partial f}{\partial \alpha}\right)_{\alpha^*} \\
f_{\alpha^*} &= \left(\frac{\partial f}{\partial \alpha^*}\right)_{\alpha} \\
f_{\alpha^*}^* &= \left(\frac{\partial f^*}{\partial \alpha^*}\right)_{\alpha} \\
f_\alpha^* &= \left(\frac{\partial f^*}{\partial \alpha}\right)_{\alpha^*}
\end{aligned}\right\} \tag{4.11}$$

then substituting eqn. (4.10) into eqn. (4.9), one can write:

$$\left\{\alpha_f , \alpha_f^*\right\}_{\alpha,\alpha^*} = f\, f^* + \alpha\, f^* f_\alpha + \alpha^* f\, f_{\alpha^*}^* + |\alpha|^2 \left(f_\alpha\, f_{\alpha^*}^* - f_\alpha^*\, f_{\alpha^*} \right) \quad (4.12)$$

For the special case of real function f, eqn. (4.12) becomes,

$$\left\{\alpha_f , \alpha_f^*\right\}_{\alpha,\alpha^*} = f^2 + f \cdot \left(\alpha\, f_\alpha + \alpha^* f_{\alpha^*} \right) \quad (4.13)$$

Now, letting

$$\chi_f\,(\alpha,\alpha^*) = f^2 + f \cdot \left(\alpha\, f_\alpha + \alpha^* f_{\alpha^*} \right) \quad (4.14)$$

then, one can write:

$$\left\{\alpha_f , \alpha_f^*\right\}_{\alpha,\alpha^*} = \chi_f\,(\alpha,\alpha^*) \quad (4.15)$$

Substituting for $\left\{\alpha , \alpha^*\right\}$ from eqn. (2.6) and using eqn. (4.15) in eqn. (4.8), then:

$$\left\{\alpha_f , \alpha_f^*\right\} = -\left(\frac{i}{\hbar}\right) \chi_f\,(\alpha,\alpha^*) \quad (4.16)$$

Eqn. (4.16) represents the general formula for the Poisson bracket of the f-deformed classical harmonic oscillator.

For the special case of the q-deformed classical oscillator, where the function f is given by eqns. (4.3), the Poisson bracket $\left\{\alpha_f , \alpha_f^*\right\}$ in eqn. (4.16) becomes $\left\{\alpha_q , \alpha_q^*\right\}$ and can then take the forms:

(A) $\left\{\alpha_q , \alpha_q^*\right\}$ in the α –Representation

It is easy to show that for f defined by eqn. (4.3a):

$$f_\alpha = \left(\frac{1}{2f}\right) \left\{ \frac{\alpha^* \left[\lambda\, |\alpha|^2 \cosh\left(\lambda |\alpha|^2\right) - \sinh\left(\lambda |\alpha|^2\right) \right]}{|\alpha|^4 \sinh(\lambda)} \right\} \quad (4.17)$$

and,

$$f_{\alpha^*} = \left(\frac{1}{2f}\right)\left\{\frac{\alpha\left[\lambda\,|\alpha|^2\cosh\left(\lambda|\alpha|^2\right) - \sinh\left(\lambda|\alpha|^2\right)\right]}{|\alpha|^4\sinh(\lambda)}\right\} \tag{4.18}$$

Then,

$$\alpha f_\alpha = \alpha^* f_{\alpha^*} = \left(\frac{1}{2f}\right)\left\{\frac{\lambda\cosh\left(\lambda|\alpha|^2\right)}{\sinh(\lambda)} - f^2\right\} \tag{4.19}$$

Similarly, for f defined by eqn. (4.3b):

$$f_\alpha = \left(\frac{1}{2f}\right)\left\{\frac{\alpha^*\left[\lambda\,|\alpha|^2 e^{\lambda|\alpha|^2} - \left(e^{\lambda|\alpha|^2} - 1\right)\right]}{|\alpha|^4\left(e^{\lambda} - 1\right)}\right\} \tag{4.20}$$

$$f_{\alpha^*} = \left(\frac{1}{2f}\right)\left\{\frac{\alpha\left[\lambda\,|\alpha|^2 e^{\lambda|\alpha|^2} - \left(e^{\lambda|\alpha|^2} - 1\right)\right]}{|\alpha|^4\left(e^{\lambda} - 1\right)}\right\} \tag{4.21}$$

Hence,

$$\alpha f_\alpha = \alpha^* f_{\alpha^*} = \left(\frac{1}{2f}\right)\left\{\frac{\lambda e^{\lambda|\alpha|^2}}{\left(e^{\lambda} - 1\right)} - f^2\right\} \tag{4.22}$$

Substituting eqns. (4.19) and (4.22) into eqn. (4.14), one obtains:

$$\chi_q(\alpha,\alpha^*) = \left\{\frac{\lambda\cosh\left(\lambda|\alpha|^2\right)}{\sinh(\lambda)}\right\} \tag{4.23a}$$

and,

$$\chi_q(\alpha,\alpha^*) = \left\{\frac{\lambda e^{\lambda|\alpha|^2}}{\left(e^{\lambda} - 1\right)}\right\} \tag{4.23b}$$

respectively.

When these two expressions for $\chi_q(\alpha,\alpha^*)$ are substituted in eqn. (4.16), they give

$$\left\{\alpha_q,\alpha_q^*\right\}=-\left(\frac{i}{\hbar}\right)\left\{\frac{\lambda\cosh\left(\lambda|\alpha|^2\right)}{\sinh(\lambda)}\right\} \tag{4.24a}$$

and,

$$\left\{\alpha_q,\alpha_q^*\right\}=-\left(\frac{i}{\hbar}\right)\left\{\frac{\lambda e^{\lambda|\alpha|^2}}{\left(e^{\lambda}-1\right)}\right\} \tag{4.24b}$$

respectively.

Eqns. (4.24) represent the Poisson brackets for the q-deformed harmonic oscillator in the α–representation for two definitions of the function f. It is observed that eqn. (4.24a) is the same as that introduced by Man'ko [91].

(B) $\left\{\alpha_q,\alpha_q^*\right\}$ in the α_q–Representation

To obtain the Poisson bracket $\left\{\alpha_f,\alpha_f^*\right\}$ in the α_q–representation one needs the inverse function $f^{-1}=f^{-1}\left(\alpha_q,\alpha_q^*\right)$. Substituting the definitions of the function f from eqns. (4.3) in the non-linear transformation represented by eqns. (4.1) and (4.2), then multiplying these two equations with each other, one gets:

$$|\alpha|^2=\frac{\sinh^{-1}\left(|\alpha_q|^2\sinh(\lambda)\right)}{\lambda} \tag{4.25a}$$

and,

$$|\alpha|^2=\frac{\ln\left(1-|\alpha_q|^2\left(1-e^{\lambda}\right)\right)}{\lambda} \tag{4.25b}$$

respectively.

The inverse of the non-linear transformation represented by eqns. (4.1) and (4.2) is given as:

$$\alpha = f^{-1}\alpha_f \tag{4.26}$$

$$\alpha^* = \left(f^*\right)^{-1}\alpha_f^* \tag{4.27}$$

Then, multiplying these two equations one gets:

$$|\alpha|^2 = \left(f^{-1}\right)^2\left|\alpha_q\right|^2 \tag{4.28}$$

Substituting eqns. (4.25) into eqn. (4.28) and solving for f^{-1}, one obtains:

$$f^{-1} = \sqrt{\frac{\ln\left(|\alpha_q|^2\sinh(\lambda)+\sqrt{1+|\alpha_q|^4\sinh^2(\lambda)}\right)}{\lambda|\alpha_q|^2}} \tag{4.29a}$$

and

$$f^{-1} = \sqrt{\frac{\ln\left(1-|\alpha_q|^2\left(1-e^{\lambda}\right)\right)}{\lambda|\alpha_q|^2}} \tag{4.29b}$$

respectively.

Eqn. (4.29a) is the same as that introduced by Man'ko [61]. Substituting eqns. (4.29) into eqn. (4.28) and substituting the result into eqns. (4.23) and after some mathematical manipulations one obtains:

$$\chi_q\left(\alpha_q,\alpha_q^*\right) = \left\{\frac{\lambda\sqrt{1+|\alpha_q|^4\sinh^2(\lambda)}}{\sinh(\lambda)}\right\} \tag{4.30a}$$

and,

$$\chi_q\left(\alpha_q,\alpha_q^*\right) = \left\{\frac{\lambda\left[1-|\alpha_q|^2\left(1-e^{\lambda}\right)\right]}{\left(e^{\lambda}-1\right)}\right\} \tag{4.30b}$$

respectively.

Finally, substituting eqns. (4.30) into eqn. (4.16), the Poisson brackets in the α_q –representation become:

$$\left\{\alpha_q,\alpha_q^*\right\}=-\left(\frac{i}{\hbar}\right)\left\{\frac{\lambda\left[\sqrt{1+|\alpha_q|^4\sinh^2(\lambda)}\right]}{\sinh(\lambda)}\right\} \tag{4.31a}$$

and,

$$\left\{\alpha_q,\alpha_q^*\right\}=-\left(\frac{i}{\hbar}\right)\left\{\frac{\lambda\left[1-|\alpha_q|^2\left(1-e^{\lambda}\right)\right]}{\left(e^{\lambda}-1\right)}\right\} \tag{4.31b}$$

respectively.

It is noticed that eqn. (4.31a) is the same as that introduced by Man’ko [91].

4.4 The Equation of Motion

The equation of motion for the f–deformed harmonic oscillator can be obtained in the α -representation and α_q -representation as:

(A) α –Representation:

The equation of motion in this representation is:

$$\dot{\alpha}(t)=\left\{\alpha,\mathbb{H}_f\left(\alpha,\alpha^*\right)\right\} \tag{4.32}$$

But since, (see Appendix-B, eqn. (B.7))

$$\left\{\alpha,\mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}=\left\{\alpha,\alpha^*\right\}\cdot\left\{\alpha,\mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} \tag{4.33}$$

then, simplifying $\left\{\alpha, \mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*}$ by using eqn. (2.13), to obtain

$$\left\{\alpha, \mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \left(\frac{\partial \mathbb{H}_f\left(\alpha,\alpha^*\right)}{\partial \alpha^*}\right)_{\alpha} \tag{4.34}$$

and substituting $\mathbb{H}_f\left(\alpha,\alpha^*\right)$ from eqn. (4.4) into eqn. (4.34), this equation becomes:

$$\left\{\alpha, \mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \hbar\omega \left\{f\, f^* + \alpha^* f^* f_{\alpha^*} + \alpha^* f\, f^*_{\alpha^*}\right\}\alpha \tag{4.35}$$

Observing again that $f = f^*$, and defining:

$$\eta_f\left(\alpha,\alpha^*\right) = f^2 + 2\alpha^* f\, f_{\alpha^*}\,, \tag{4.36}$$

eqn. (4.35) takes the form:

$$\left\{\alpha, \mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \hbar\omega\, \eta_f\left(\alpha,\alpha^*\right)\alpha \tag{4.37}$$

Defining

$$\omega_f = \omega\, \eta_f\left(\alpha,\alpha^*\right) \tag{4.38}$$

eqn. (4.37) becomes:

$$\left\{\alpha, \mathbb{H}_f\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \hbar\omega_f\, \alpha \tag{4.39}$$

where ω_f can be considered as the frequency of the f–deformed classical harmonic oscillator in the α–representation.

Substituting $\left\{\alpha,\alpha^*\right\}$ from eqn. (2.6) and using eqn. (4.39) in eqn. (4.33), eqn. (4.32) can be cast in the form:

$$\dot{\alpha}(t) = -i\,\omega_f\,\alpha \tag{4.40}$$

Eqn. (4.40) represents the equation of motion for the f-deformed classical harmonic oscillator in the α –representation.

For the special case of the q-deformed classical oscillator, where the function f is given by eqns. (4.3), substituting for $\alpha^* f_{\alpha^*}$ from eqns. (4.19) and (4.22) into eqn. (4.36), the result becomes:

$$\eta_q(\alpha,\alpha^*) = \left\{ \frac{\lambda \cosh\left(\lambda|\alpha|^2\right)}{\sinh(\lambda)} \right\} \tag{4.41a}$$

and,

$$\eta_q(\alpha,\alpha^*) = \left\{ \frac{\lambda e^{\lambda|\alpha|^2}}{\left(e^{\lambda}-1\right)} \right\} \tag{4.41b}$$

respectively.

Then, substituting eqns. (4.41) into eqn. (4.38), the frequency of the q-deformed classical oscillator, $\omega_q^{(\mu)}$, becomes:

$$\omega_q^{(1)} = \omega \left\{ \frac{\lambda \cosh\left(\lambda|\alpha|^2\right)}{\sinh(\lambda)} \right\} \tag{4.42a}$$

and,

$$\omega_q^{(2)} = \omega \left\{ \frac{\lambda e^{\lambda|\alpha|^2}}{\left(e^{\lambda}-1\right)} \right\} \tag{4.42b}$$

respectively.

Again, it is noticed that eqn. (4.42a) is the same as that introduced by Man’ko [91].

Substituting eqns. (4.42) into eqn. (4.40), leads to:

$$\dot{\alpha}(t) = -i\,\omega_q^{(\mu)}\,\alpha \qquad ; \quad \mu = 1, 2. \tag{4.43}$$

and the complex conjugate of eqn. (4.43) is:

$$\dot{\alpha}^*(t) = i\,\omega_q^{(\mu)}\,\alpha^* \qquad ; \quad \mu = 1, 2. \tag{4.44}$$

where $\omega_q^{(\mu)}$ is obviously real.

Eqns. (4.43) and (4.44) represent the equations of motion for the q-deformed classical harmonic oscillator in the α-representation.

Solving these equations of motion, gives the equations of trajectories for the q-deformed classical harmonic oscillator in the complex (α,α^*) phase space:

$$\left.\begin{aligned} \alpha(t) &= \alpha(0)\, e^{-i\,\omega_q^{(\mu)} t} \\ \alpha^*(t) &= \alpha^*(0)\, e^{i\,\omega_q^{(\mu)} t} \end{aligned}\right\} \quad ; \quad \mu = 1,2. \tag{4.45}$$

where, $\alpha(0)$ and $\alpha^*(0)$ are initial trajectory points at $t=0$.

(B) α_f –Representation:

In this case, the equation of motion is given by:

$$\dot{\alpha}_f(t) = \left\{\alpha_f, \mathcal{H}_f\left(\alpha_f, \alpha_f^*\right)\right\} \tag{4.46}$$

where, (see Appendix-B, eqn. (B.8))

$$\left\{\alpha_f, \mathcal{H}_f\left(\alpha_f, \alpha_f^*\right)\right\} = \left\{\alpha_f, \alpha_f^*\right\} \cdot \left\{\alpha_f, \mathcal{H}_f\left(\alpha_f, \alpha_f^*\right)\right\}_{\alpha_f, \alpha_f^*} \tag{4.47}$$

Also,

$$\left\{\alpha_f, \mathcal{H}_f\left(\alpha_f, \alpha_f^*\right)\right\}_{\alpha_f, \alpha_f^*} = \left(\frac{\partial \mathcal{H}_f\left(\alpha_f, \alpha_f^*\right)}{\partial \alpha_f^*}\right)_{\alpha_f} \tag{4.48}$$

where the counterparts of eqns. (2.13) for α_f and α_f^*, i.e.,

$$\left.\begin{aligned}\left(\frac{\partial\alpha_f}{\partial\alpha_f}\right)_{\alpha_f^*} &= \left(\frac{\partial\alpha_f^*}{\partial\alpha_f^*}\right)_{\alpha_f} = 1\\ \left(\frac{\partial\alpha_f}{\partial\alpha_f^*}\right)_{\alpha_f} &= \left(\frac{\partial\alpha_f^*}{\partial\alpha_f}\right)_{\alpha_f^*} = 0\end{aligned}\right\} \tag{4.49}$$

were used to obtain eqn.(4.48).

Substituting $\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right)$ from eqn. (4.6) into eqn. (4.48), the result is:

$$\left\{\alpha_f,\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right)\right\}_{\alpha_f,\alpha_f^*} = \hbar\omega\,\alpha_f \tag{4.50}$$

Then, finally the substitution of $\left\{\alpha_f,\alpha_f^*\right\}$ from eqn. (4.16) and the Poisson bracket of eqn. (4.50) into eqn. (4.47), and using the result in eqn. (4.46), yields:

$$\dot{\alpha}_f\left(t\right) = -i\,\omega_f\;\alpha_f \tag{4.51}$$

where,

$$\omega_f = \omega\,\chi_f\;\left(\alpha,\alpha^*\right) \tag{4.52}$$

It is worth mentioning that $\left|\alpha\right|^2$ represents a constant of the motion for the undeformed classical oscillator. This can be proved by evaluating the Poisson bracket for the undeformed classical oscillator $\left\{\alpha\alpha^*,H\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*}$ as:

$$\left\{\alpha\alpha^*,H\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \left(\frac{\partial\left(\alpha\alpha^*\right)}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial H\left(\alpha,\alpha^*\right)}{\partial\alpha^*}\right)_{\alpha} - \left(\frac{\partial\left(\alpha\alpha^*\right)}{\partial\alpha^*}\right)_{\alpha}\left(\frac{\partial H\left(\alpha,\alpha^*\right)}{\partial\alpha}\right)_{\alpha^*} \tag{4.53}$$

Therfore, using eqns. (2.13) in eqn. (4.53) leads to:

$$\left\{\alpha\alpha^*, H\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = \alpha^* \left(\frac{\partial H\left(\alpha,\alpha^*\right)}{\partial \alpha^*}\right)_{\alpha} - \alpha \left(\frac{\partial H\left(\alpha,\alpha^*\right)}{\partial \alpha}\right)_{\alpha^*} \quad (4.54)$$

Substituting the Hamiltonian from eqn. (2.9) in eqn. (4.54) results in:

$$\left\{\alpha\alpha^*, H\left(\alpha,\alpha^*\right)\right\}_{\alpha,\alpha^*} = 0 \quad (4.55)$$

which proves that $|\alpha|^2$ is a constant of the motion for the undeformed oscillator. A similar result can be obtained for $|\alpha_f|^2$ and, hence, for the special case $|\alpha_q|^2$. Therefore, the functions $\chi_f\left(\alpha,\alpha^*\right)$ and $\eta_f\left(\alpha,\alpha^*\right)$ also represent constants of the motion, where one can prove this by noticing that $f = f\left(|\alpha|^2\right)$, then substituting into eqn.(4.14), and after some mathematical manipulations, the result is:

$$\chi_f\left(|\alpha|^2\right) = \left\{f\left(|\alpha|^2\right)\right\}^2 + 2f\left(|\alpha|^2\right)\frac{\partial f\left(|\alpha|^2\right)}{\partial |\alpha|^2} \quad (4.56)$$

It is obvious that the f is a function of $|\alpha|^2$. Then, the function $\chi_f\left(\alpha,\alpha^*\right)$, that is given by eqn. (4.14), reduces to the function $\chi_f\left(|\alpha|^2\right)$. Similarly, one can prove that the function $\eta_f\left(\alpha,\alpha^*\right)$ also represents a constant of the motion.

In the same context, the frequency of the f-deformed classical oscillator, ω_f, which is a function of $|\alpha|^2$, is a constant of the motion and depends on the energy of the oscillator orbit.

The equation of motion for the special case of the q-deformed classical harmonic oscillator is obtained by substituting the function $\chi_q\left(\alpha_q,\alpha_q^*\right)$ from eqns. (4.30) into eqn. (4.52). Then, the frequencies of this oscillator are given as:

$$\omega_q^{(3)} = \omega \left\{ \frac{\lambda\left[\sqrt{1+|\alpha_q|^4 \sinh^2(\lambda)}\ \right]}{\sinh(\lambda)} \right\} \tag{4.57a}$$

and,

$$\omega_q^{(4)} = \omega \left\{ \frac{\lambda\left[1-|\alpha_q|^2\left(1-e^{\lambda}\right)\right]}{\left(e^{\lambda}-1\right)} \right\} \tag{4.57b}$$

respectively.

It is noticed that eqn. (4.57a) is the same as that introduced by Man'ko [91].

Substituting eqns. (4.57) into eqn. (4.51), one gets:

$$\dot{\alpha}_q(t) = -\,i\,\omega_q^{(\mu)}\,\alpha_q \qquad ; \quad \mu = 3, 4. \tag{4.58}$$

The complex conjugate of eqn. (4.58) becomes:

$$\dot{\alpha}_q^*(t) = \ i\,\omega_q^{(\mu)}\,\alpha_q^* \qquad ; \quad \mu = 3, 4. \tag{4.59}$$

Eqns. (4.58) and (4.59) represent the equations of motion for the q-deformed classical harmonic oscillator in the α_q-representation.

Again, solving these equations of motion, gives the equations of trajectories for the q-deformed classical harmonic oscillator in the complex $\left(\alpha_q,\alpha_q^*\right)$ phase space:

$$\left.\begin{aligned} \alpha_q(t) &= \alpha_q(0)\, e^{-i\,\omega_q^{(\mu)} t} \\ \alpha_q^*(t) &= \alpha_q^*(0)\, e^{i\,\omega_q^{(\mu)} t} \end{aligned}\right\} \quad ; \quad \mu = 3, 4. \tag{4.60}$$

where $\alpha_q(0)$ and $\alpha_q^*(0)$ are initial trajectory points at $t = 0$.

4.5 The Classical Liouville Equation

In this section, the classical Liouville equations for the f-deformed classical harmonic oscillator in the two complex phase space representations are derived by using the classical Liouville equation for the undeformed classical system that was introduced in ref. [100]. The q-deformed case is also treated as a special case.

(A) α–Representation

In this case, and using the Hamiltonian $\mathbb{H}_f(\alpha, \alpha^*)$ of eqn. (4.4), the classical Liouville equation is given as:

$$\frac{\partial \mathcal{P}_{CL}^{f}(\alpha, \alpha^*; t)}{\partial t} = \left\{ \mathbb{H}_f(\alpha, \alpha^*), \mathcal{P}_{CL}^{f}(\alpha, \alpha^*; t) \right\} \tag{4.61}$$

where $P_{CL}^{f}(\alpha, \alpha^*; t)$ represents the classical probability distribution function for the f-deformed classical oscillator in the α-representation.
But since, (see Appendix-C, eqn. (C.9))

$$\left\{ \mathbb{H}_f(\alpha, \alpha^*), \mathcal{P}_{CL}^{f}(\alpha, \alpha^*; t) \right\} = \left\{ \alpha, \alpha^* \right\} \cdot \left\{ \mathbb{H}_f(\alpha, \alpha^*), \mathcal{P}_{CL}^{f}(\alpha, \alpha^*; t) \right\}_{\alpha, \alpha^*} \tag{4.62}$$

then, using the definition of the Hamiltonian $\mathbb{H}_f(\alpha,\alpha^*)$ in the α –representation as in eqn. (4.4), and assuming $f = f^*$, one obtains:

$$\left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial \alpha}\right)_{\alpha^*} = \hbar\omega\left\{f^2 + 2\alpha f\, f_\alpha\right\}\alpha^* \tag{4.63}$$

and,

$$\left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial \alpha^*}\right)_{\alpha} = \hbar\omega\left\{f^2 + 2\alpha^* f\, f_{\alpha^*}\right\}\alpha \tag{4.64}$$

Furthermore, the Poisson bracket $\left\{\mathbb{H}_f(\alpha,\alpha^*), \mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}_{\alpha,\alpha^*}$ is given by:

$$\left\{\mathbb{H}_f(\alpha,\alpha^*), \mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}_{\alpha,\alpha^*} =$$
$$\left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial \alpha}\right)_{\alpha^*}\left(\frac{\partial \mathcal{P}_{CL}^f(\alpha,\alpha^*;t)}{\partial \alpha^*}\right) - \left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial \alpha^*}\right)_{\alpha}\left(\frac{\partial \mathcal{P}_{CL}^f(\alpha,\alpha^*;t)}{\partial \alpha}\right) \tag{4.65}$$

Now, substituting eqns. (4.63) and (4.64) into eqn. (4.65), the latter takes the form:

$$\left\{\mathbb{H}_f(\alpha,\alpha^*), \mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}_{\alpha,\alpha^*} =$$
$$\hbar\omega\left\{(f^2 + 2\alpha f\, f_\alpha)\alpha^*\frac{\partial}{\partial\alpha^*} - \left(f^2 + 2\alpha^* f\, f_{\alpha^*}\right)\alpha\frac{\partial}{\partial\alpha}\right\}\mathcal{P}_{CL}^f(\alpha,\alpha^*;t) \tag{4.66}$$

Then, substituting eqns. (4.66) and the Poisson bracket $\left\{\alpha,\alpha^*\right\}$ from eqn. (2.6) into eqn. (4.62), one can write:

$$\left\{\mathbb{H}_f(\alpha,\alpha^*),\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}=$$
$$-i\omega\left\{(f^2+2\alpha f\ f_\alpha)\alpha^*\frac{\partial}{\partial\alpha^*}-\left(f^2+2\alpha^* f\ f_{\alpha^*}\right)\alpha\frac{\partial}{\partial\alpha}\right\}\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)$$
(4.67)

Using eqn. (4.67) in eqn. (4.61), one obtains:

$$\frac{\partial\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)}{\partial t}=$$
$$-i\omega\left\{(f^2+2\alpha f\ f_\alpha)\alpha^*\frac{\partial}{\partial\alpha^*}-\left(f^2+2\alpha^* f\ f_{\alpha^*}\right)\alpha\frac{\partial}{\partial\alpha}\right\}\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)$$
(4.68)

Noting here that if the function $f=f(\alpha,\alpha^*)$ is a function of $|\alpha|^2$ (i.e.; $f=f\left(|\alpha|^2\right)$), then $\alpha f_\alpha=\alpha^* f_{\alpha^*}$. Also, $\chi_f(\alpha,\alpha^*)=\eta_f(\alpha,\alpha^*)$ (see eqns. (4.14) and (4.36)). Therefore, eqn. (4.68) becomes

$$\frac{\partial\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)}{\partial t}=-i\omega_f\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)\mathcal{P}_{CL}^f(\alpha,\alpha^*;t) \qquad (4.69)$$

where $\omega_f=\omega\chi_f(\alpha,\alpha^*)$ and $\chi_f(\alpha,\alpha^*)$ is as given by eqn. (4.14).

Eqn. (4.69) represents the classical Liouville equation for the f-deformed classical oscillator in the α-representation.

For the special case of the q-deformed classical oscillator, the classical Liouville equation (4.69) for the f-deformed oscillator becomes:

$$\frac{\partial \mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t)}{\partial t} = -i\,\omega_q^{(\mu)}\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t) \qquad ; \ \mu = 1,2.$$

(4.70)

where, $\omega_q^{(\mu)}$ is given by eqns. (4.42).

Eqn. (4.70) represents the classical Liouville equation for the q-deformed classical oscillator in the α-representation.

(B) α_f-Representation

For this case, the classical Liouville equation is given as:

$$\frac{\partial P_{CL}^{f}(\alpha_f,\alpha_f^*;t)}{\partial t} = \left\{\mathcal{H}_f(\alpha_f,\alpha_f^*), P_{CL}^{f}(\alpha_f,\alpha_f^*;t)\right\} \tag{4.71}$$

where $P_{CL}^{f}(\alpha_f,\alpha_f^*;t)$ represents the classical probability distribution function for the f-deformed classical harmonic oscillator in the α_f-representation.

But since, (see Appendix-C, eqn. (C.8))

$$\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*), P_{CL}^{f}(\alpha_f,\alpha_f^*;t)\right\} = \left\{\alpha_f,\alpha_f^*\right\}\cdot\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*), P_{CL}^{f}(\alpha_f,\alpha_f^*;t)\right\}_{\alpha_f,\alpha_f^*}$$

(4.72)

then, using the definition of the Hamiltonian $\mathcal{H}_f(\alpha_f,\alpha_f^*)$ in the α_f-representation as in eqn. (4.6), and assuming $f = f^*$, one obtains:

$$\left(\frac{\partial \mathcal{H}_f(\alpha_f,\alpha_f^*)}{\partial \alpha_f}\right)_{\alpha_f^*} = \hbar\,\omega\,\alpha_f^* \tag{4.73}$$

and,

$$\left(\frac{\partial \mathcal{H}_f(\alpha_f,\alpha_f^*)}{\partial \alpha_f^*}\right)_{\alpha_f} = \hbar\omega\alpha_f \tag{4.74}$$

Also, since the Poisson bracket $\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*),P_{CL}^f(\alpha_f,\alpha_f^*;t)\right\}_{\alpha_f,\alpha_f^*}$ is defined as:

$$\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*),P_{CL}^f(\alpha_f,\alpha_f^*;t)\right\}_{\alpha_f,\alpha_f^*} = \left(\frac{\partial \mathcal{H}_f(\alpha_f,\alpha_f^*)}{\partial \alpha_f}\right)_{\alpha_f^*}$$

$$\cdot\left(\frac{\partial P_{CL}^f(\alpha_f,\alpha_f^*;t)}{\partial \alpha_f^*}\right) - \left(\frac{\partial \mathcal{H}_f(\alpha_f,\alpha_f^*)}{\partial \alpha_f^*}\right)_{\alpha_f}\left(\frac{\partial P_{CL}^f(\alpha_f,\alpha_f^*;t)}{\partial \alpha_f}\right) \tag{4.75}$$

then, substituting eqns. (4.73) and (4.74) into (4.75) and re-arranging, the result takes the form:

$$\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*),P_{CL}^f(\alpha_f,\alpha_f^*;t)\right\}_{\alpha_f,\alpha_f^*} =$$

$$\hbar\omega\left(\alpha_f^*\frac{\partial}{\partial\alpha_f^*} - \alpha_f\frac{\partial}{\partial\alpha_f}\right)P_{CL}^f(\alpha_f,\alpha_f^*;t) \tag{4.76}$$

Now, substituting eqn. (4.76) and the Poisson bracket $\left\{\alpha_f,\alpha_f^*\right\}$ of eqn. (4.16) into eqn. (4.72), one obtains:

$$\left\{\mathcal{H}_f(\alpha_f,\alpha_f^*),P_{CL}^f(\alpha_f,\alpha_f^*;t)\right\} =$$

$$-i\omega_f\left(\alpha_f^*\frac{\partial}{\partial\alpha_f^*} - \alpha_f\frac{\partial}{\partial\alpha_f}\right)P_{CL}^f(\alpha_f,\alpha_f^*;t) \tag{4.77}$$

where $\omega_f = \omega\chi_f(\alpha,\alpha^*)$ and $\chi_f(\alpha,\alpha^*)$ are as defined in eqn. (4.14).

Inserting eqn. (4.77) into eqn. (4.71) one obtains:

$$\frac{\partial P_{CL}^{f}\left(\alpha_f,\alpha_f^*;t\right)}{\partial t}=-i\omega_f\left(\alpha_f^*\frac{\partial}{\partial\alpha_f^*}-\alpha_f\frac{\partial}{\partial\alpha_f}\right)P_{CL}^{f}\left(\alpha_f,\alpha_f^*;t\right) \qquad (4.78)$$

Eqn. (4.78) represents the classical Liouville equation for the f-deformed classical harmonic oscillator in the α_f -representation.

Also, for the special case of the q-deformed classical oscillator, the classical Liouville equation (4.78) becomes:

$$\frac{\partial P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)}{\partial t}=-i\omega_q^{(\mu)}\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right)P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right) \quad ; \quad \mu=3,4. \qquad (4.79)$$

where, $\omega_q^{(\mu)}$ is given by eqns. (4.57). Eqn. (4.79) represents the classical Liouville equation for the q-deformed classical oscillator in the α_q-representation.

It is obvious that in the limit $f\rightarrow 1$, the classical Liouville eqns. (4.69) and (4.78) for the case of the f-deformed classical harmonic oscillator in the α- and α_f - representation respectively, reduce to the classical Liouville equation of the undeformed simple harmonic oscillator (i.e., eqn. (2.69)) in the α - representation as expected.

Similarly, the same result can be attainable in the limit $q\rightarrow 1$for the classical Liouville eqns. (4.70) and (4.79) of the q-deformed classical harmonic oscillator in the α - and α_q - representation respectively.

4.6 Solution of the Classical Liouville Equation

Solutions of the classical Liouville equations of the q-deformed harmonic oscillator, (i.e., eqns. (4.70) and (4.79)) can be obtained by using the method of characteristics [77] in the same manner performed by Milburn [77] for the anharmonic oscillator. Hence, assuming the following initial solution at $t=0$:

$$\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};0\right)=e^{-\left|\alpha-\alpha(0)\right|^{2}} \quad \text{(in the } \alpha \text{-representation)} \tag{4.80}$$

then, the time-evolution for each point (α,α^{*}) in complex phase space can be obtained by replacing α by $\alpha(t)$ given by [77]:

$$\alpha(t)=\alpha e^{-i\omega_{q}^{(\mu)}t} \tag{4.81}$$

Substituting eqn. (4.81) into eqn. (4.80), one obtains:

$$\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)=e^{-\left|\alpha e^{-i\omega_{q}^{(\mu)}t}-\alpha(0)\right|^{2}} \quad ; \quad \mu=1,2. \tag{4.82}$$

Similarly, in the α_q -representation:

$$P_{CL}^{q}\left(\alpha_{q},\alpha_{q}^{*};t\right)=e^{-\left|\alpha_{q}e^{-i\omega_{q}^{(\mu)}t}-\alpha_{q}(0)\right|^{2}} \quad ; \quad \mu=3,4. \tag{4.83}$$

Direct insertion of the solutions given in eqns. (4.82) and (4.83) into the classical Liouville eqns. (4.70) and (4.79) has verified that these solutions indeed satisfy these equations.

4.7 Computer Visualizations of the q-Deformed Classical Harmonic Oscillator

The computer visualization method introduced in ref. [77] was utilized by Milburn to investigate the time-evolution of the probability distribution function for the anharmonic oscillator in phase-space using the Q-function which was defined in Sec. (1.4). It is noted that the frequency of the anhrmonic oscillator treated by Milburn [77] is a function of $|\alpha|^{2}$, which is the same case as that of the q-deformed oscillator where the frequency of this oscillator is also a function of $|\alpha|^2$ or $|\alpha_q|^2$ depending on the representation. This motivates using the same method as that used by Milburn [77] to investigate the behavior of the classical probability distribution functions $\mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t)$ and $P_{CL}^{q}(\alpha_q,\alpha_q^*;t)$ in the case of the q-deformed oscillator in the α and α_q representations respectively.

In the present work, the computer visualization method was implemented by writing a computer program using Mathematica® [125]. The complex coordinate α is given by eqn. (2.4) and the mass of the oscillator, the initial value for the complex coordinate, $\alpha(0)$, and the momentum p_0 are taken as 1, 0.5 and 0 respectively. The position coordinate is measured in units of $\sqrt{\frac{2\hbar}{m\omega}}$ and the momentum coordinate in units of $\sqrt{2\hbar m\omega}$. It should also be mentioned that all the computer visualizations are performed in the rotating frame [77].

To verify the reliability of this program, the same results that were obtained by Milburn [77] were reproduced by applying the computer program using eqn. (4.82) with $Q=\mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t)$. The results obtained are illustrated in Fig. (4.1).

Hence, the time evolution of the classical probability distribution functions $\mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t)$ and $P_{CL}^{q}(\alpha_q,\alpha_q^*;t)$ can be represented in phase space through the behavior of two particular initial contours $|\alpha-\alpha(0)|=\frac{1}{2}$ and $|\alpha_q-\alpha_q(0)|=\frac{1}{2}$ centered at $\alpha(0)$ and $\alpha_q(0)=0.5$ respectively [77]. In time $(\tau=\omega t)$, each point on an initial contour will move according to eqns. (4.45) and (4.60), and the evolution of this initial contour in the time interval $0\le\tau\le2\pi$ in the phase space region $-1 \le q \le 1$ and $-1 \le p \le 1$ is followed. The results of such a procedure are depicted in Figs. (4.2) and (4.3) where the time-evolution of the 2-D probability distribution is shown. These figures exhibit whorl shapes and can be compared with those obtained by Milburn [77] for the anharmonic oscillator as shown in Fig. (4.1). Also, it is obvious that these whorl shapes become finer as $t \to \infty$.

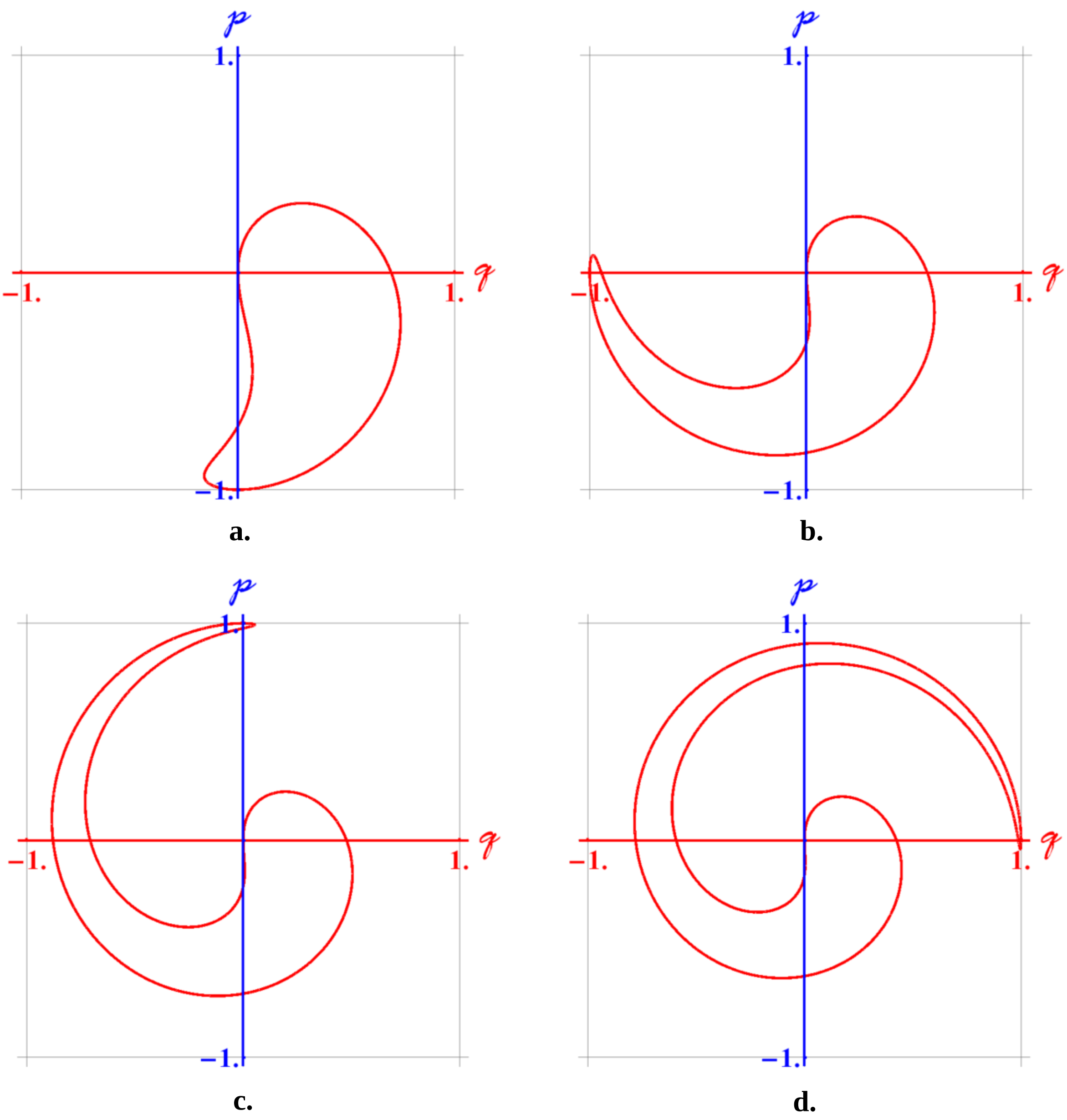

Fig. (4.1): The 2-D time-evolution contours of the classical probability distribution function Q for the anharmonic oscillator in phase space, for different values of time (τ): (a) $\tau = \pi/2$, (b) $\tau = \pi$, (c) $\tau = 3\pi/2$, and (d) $\tau = 2\pi$.

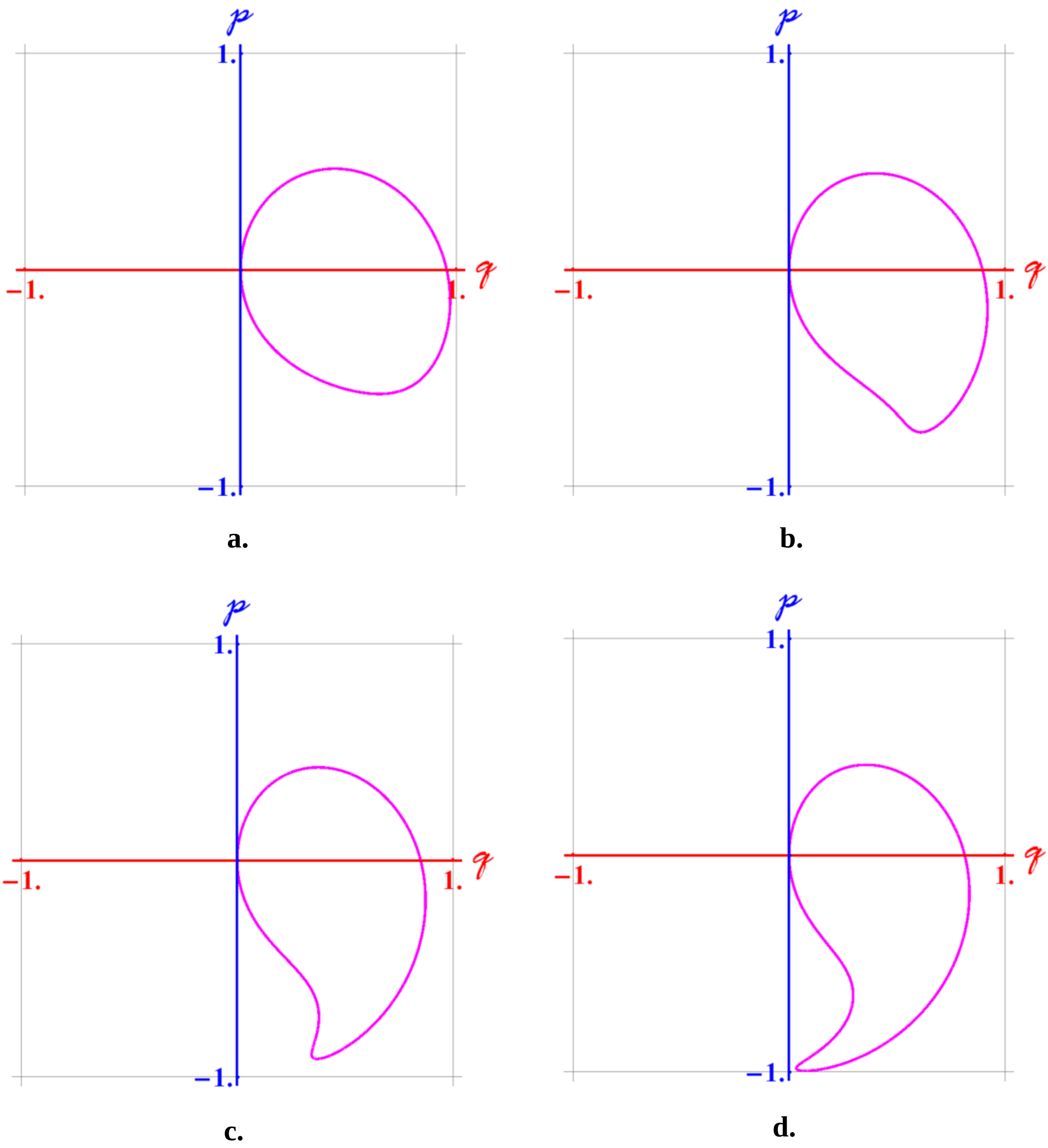

Fig. (4.2): The 2-D time-evolution contours of the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_{q}^{(1)}$ given by eqn. (4.42a) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

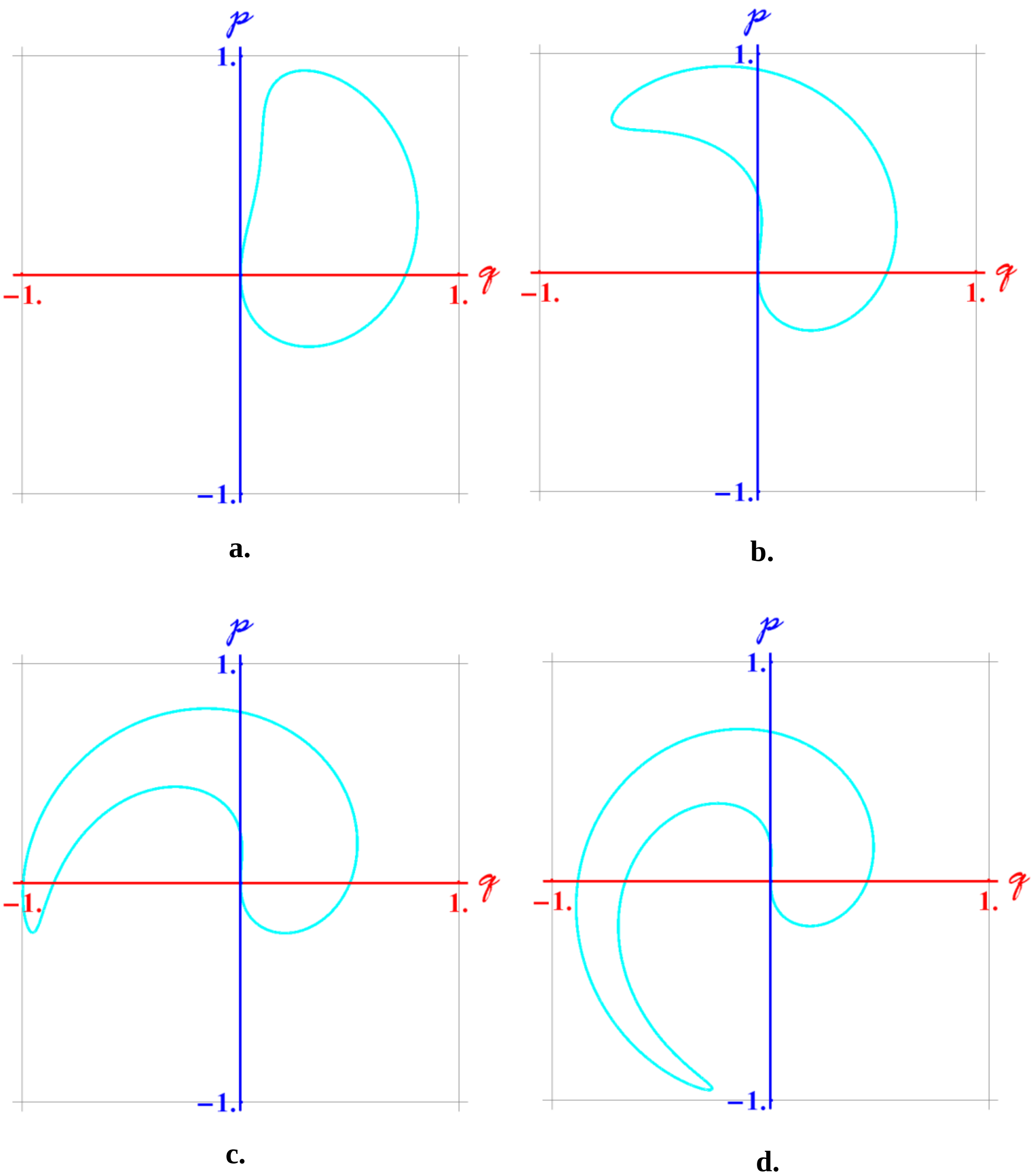

Fig. (4.3): The 2-D time-evolution contours of the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_{q}^{(2)}$ given by eqn. (4.42b) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

It is observed that the behavior of the classical probability distribution function $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ in phase space is similar to the behavior shown in Figs. (4.2) and (4.3) for the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$. This similarity is a result of the fact that the expressions for the frequencies $\omega_q^{(1)}$ and $\omega_q^{(2)}$ in α-representation, are equivalent to the expressions for the frequencies $\omega_q^{(3)}$ and $\omega_q^{(4)}$ in the α_q-representation (see eqns. (4.42) and (4.57)).

In Figs. (4.4) - (4.6), results of 3-D time-evolution of the same classical probability distributions are presented. Fig. (4.4) shows the Q function which corresponds to Fig. (4.1) presented as 3-D plot. The eqns. (4.82) and (4.83) have been used to calculate the values of the classical probability distributions $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$ and $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$. All of these figures are presented for $-4\leq \mathscr{q} \leq 4$ and $-4\leq \mathscr{p} \leq 4$. It is clear that from all of these figures that the peaks of the q-deformed Gaussian for the classical probability distributions Q and $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$ do not change with time and are equal to the maximum value (i.e., 1). These peaks follow the classical trajectories shown in Figs. (4.1) - (4.3), for the probability distribution functions.

Another noticeable feature is the observation that the Gaussian shapes of these distributions become more convoluted around themselves as $t\to\infty$, which is clear in Figs. (4.4) - (4.6).

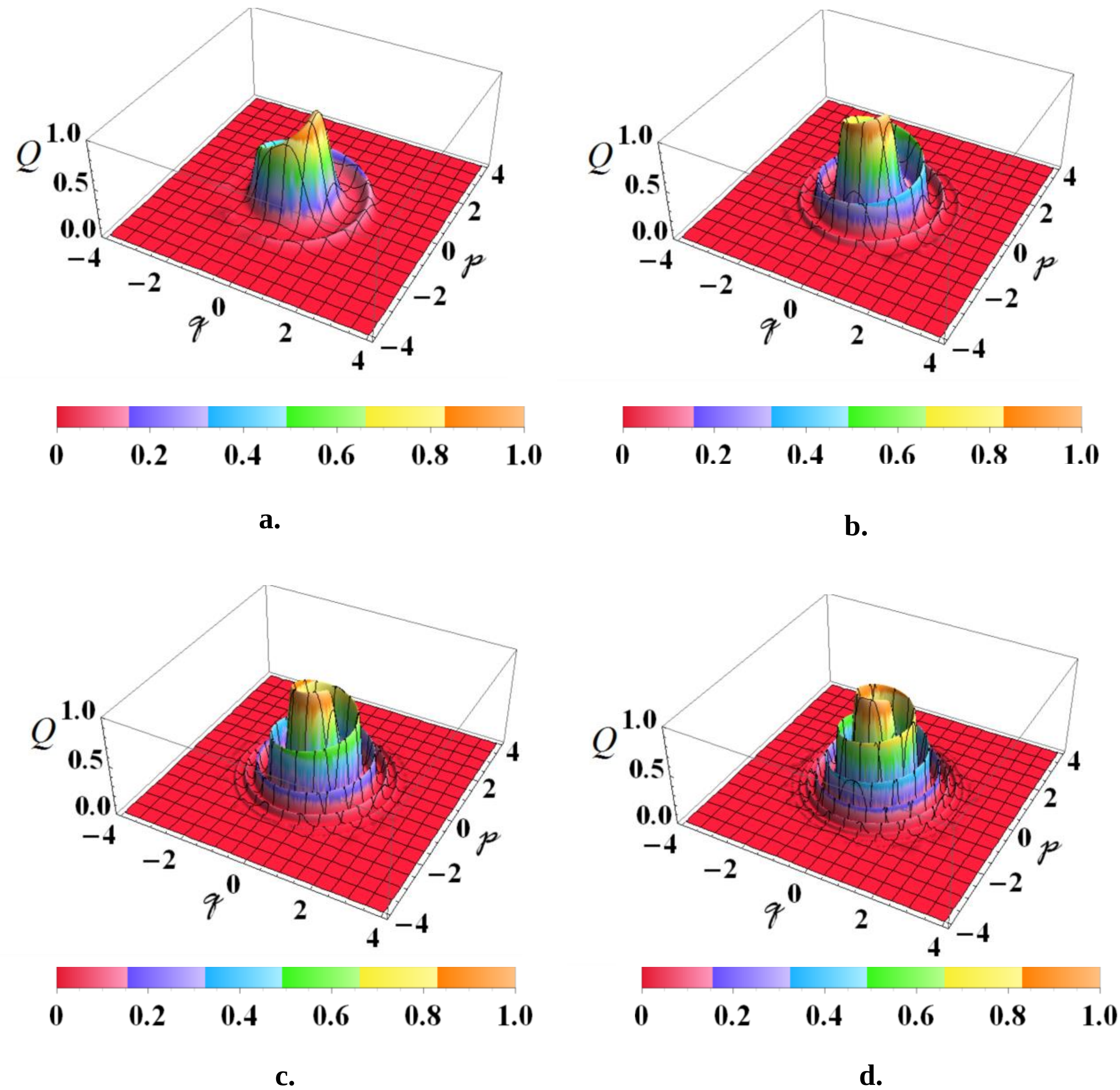

Fig. (4.4): The 3-D time-evolution of the classical probability distribution function Q for the anharmonic oscillator in phase space, for different values of time (τ): (a) $\tau = \pi/2$, (b) $\tau = \pi$, (c) $\tau = 3\pi/2$, and (d) $\tau = 2\pi$.

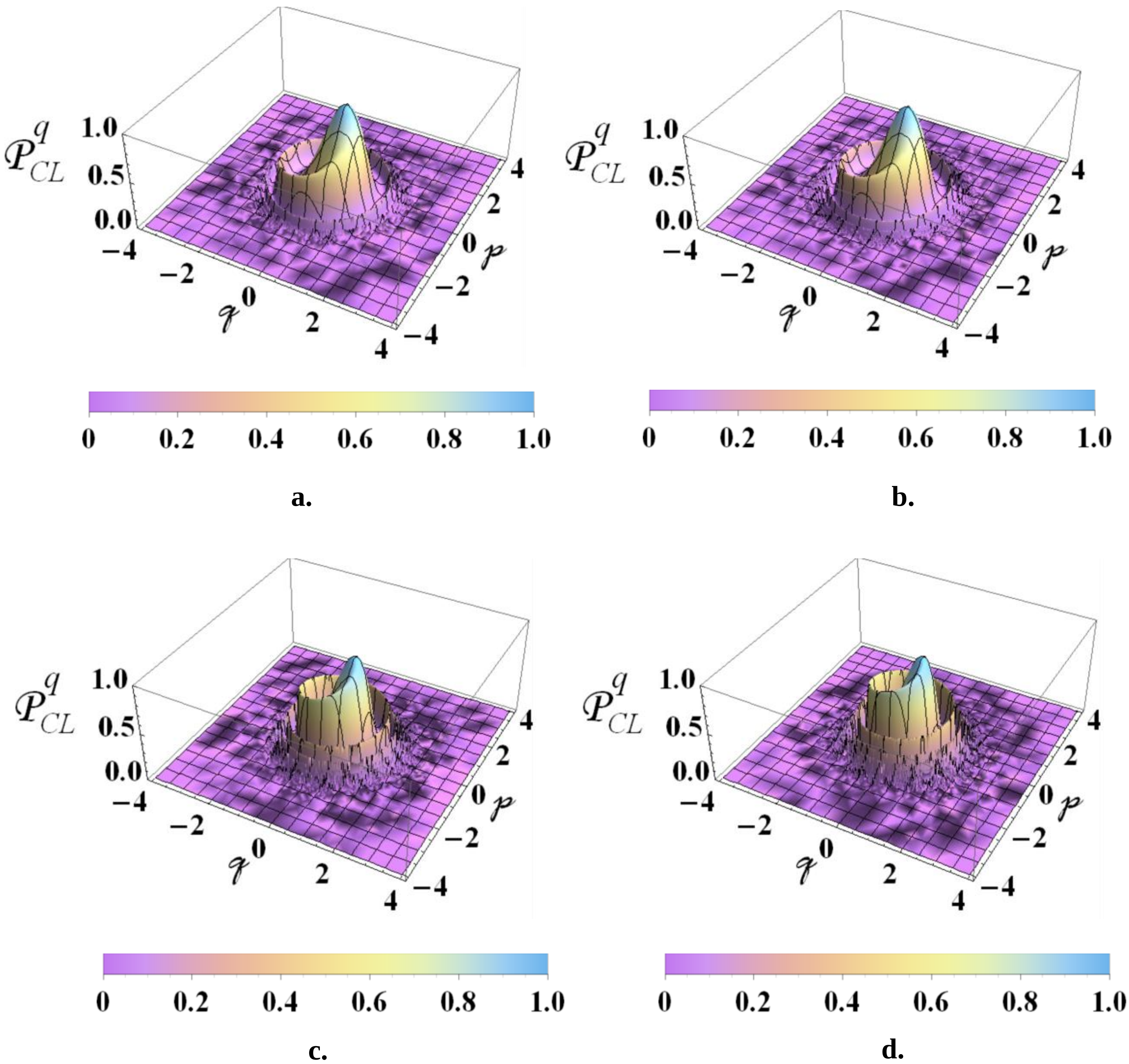

Fig. (4.5): The 3-D time-evolution of the classical probability distribution function $\mathcal{P}_{CL}^{q}(\alpha,\alpha^{*};t)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(1)}$ given by eqn. (4.42a) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

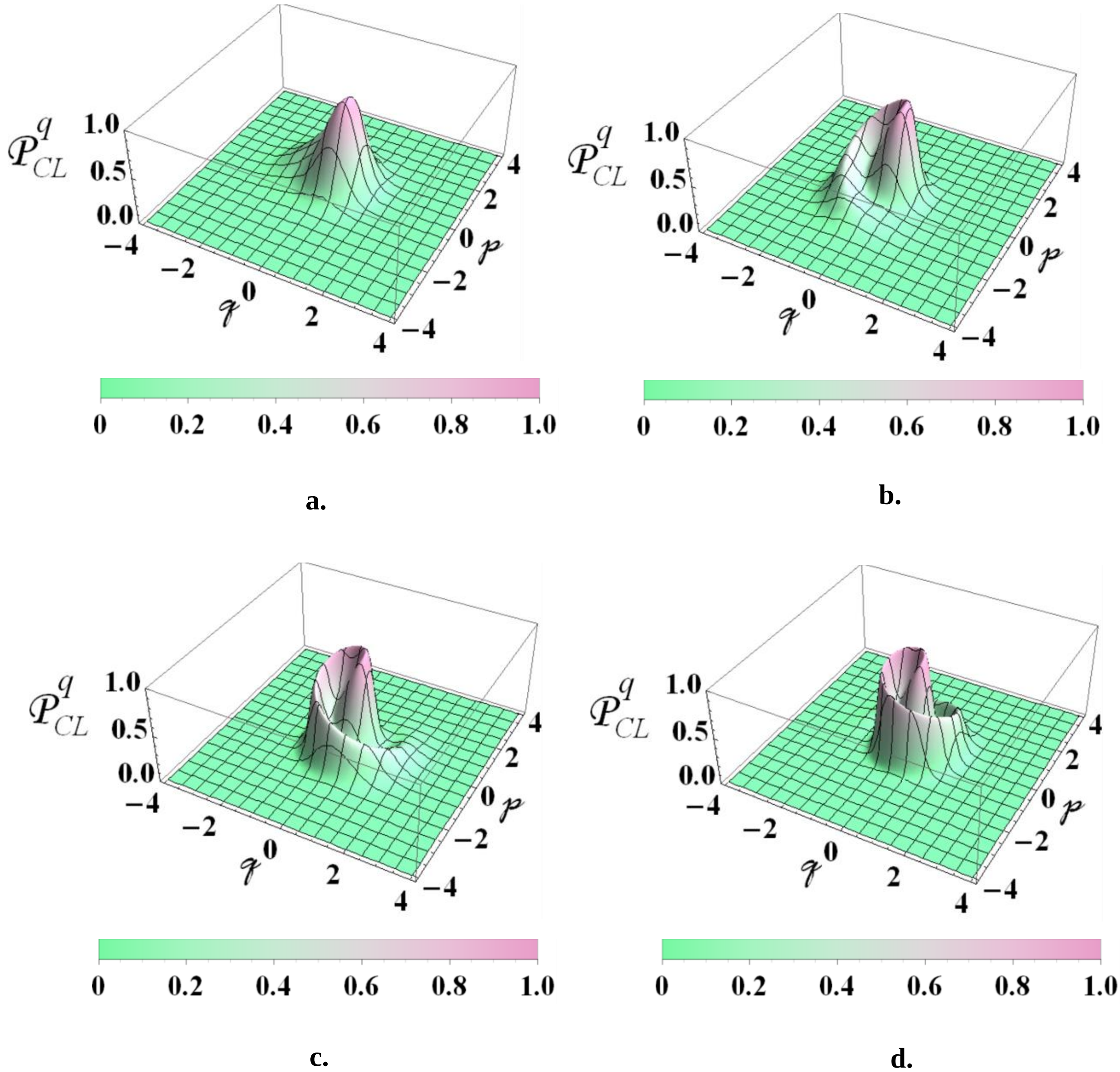

Fig. (4.6): The 3-D time-evolution of the classical probability distribution function $\mathcal{P}_{CL}^{q}(\alpha,\alpha^{*};t)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(2)}$ given by eqn. (4.42b) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

Chapter Five

The Classical Limit of the Quantum q-Deformed Harmonic Oscillator

The Classical Limit of the Quantum q-Deformed Harmonic Oscillator

This chapter begins with deriving the quantum Liouville equation for the f-deformed 1-D quantum harmonic oscillator in the Heisenberg picture by generalizing the Glauber-Sudarshan P-representation that was presented in refs. [40, 41] for the density operator in the α-representation. The q-deformation then follows as a special case arising from the general case of f-deformation. In a similar manner, the quantum Liouville equation is also derived in the α_q-representation. Then, the classical limits of these quantum Liouville equations are investigated in α- and α_q-representations. Finally, a computer visualization method similar to that adopted in Chapter 4 is used to investigate the solution of the resulting Liouville equations in the classical limit.

5.1 The Quantum Liouville Equation of the f-Deformed 1-D Quantum Harmonic Oscillator in the α-Representation

The quantum Liouville equation of the f-deformed harmonic oscillator can be derived in general in the two representations of the complex coordinates α and α_f. The q-deformed harmonic oscillator follows as a special case from the f-deformed oscillator as shown previously in Chapter 4.

The Heisenberg equation of motion for the density operator $\hat{\rho}$ is given by eqn. (2.57), where the Hamiltonian $\hat{\mathbb{H}}_f$ is defined as in eqn. (3.31) and $\hat{\rho}$ is as defined in eqn. (2.55). Substituting $\hat{\mathbb{H}}_f$ in the equation for the commutator $\left[\hat{\mathbb{H}}_f, \hat{\rho}\right]$ and then inserting the result into the Heisenberg equation of motion given by eqn. (2.57), one obtains

$$\frac{\partial \hat{\rho}}{\partial t} = -\left(\frac{i\omega}{2}\right)\left\{\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)\hat{\rho}+\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\hat{\rho}\right.$$
$$\left.-\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)-\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\right\} \quad (5.1)$$

where $f(\hat{a}^{\dagger}\hat{a})$ represents a boson operator function with normal ordering.

After lengthy mathematical manipulations, one obtains (see Appendix-D, eqns. (D.25) and (D.26)):

$$\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\hat{\rho}=\int d^{2}\alpha\,|\alpha\rangle\langle\alpha|\left\{\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)\right.$$
$$\left.\cdot\, f^{2}\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*})\right\} \quad (5.2)$$

and,

$$\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)=\int d^{2}\alpha\,|\alpha\rangle\langle\alpha|\left\{f^{2}\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\right.$$
$$\left.\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*})\right\}$$

(5.3)

In a similar manner, equations for $\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)\hat{\rho}$ and $\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)$ can be obtained.

Eqns. (5.2) and (5.3) can be understood to imply the one-to-one correspondence,

$$\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\hat{\rho}\rightarrow\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)$$
$$\cdot\, f^{2}\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*})$$

(5.4)

$$\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\rightarrow f^{2}\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)$$
$$\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*}) \quad (5.5)$$

Similarly, one-to-one correspondence relations for $\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)\hat{\rho}$ and $\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}+1\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a}+1)\right)$ can be obtained. Substituting the transformations of eqn. (5.4) and eqn. (5.5) into eqn. (5.1), using the definition of the density operator $\hat{\rho}$ from eqn. (2.55) and replacing $\varphi^{(s)}(\alpha,\alpha^{*};t)$ by $\varphi_{q}^{(s)}(\alpha,\alpha^{*};t)$ then simplifying the result, one obtains:

$$\frac{\partial\varphi_{q}^{(s)}(\alpha,\alpha^{*};t)}{\partial t}=-\left(\frac{i\omega}{2}\right)\Bigg\{\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]+1\right)$$
$$\cdot f^{2}\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]+1\right)$$
$$-f^{2}\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)$$
$$\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)$$
$$+\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)$$
$$\cdot f^{2}\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)$$
$$-f^{2}\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)$$
$$\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\Bigg\}\varphi_{q}^{(s)}(\alpha,\alpha^{*};t) \quad (5.6)$$

The function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ represents the q-analog of the non-deformed quasiprobability distribution function $\varphi^{(s)}(\alpha,\alpha^*)$ (see Sec. (3.3.2)), where in the limit $q \to 1$ the function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ reduces to $\varphi^{(s)}(\alpha,\alpha^*;t)$.

Eqn. (5.6) represents the quantum Liouville equation for the f-deformed 1-D quantum harmonic oscillator in terms of the q-deformed quasiprobability distribution function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ in the α-representation.

For the special case of the q-deformed oscillator, the definitions of the function f from Table (4.1) and the q-number from eqn. (4.3a) can be used to write:

$$f^2\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+1\right)= \frac{\sinh\left(\lambda\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+1\right)\right)}{\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+1\right)\sinh(\lambda)} \qquad (5.7)$$

The complex conjugate of eqn. (5.7) is:

$$f^2\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)= \frac{\sinh\left(\lambda\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)\right)}{\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)\sinh(\lambda)} \qquad (5.8)$$

Then, substituting for the sine hyperbolic function in terms of the exponential function into eqns. (5.7) and (5.8), one obtains:

$$f^2\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+1\right)=$$
$$\left\{2\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+1\right)\sinh(\lambda)\right\}^{-1}$$
$$\cdot\left\{e^{\lambda\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]+\lambda}-e^{-\lambda\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]-\lambda}\right\}\tag{5.9}$$

The complex conjugate of eqn. (5.9) is:

$$f^2\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)=$$
$$\left\{2\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+1\right)\sinh(\lambda)\right\}^{-1}$$
$$\cdot\left\{e^{\lambda\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]+\lambda}-e^{-\lambda\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]-\lambda}\right\}\tag{5.10}$$

Substituting eqns. (5.9) and (5.10) together with their complex conjugates into eqn. (5.6), and after some lengthy mathematical manipulations, one obtains:

$$\frac{\partial \varphi_q^{(s)}(\alpha,\alpha^*;t)}{\partial t} = \Bigg\{ -i\omega\{4\sinh(\lambda)\}^{-1}$$

$$\cdot\Bigg[\left(1+e^{\lambda}\right)e^{\lambda\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]}$$

$$+\left(1+e^{-\lambda}\right)e^{-\lambda\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]}\Bigg] + c.c.\Bigg\}\,\varphi_q^{(s)}(\alpha,\alpha^*;t)$$

(5.11)

where the abbreviation $c.c.$ represents the complex conjugate of the first term in the braces.

Simplifying eqn. (5.11), the final result takes the form:

$$\frac{\partial \varphi_q^{(s)}(\alpha,\alpha^*;t)}{\partial t} = \Bigg\{ -i\omega\left\{4\sinh(\lambda)\right\}^{-1}\Bigg[\left(1+e^{\lambda}\right)e^{\lambda\left(\frac{s-1}{2}\right)}$$

$$\cdot\, e^{\lambda\left[|\alpha|^2+\left(\frac{s+1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*}+\left(\frac{s-1}{2}\right)\alpha\frac{\partial}{\partial\alpha}+\left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]}$$

$$+\left(1+e^{-\lambda}\right)e^{-\lambda\left(\frac{s-1}{2}\right)}$$

$$\cdot\, e^{-\lambda\left[|\alpha|^2+\left(\frac{s+1}{2}\right)\alpha\frac{\partial}{\partial\alpha}+\left(\frac{s-1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*}+\left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha^*\,\partial\alpha}\right]}\Bigg]$$

$$+\, c.c.\Bigg\}\;\varphi_q^{(s)}(\alpha,\alpha^*;t)$$

(5.12)

Eqn. (5.12) represents the quantum Liouville equation for the q-deformed 1-D quantum harmonic oscillator in terms of the q-deformed quasiprobability distribution function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ in the α-representation. The exponential functions appearing in this equation are operator functions because they contain the differential operators $\alpha\frac{\partial}{\partial\alpha}$ and $\alpha^*\frac{\partial}{\partial\alpha^*}$.

5.1.1 Solution of the Disentanglement Problem for the Operator Functions

In general, the disentanglement problem is the problem of how to express the exponential of a sum of two operators in terms of the product of exponentials of these operators [50,110,126].

Assuming that $\hat{A}$ and $\hat{B}$ are two given operators, then the problem of disentanglement consists in finding operators $\hat{C}_1, \hat{C}_2, \cdots$ such that [50,110,126]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} \cdot e^{\hat{B}} \cdot e^{\hat{C}_1} \cdot e^{\hat{C}_2} \cdots e^{\hat{C}_\infty} \tag{5.13}$$

The $\hat{C}_m$s are the combinations of repeated commutators of $\hat{A}$ and $\hat{B}$.

If these combinations of commutators satisfy:

$$\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]=\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right]=\left[\hat{B},\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]\right]=\cdots=0 \tag{5.14}$$

where the dots $\cdots$ represent all the commutators of order higher than $\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right]$ and $\left[\hat{B},\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]\right]$, then eqn. (5.13) reduces to:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} \cdot e^{\hat{B}} \cdot e^{\frac{1}{2}\left[\hat{A},\hat{B}\right]} \tag{5.15}$$

Eqn. (5.15) is known as the Baker-Campbell-Hausdorff (BCH) formula [50,110,126].

Therefore, letting

$$\hat{A}_{\pm} = \pm\lambda\,|\alpha|^2 \tag{5.16}$$

$$\hat{B}_1 = \lambda\left\{\left(\frac{s+1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left(\frac{s-1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right\} \tag{5.17}$$

and,

$$\hat{B}_2 = -\,\hat{B}_1^* \tag{5.18}$$

and substituting for the operators $\hat{A}$ and $\hat{B}$ from eqns. (5.16), (5.17) and (5.18) into eqn. (5.14), the results are:

$$\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right] = \left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right] = 0 \tag{5.19}$$

and,

$$\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right] = \left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right] = 0 \tag{5.20}$$

Eqns. (5.19) and (5.20) represent the required conditions that should be satisfied to apply the BCH-formula (i.e., eqn. (5.15)).

The commutators $\left[\hat{A}_+,\hat{B}_1\right]$ and $\left[\hat{A}_-,\hat{B}_2\right]$ can be calculated by substituting the operators $\hat{A}_{\pm}$, $\hat{B}_1$ and $\hat{B}_2$ from eqns. (5.16), (5.17) and (5.18) into the expansion of these commutators to obtain (see Appendix-E):

$$\left[\hat{A}_+,\hat{B}_1\right] = \left[\hat{A}_-,\hat{B}_2\right] = -\lambda^2\left\{s\,|\alpha|^2 + \left(\frac{s^2-1}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + 1\right]\right\} \tag{5.21}$$

Similarly, the commutators $\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$, $\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$, $\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]$ and $\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]$ can be calculated to obtain (see Appendix-E, eqns. (E.9), (E.10), (E.14) and (E.15)):

$$\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]=\left(\frac{s^2-1}{2}\right)\lambda^3|\alpha|^2 \tag{5.22}$$

$$\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]=-\lambda^3\left\{s^2|\alpha|^2+\left(\frac{s\,(s^2-1)}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*}+\alpha\frac{\partial}{\partial\alpha}+1\right]\right.$$
$$\left.+\left[\frac{(s+1)^2(s-1)^2}{8}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right\} \tag{5.23}$$

and,

$$\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]=-\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right] \tag{5.24}$$

$$\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]=-\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right] \tag{5.25}$$

The expressions for the commutators $\left[\hat{A}_+,\hat{B}_1\right]$, $\left[\hat{A}_-,\hat{B}_2\right]$, $\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$, $\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]$, $\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$ and $\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]$ for different values of the ordering parameter s are illustrated in Tables (5.1) and (5.2), where eqn. (5.16) and the eqns. (5.21)-(5.25) have been used to obtain the results:

Table (5.1)

The expressions for the commutators $\left[\hat{A}_+,\hat{B}_1\right]$ and $\left[\hat{A}_-,\hat{B}_2\right]$ for different values of the ordering parameter s.

s	$\left[\hat{A}_+,\hat{B}_1\right]$	$\left[\hat{A}_-,\hat{B}_2\right]$
1	$-\lambda\hat{A}_+$	$\lambda\hat{A}_-$
0	$\left(\frac{\lambda^2}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*}+\alpha\frac{\partial}{\partial\alpha}+1\right]$	$\left(\frac{\lambda^2}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*}+\alpha\frac{\partial}{\partial\alpha}+1\right]$
-1	$\lambda\hat{A}_+$	$-\lambda\hat{A}_-$

Table (5.2)

The expressions for the commutators $\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$, $\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]$, $\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$ and $\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]$ for different values of the ordering parameter s.

s	$\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$	$\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$	$\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]$	$\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]$
1	0	$-\lambda^2\hat{A}_+$	0	$-\lambda^2\hat{A}_-$
0	$-\left(\frac{\lambda^2}{2}\right)\hat{A}_+$	$-\left(\frac{\lambda^3}{8}\right)\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}$	$-\left(\frac{\lambda^2}{2}\right)\hat{A}_-$	$\left(\frac{\lambda^3}{8}\right)\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}$
-1	0	$-\lambda^2\hat{A}_+$	0	$-\lambda^2\hat{A}_-$

The fact that the expressions for the commutators $\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$, $\left[\hat{A}_-,\left[\hat{A}_-,\hat{B}_2\right]\right]$, $\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$ and $\left[\hat{B}_2,\left[\hat{A}_-,\hat{B}_2\right]\right]$ shown in Table (5.2) are non-vanishing for all three values of the ordering parameter s means that the conditions given in eqns. (5.19) and (5.20) are not satisfied. In other words, the BCH-formula cannot be applied to solve the problem of the disentanglement of the operators appearing in eqn. (5.12).

An alternative solution to this operator disentanglement problem could be through the Zassenhous formula [127]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} \cdot e^{\hat{B}} \cdot e^{\hat{C}} \tag{5.26}$$

Again, the operators $\hat{A}$ and $\hat{B}$ are substituted from eqns. (5.16) and (5.17) respectively and the operator $e^{\hat{C}}$ can be calculated by employing the method introduced in ref. [128] since the same conditions of ref. [128] apply to our case, i.e.,

$$\left[\hat{A}_+,\hat{B}_1\right] = -f(Z)\hat{A}_+ \tag{5.27}$$

$$\left[\hat{A}_-,\hat{B}_2\right] = -f(Z)\,\hat{A}_- \tag{5.28}$$

where $f(Z)$ represents any real function of arbitrary variable $Z \in \mathbb{R}$.

In our case, the function $f(Z)$, denoted here for convenience by $\gamma_q^{(s)}$, can be calculated according to eqns. (5.27) and (5.28) as:

$$\gamma_q^{(s)} = \begin{cases} \dfrac{-\left[\hat{A}_+,\hat{B}_1\right]}{\hat{A}_+} & \text{associated with the operator } \hat{A}_+ \\ \dfrac{-\left[\hat{A}_-,\hat{B}_2\right]}{\hat{A}_-} & \text{associated with the operator } \hat{A}_- \end{cases} \tag{5.29}$$

Thus, the operator $e^{\hat{C}}$ can be written as [128]:

$$e^{\hat{C}} = e^{\hat{C}_2}\cdot e^{\hat{C}_3}\cdots e^{\hat{C}_\infty} = e^{\left[\sum_{m=1}^{\infty}\left(\frac{1-m}{m!}\right)\left(-\gamma_q^{(s)}\right)^{m-1}\right]\hat{A}_\pm} \tag{5.30}$$

and the subscript q and superscript s in $\gamma_q^{(s)}$ refer respectively to the q-deformation and the associated type of ordering, i.e.; $s = 1, 0, -1$.

But since [128]:

$$\sum_{m=1}^{\infty}\left(\frac{1-m}{m!}\right)\left(-\gamma_q^{(s)}\right)^{m-1} = \left(-\gamma_q^{(s)}\right)^{-1}\left(e^{-\gamma_q^{(s)}} - 1\right) - e^{-\gamma_q^{(s)}} \tag{5.31}$$

then, inserting eqn. (5.31) into eqn. (5.30) results in:

$$e^{\hat{C}} = e^{\hat{C}_2}\cdot e^{\hat{C}_3}\cdots e^{\hat{C}_\infty} = e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}} - 1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_\pm}$$

(5.32)

Substituting $\hat{A}_\pm$ from eqn. (5.16) and the commutators $\left[\hat{A}_+,\hat{B}_1\right]$ and $\left[\hat{A}_-,\hat{B}_2\right]$ from Table (5.1) into eqn. (5.29) leads to three expressions for $\gamma_q^{(s)}$ as given in Table (5.3).

Table (5.3)

The expressions for $\gamma_q^{(s)}$ corresponding to different values of the ordering parameter s.

s	$\gamma_q^{(s)}$
1	$\gamma_q^{(1)} = \begin{cases} \lambda & \text{associated with the operator } \hat{A}_+ \\ -\lambda & \text{associated with the operator } \hat{A}_- \end{cases}$
0	$\gamma_q^{(0)} = \begin{cases} -\left(\frac{\lambda}{4}\right)\lvert\alpha\rvert^{-2}\left(\alpha^* \frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + 1\right) & \text{associated with the operator } \hat{A}_+ \\ \left(\frac{\lambda}{4}\right)\lvert\alpha\rvert^{-2}\left(\alpha^* \frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + 1\right) & \text{associated with the operator } \hat{A}_- \end{cases}$
-1	$\gamma_q^{(-1)} = \begin{cases} -\lambda & \text{associated with the operator } \hat{A}_+ \\ \lambda & \text{associated with the operator } \hat{A}_- \end{cases}$

Inserting the expressions for $\gamma_q^{(s)}$ from Table (5.3) into eqn. (5.32) produces different expressions for $e^{\hat{C}}$ associated with different values of s. These results are summarized in Table (5.4).

Table (5.4)

The expressions for $e^{\hat{C}}$ corresponding to different s values.

s	$e^{\hat{C}}$
1	$e^{\hat{C}} = e^{\left[\left(-\gamma_q^{(1)}\right)^{-1}\left\{e^{-\gamma_q^{(1)}} - 1\right\} - e^{-\gamma_q^{(1)}}\right]\hat{A}_\pm}$
0	$e^{\hat{C}} = e^{\left[\left(-\gamma_q^{(0)}\right)^{-1}\left\{e^{-\gamma_q^{(0)}} - 1\right\} - e^{-\gamma_q^{(0)}}\right]\hat{A}_\pm}$
-1	$e^{\hat{C}} = e^{\left[\left(-\gamma_q^{(-1)}\right)^{-1}\left\{e^{-\gamma_q^{(-1)}} - 1\right\} - e^{-\gamma_q^{(-1)}}\right]\hat{A}_\pm}$

Substituting $e^{\hat{C}}$ from eqn. (5.32) into eqn. (5.26) gives:

$$e^{\hat{A}_+ + \hat{B}_1} = e^{\hat{A}_+} \cdot e^{\hat{B}_1} \cdot e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}} -1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_+} \tag{5.33}$$

$$e^{\hat{A}_- + \hat{B}_2} = e^{\hat{A}_-} \cdot e^{\hat{B}_2} \cdot e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}} -1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_-} \tag{5.34}$$

where the expressions for $\gamma_q^{(s)}$ associated with $\hat{A}_+$ and $\hat{A}_-$ are given in Table (5.3) for different s values. Eqns. (5.33) and (5.34) represent the solution of the operator disentanglement problem for eqn. (5.12).

Substitution of the expressions for $\hat{A}_\pm$, $\hat{B}_1$ and $\hat{B}_2$ into eqns. (5.33) and (5.34) gives:

$$e^{\lambda\left[|\alpha|^2 + \left(\frac{s+1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left(\frac{s-1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]} =$$

$$e^{\lambda|\alpha|^2}\, e^{\lambda\left[\left(\frac{s+1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left(\frac{s-1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]}$$

$$\cdot\, e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}} -1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_+} \tag{5.35}$$

$$e^{-\lambda\left[|\alpha|^2 + \left(\frac{s+1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left(\frac{s-1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]} =$$

$$e^{-\lambda|\alpha|^2}\, e^{-\lambda\left[\left(\frac{s+1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left(\frac{s-1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]}$$

$$\cdot\, e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}} -1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_-} \tag{5.36}$$

Finally, substituting eqns. (5.35) and (5.36) into eqn. (5.12) results in:

$$\frac{\partial \varphi_q^{(s)}(\alpha,\alpha^*;t)}{\partial t} = \Bigg\{ -i\omega\{4\sinh(\lambda)\}^{-1} \Bigg[(1+e^{\lambda})\, e^{\lambda\left(\frac{s-1}{2}\right)}$$

$$\cdot\; e^{\lambda|\alpha|^2}\, e^{\lambda\left[\left(\frac{s+1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left(\frac{s-1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]}$$

$$\cdot\; e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}}-1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_+}$$

$$+\; (1+e^{-\lambda})\, e^{-\lambda\left(\frac{s-1}{2}\right)}\, e^{-\lambda|\alpha|^2}$$

$$\cdot\; e^{-\lambda\left[\left(\frac{s+1}{2}\right)\alpha\frac{\partial}{\partial\alpha} + \left(\frac{s-1}{2}\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \left[\frac{(s^2-1)}{4}\right]\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]}$$

$$\cdot\; e^{\left[\left(-\gamma_q^{(s)}\right)^{-1}\left\{e^{-\gamma_q^{(s)}}-1\right\} - e^{-\gamma_q^{(s)}}\right]\hat{A}_-} \Bigg] + c.c. \Bigg\}\, \varphi_q^{(s)}(\alpha,\alpha^*;t)$$

(5.37)

Eqn. (5.37) represents the quantum Liouville equation for the q-deformed 1-D quantum harmonic oscillator in terms of the q-deformed quasiprobability distribution function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ in α-representation. Substituting the values of the ordering parameter s (i.e.; $s=1, 0, -1$) into eqn. (5.37) leads to the q-analogs of the well-known quasiprobability functions in quantum optics, namely; the Husimi Q-function ($s=1$), the Wigner W-function ($s=0$) and the Glauber-Sudarshan P-function ($s=-1$). This thesis is concerned with the q-analog of both Q-function ($s=1$) and P-function ($s=-1$), where for these two functions, eqn. (5.37) has a well-behaved analytical solution as will be seen in the next sections.

5.1.2 The q-deformed Husimi Q_q-function ($s=1$), Glauber-Sudarshan P_q-function ($s=-1$) and the Dilatation (Shift) Operator

The q-deformed quasiprobabilitity distribution function $\varphi_q^{(s)}(\alpha,\alpha^*;t)$ that appeared in eqn. (5.37) represents the q-analog of the function $\varphi^{(s)}(\alpha,\alpha^*;t)$. Hence, for ($s=1$) this function becomes $\varphi_q^{(1)}(\alpha,\alpha^*;t)$, and from Table (2.1) this function is the q-analog of $Q\ \alpha,\alpha^*;t$ which is denoted by $Q_q(\alpha,\alpha^*;t)$. Substituting $Q_q(\alpha,\alpha^*;t)$, $\hat{A}_\pm$ and $\gamma_q^{(1)}$ into eqn. (5.37) leads to:

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t}=\Bigg\{-i\omega\{4\sinh(\lambda)\}^{-1}\Bigg[\left(1+e^{\lambda}\right)e^{\lambda|\alpha|^2}$$

$$\cdot\, e^{\lambda\,\alpha^*\frac{\partial}{\partial\alpha^*}}\, e^{\left[(-\lambda)^{-1}\left(e^{-\lambda}-1\right)-e^{-\lambda}\right]\left(\lambda|\alpha|^2\right)}+\left(1+e^{-\lambda}\right)e^{-\lambda|\alpha|^2}$$

$$\cdot\, e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\, e^{\left[(\lambda)^{-1}\left(e^{\lambda}-1\right)-e^{\lambda}\right]\left(-\lambda|\alpha|^2\right)}\Bigg]+c.c.\Bigg\}Q_q(\alpha,\alpha^*;t) \tag{5.38}$$

which can be simplified to

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t}=\Bigg\{-i\omega\{4\sinh(\lambda)\}^{-1}$$

$$\cdot\Bigg[\left(1+e^{\lambda}\right)e^{\lambda|\alpha|^2}e^{\lambda\,\alpha^*\frac{\partial}{\partial\alpha^*}}\, e^{-\left\{(1+\lambda)e^{-\lambda}-1\right\}\left(|\alpha|^2\right)}+\left(1+e^{-\lambda}\right)e^{-\lambda|\alpha|^2}$$

$$\cdot\, e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\, e^{-\left\{(1-\lambda)e^{\lambda}-1\right\}\left(|\alpha|^2\right)}\Bigg]+c.c.\Bigg\}Q_q(\alpha,\alpha^*;t) \tag{5.39}$$

Similarly, the q-deformed quasiprobability distribution function is denoted by $P_q(\alpha,\alpha^*;t)=\varphi_q^{(-1)}(\alpha,\alpha^*;t)$. As for $Q_q(\alpha,\alpha^*;t)$, insertion of $P_q(\alpha,\alpha^*;t)$, $\hat{A}_\pm$ and $\gamma_q^{(-1)}$ into eqn. (5.37) gives:

$$\frac{\partial P_q(\alpha,\alpha^*;t)}{\partial t}=\left\{-i\omega\{4\sinh(\lambda)\}^{-1}\left[\left(1+e^{\lambda}\right)e^{-\lambda}e^{\lambda|\alpha|^2}\right.\right.$$
$$\cdot e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}e^{\left\{(1-\lambda)e^{\lambda}-1\right\}\left(|\alpha|^2\right)}+\left(1+e^{-\lambda}\right)e^{\lambda}e^{-\lambda|\alpha|^2}$$
$$\left.\left.\cdot e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}e^{\left\{(1+\lambda)e^{-\lambda}-1\right\}\left(|\alpha|^2\right)}\right]+c.c.\right\}P_q(\alpha,\alpha^*;t) \tag{5.40}$$

Since the action of the dilatation (shift) operator $e^{\beta x\frac{\partial}{\partial x}}$ on a function $F\ x$ can be written as (see eqn. (3.21)):

$$e^{\beta x\frac{\partial}{\partial x}}F(x)=F\left(xe^{\beta}\right) \tag{5.41}$$

for any arbitrary constant β, then one can employ eqn. (5.41), assuming that $\beta=\pm\lambda$ and replacing $F\ x$ by $F(\alpha,\alpha^*)$ to write:

$$e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}F(\alpha,\alpha^*)=F\left(\alpha,e^{\pm\lambda}\alpha^*\right) \tag{5.42}$$

$$e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}F(\alpha,\alpha^*)=F\left(e^{\pm\lambda}\alpha,\alpha^*\right) \tag{5.43}$$

Eqns. (5.42) and (5.43) can be generalized to the case when the shift operator acts on the product of two function $F(\alpha,\alpha^*)G(\alpha,\alpha^*)$ (see Appendix-F, eqns. (F.10) and (F.11)).

Using this result leads to:

$$e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}F(\alpha,\alpha^*)G(\alpha,\alpha^*)=F\left(\alpha,e^{\pm\lambda}\alpha^*\right)e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}G(\alpha,\alpha^*) \quad (5.44)$$

and,

$$e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}F(\alpha,\alpha^*)G(\alpha,\alpha^*)=F\left(e^{\pm\lambda}\alpha,\alpha^*\right)e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}G(\alpha,\alpha^*) \quad (5.45)$$

Letting

$$F(\alpha,\alpha^*)=\begin{cases} e^{-\left\{(1+\lambda)e^{-\lambda}-1\right\}\left(|\alpha|^2\right)} \\ e^{-\left\{(1-\lambda)e^{\lambda}-1\right\}\left(|\alpha|^2\right)} \end{cases} \quad (5.46)$$

and,

$$G(\alpha,\alpha^*)=Q_q(\alpha,\alpha^*;t) \quad (5.47)$$

and substituting eqns. (5.46) and (5.47) into eqns. (5.44) and (5.45), gives:

$$e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}e^{-\left\{(1+\lambda)e^{-\lambda}-1\right\}\left(|\alpha|^2\right)}Q_q(\alpha,\alpha^*;t)=$$
$$e^{-\left\{(1+\lambda)e^{-\lambda}-1\right\}\left[e^{\pm\lambda}|\alpha|^2\right]}e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}Q_q(\alpha,\alpha^*;t) \quad (5.48)$$

and,

$$e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}e^{-\left\{(1-\lambda)e^{\lambda}-1\right\}\left(|\alpha|^2\right)}Q_q(\alpha,\alpha^*;t)=$$
$$e^{-\left\{(1-\lambda)e^{\lambda}-1\right\}\left[e^{\pm\lambda}|\alpha|^2\right]}e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}Q_q(\alpha,\alpha^*;t)$$

(5.49)

Then, substituting eqns. (5.48) and (5.49) into eqn. (5.39), this equation becomes:

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t} = \Bigg\{ -i\omega\{4\sinh(\lambda)\}^{-1}$$

$$\cdot\Bigg[\left(1+e^{\lambda}\right)e^{\lambda|\alpha|^2}e^{-\{(1+\lambda)e^{-\lambda}-1\}\left[e^{\lambda}|\alpha|^2\right]}e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}+\left(1+e^{-\lambda}\right)e^{-\lambda|\alpha|^2}$$

$$\cdot\, e^{-\{(1-\lambda)e^{\lambda}-1\}\left[e^{-\lambda}|\alpha|^2\right]}e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\Bigg]+c.c.\Bigg\}Q_q(\alpha,\alpha^*;t)$$

(5.50)

But since,

$$e^{-\{(1+\lambda)e^{-\lambda}-1\}\left[e^{\lambda}|\alpha|^2\right]}=e^{-|\alpha|^2}e^{-\lambda|\alpha|^2}e^{\left(e^{\lambda}|\alpha|^2\right)} \qquad (5.51)$$

and,

$$e^{-\{(1-\lambda)e^{\lambda}-1\}\left[e^{-\lambda}|\alpha|^2\right]}=e^{-|\alpha|^2}e^{\lambda|\alpha|^2}e^{\left(e^{-\lambda}|\alpha|^2\right)} \qquad (5.52)$$

then, substituting eqns. (5.51) and (5.52) into eqn. (5.50) gives:

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t} = \Bigg\{ -i\omega\{4\sinh(\lambda)\}^{-1}\Bigg[\left(1+e^{\lambda}\right)e^{-|\alpha|^2}e^{\left(e^{\lambda}|\alpha|^2\right)}e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}$$

$$+\left(1+e^{-\lambda}\right)e^{-|\alpha|^2}e^{\left(e^{-\lambda}|\alpha|^2\right)}e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\Bigg]+c.c.\Bigg\}Q_q(\alpha,\alpha^*;t)$$

(5.53)

After substituting the complex conjugate terms and re-arranging, eqn. (5.53) becomes:

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t} = -i\omega\{4\sinh(\lambda)\}^{-1}\ e^{-|\alpha|^2}$$

$$\cdot\left\{(1+e^{\lambda})\, e^{\left(e^{\lambda}|\alpha|^2\right)}\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} - e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right.$$

$$\left. -\left(1+e^{-\lambda}\right) e^{\left(e^{-\lambda}|\alpha|^2\right)}\left[e^{-\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} - e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\} Q_q(\alpha,\alpha^*;t)$$

(5.54)

Eqn. (5.54) represents the quantum Liouville equation for the 1-D q-deformed quantum harmonic oscillator where the probability distribution function is the Husimi function $Q_q(\alpha,\alpha^*;t)$.

Similarly, one can simplify the Glauber-Sudarshan eqn. (5.40) for quasiprobability $P_q(\alpha,\alpha^*;t)$ to obtain:

$$\frac{\partial P_q(\alpha,\alpha^*;t)}{\partial t} = -i\omega\{4\sinh(\lambda)\}^{-1}\ e^{|\alpha|^2}$$

$$\cdot\left\{(1+e^{-\lambda})\, e^{\lambda}\ e^{\left(-e^{\lambda}|\alpha|^2\right)}\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} - e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right.$$

$$\left. -\left(1+e^{\lambda}\right) e^{-\lambda}\, e^{\left(-e^{-\lambda}|\alpha|^2\right)}\left[e^{-\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} - e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\} P_q(\alpha,\alpha^*;t)$$

(5.55)

5.1.3 Zaslavskii's Method for Deriving the Quantum Liouville Equation for the q-Deformed Oscillator in the α-Representation

According to Zaslavskii [129], the equation of motion of an averaged physical quantity $\xi(\alpha,\alpha^*;t)$ is defined as [129]:

$$\frac{\partial \xi(\alpha,\alpha^*;t)}{\partial t} = -i\hat{\mathrm{K}}\xi(\alpha,\alpha^*;t) \tag{5.56}$$

where,

$$\hat{\mathrm{K}} = \left(\frac{1}{\hbar}\right) e^{-|\alpha|^2} \left\{\hat{\mathbb{H}}_q\left(\alpha^*, \frac{\partial}{\partial \alpha^*}\right) - \hat{\mathbb{H}}_q\left(\alpha, \frac{\partial}{\partial \alpha}\right)\right\} e^{|\alpha|^2} \tag{5.57}$$

The Hamiltonian of the q-deformed quantum harmonic oscillator is given by eqn. (3.34). And according to Sudarshan [102], one has the following correspondence:

$$\left.\begin{aligned} \hat{a}^\dagger &\rightarrow \alpha^* \\ \hat{a} &\rightarrow \frac{\partial}{\partial \alpha^*} \end{aligned}\right\} \text{ for } \hat{\mathbb{H}}_q\left(\alpha^*, \frac{\partial}{\partial \alpha^*}\right) \tag{5.58}$$

and,

$$\left.\begin{aligned} \hat{a} &\rightarrow \alpha \\ \hat{a}^\dagger &\rightarrow \frac{\partial}{\partial \alpha} \end{aligned}\right\} \text{ for } \hat{\mathbb{H}}_q\left(\alpha, \frac{\partial}{\partial \alpha}\right) \tag{5.59}$$

Substituting eqns. (5.58) and (5.59) into the expressions for $\left[\hat{N}\right]_q$ and $\left[\hat{N}+1\right]_q$ appearing in $\hat{\mathbb{H}}_q\left(\alpha^*, \frac{\partial}{\partial \alpha^*}\right)$ by using eqns. (3.35), leads to:

$$\left.\begin{aligned}\left[\hat{N}\right]_q &= \frac{\sinh\left(\lambda\alpha^*\dfrac{\partial}{\partial\alpha^*}\right)}{\sinh\lambda} \\ \left[\hat{N}+1\right]_q &= \frac{\sinh\left(\lambda\left[\alpha^*\dfrac{\partial}{\partial\alpha^*}+1\right]\right)}{\sinh\lambda}\end{aligned}\right\} \quad \text{for } \left[\hat{N}\right]_q = \frac{q^{\hat{N}} - q^{-\hat{N}}}{q - q^{-1}} \tag{5.60}$$

and,

$$\left.\begin{aligned}\left[\hat{N}\right]_q &= \frac{e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} - 1}{e^{\lambda}-1} \\ \left[\hat{N}+1\right]_q &= \frac{e^{\lambda\left[\alpha^*\frac{\partial}{\partial\alpha^*}+1\right]} - 1}{e^{\lambda}-1}\end{aligned}\right\} \quad \text{for } \left[\hat{N}\right]_q = \frac{q^{\hat{N}} - 1}{q - 1} \tag{5.61}$$

The same method can be used for $\left[\hat{N}\right]_q$ and $\left[\hat{N}+1\right]_q$ appearing in $\hat{\mathbb{H}}_q\left(\alpha, \frac{\partial}{\partial\alpha}\right)$.

Now, insertion of $\left[\hat{N}\right]_q$ and $\left[\hat{N}+1\right]_q$ from eqns. (5.60) and (5.61) into eqn. (3.34), gives:

$$\hat{\mathbb{H}}_q\left(\alpha^*, \frac{\partial}{\partial\alpha^*}\right) = \left(\frac{\hbar\omega}{2}\right)\left\{\frac{\sinh\left(\lambda\alpha^*\dfrac{\partial}{\partial\alpha^*}\right)}{\sinh\lambda} + \frac{\sinh\left(\lambda\left[\alpha^*\dfrac{\partial}{\partial\alpha^*}+1\right]\right)}{\sinh\lambda}\right\} \tag{5.62a}$$

hence,

$$\hat{\mathbb{H}}_q\left(\alpha^*, \frac{\partial}{\partial\alpha^*}\right) = \left(\frac{\hbar\omega}{2}\right)\left\{\left(\frac{e^{\alpha^*\frac{\partial}{\partial\alpha^*}} - 1}{e^{\lambda}-1}\right) + \left(\frac{e^{\lambda\left[\alpha^*\frac{\partial}{\partial\alpha^*}+1\right]} - 1}{e^{\lambda}-1}\right)\right\} \tag{5.62b}$$

and,

$$\hat{\mathbb{H}}_q \ \alpha, \frac{\partial}{\partial\alpha} = \left(\frac{\hbar\omega}{2}\right)\left\{\frac{\sinh\ \lambda\alpha\frac{\partial}{\partial\alpha}}{\sinh\ \lambda} + \frac{\sinh\ \lambda\left[\alpha\frac{\partial}{\partial\alpha}+1\right]}{\sinh\ \lambda}\right\} \tag{5.63a}$$

hence,

$$\hat{\mathbb{H}}_q \ \alpha, \frac{\partial}{\partial\alpha} = \left(\frac{\hbar\omega}{2}\right)\left\{\left(\frac{e^{\alpha\frac{\partial}{\partial\alpha}} -1}{e^{\lambda}-1}\right) + \left(\frac{e^{\lambda\left[\alpha\frac{\partial}{\partial\alpha}+1\right]}-1}{e^{\lambda}-1}\right)\right\} \tag{5.63b}$$

respectively.

Using eqns. (5.62) and (5.63) in eqn. (5.57), leads to:

$$\hat{\mathrm{K}} = \omega \ 2\sinh\ \lambda \ ^{-1} e^{-|\alpha|^2}\left\{\sinh\ \lambda\alpha^*\frac{\partial}{\partial\alpha^*} + \sinh\ \lambda\left[\alpha^*\frac{\partial}{\partial\alpha^*}+1\right]\right.$$
$$\left. -\sinh\ \lambda\alpha\frac{\partial}{\partial\alpha} - \sinh\ \lambda\left[\alpha\frac{\partial}{\partial\alpha}+1\right]\ \right\} e^{|\alpha|^2} \tag{5.64a}$$

and,

$$\hat{\mathrm{K}} = \omega \ 2 \ e^{\lambda}-1 \ ^{-1} e^{-|\alpha|^2}\left\{e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}} + e^{\lambda\left[\alpha^*\frac{\partial}{\partial\alpha^*}+1\right]} - e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right.$$
$$\left. - \ e^{\lambda\left[\alpha\frac{\partial}{\partial\alpha}+1\right]}\right\} e^{|\alpha|^2} \tag{5.64b}$$

Then, using the definition of the sine hyperbolic function and re-arranging terms, eqn. (5.64a) can be cast in the form:

$$\hat{K}=\omega\,(4\sinh\lambda)^{-1}e^{-|\alpha|^2}\left\{(1+e^{\lambda})\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right.$$

$$\left.-\,(1+e^{-\lambda})\left[e^{-\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}e^{|\alpha|^2} \qquad (5.65a)$$

Similarly, eqn. (5.64b) becomes:

$$\hat{K}=\omega\,\left(2(e^{\lambda}-1)\right)^{-1}e^{-|\alpha|^2}\left\{(1+e^{\lambda})\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}e^{|\alpha|^2} \qquad (5.65b)$$

Eqns. (5.65a) and (5.65b) can be substituted in eqn. (5.56) and after replacing $\xi(\alpha,\alpha^*;t)$ by $\xi_q(\alpha,\alpha^*;t)$, one obtains:

$$\frac{\partial\xi_q(\alpha,\alpha^*;t)}{\partial t}=-i\omega\,(4\sinh\lambda)^{-1}e^{-|\alpha|^2}\left\{(1+e^{\lambda})\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right.$$

$$\left.-\,(1+e^{-\lambda})\left[e^{-\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}e^{|\alpha|^2}\xi_q(\alpha,\alpha^*;t) \qquad (5.66a)$$

and,

$$\frac{\partial\xi_q(\alpha,\alpha^*;t)}{\partial t}=-i\omega\,\left(2(e^{\lambda}-1)\right)^{-1}e^{-|\alpha|^2}\left\{(1+e^{\lambda})\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}$$

$$\cdot\, e^{|\alpha|^2}\xi_q(\alpha,\alpha^*;t)$$

$$(5.66b)$$

respectively, where the function $\xi_q\left(\alpha,\alpha^*;t\right)$ represents the q-analog of the averaged physical quantity $\xi\left(\alpha,\alpha^*;t\right)$ appearing in ref. [129]. Also, it is noted that the function $\xi_q\left(\alpha,\alpha^*;t\right)$ reduces to $\xi\left(\alpha,\alpha^*;t\right)$ in the limit $q\to 1$.

Finally, eqn. (5.66a) and (5.66b) can be simplified by using eqns. (5.44) and (5.45) with $F\left(\alpha,\alpha^*\right)=e^{|\alpha|^2}$, $G\left(\alpha,\alpha^*\right)=\xi_q\left(\alpha,\alpha^*;t\right)$ and re-arranging to get:

$$\frac{\partial\xi_q\left(\alpha,\alpha^*;t\right)}{\partial t}=-i\omega\left(4\sinh\lambda\right)^{-1}e^{-|\alpha|^2}\left\{\left(1+e^{\lambda}\right)e^{e^{\lambda}|\alpha|^2}\cdot\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]-\left(1+e^{-\lambda}\right)e^{e^{-\lambda}|\alpha|^2}\cdot\left[e^{-\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{-\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}\xi_q\left(\alpha,\alpha^*;t\right)$$

(5.67a)

and,

$$\frac{\partial\xi_q\left(\alpha,\alpha^*;t\right)}{\partial t}=-i\omega\left(2\left(e^{\lambda}-1\right)\right)^{-1}e^{-|\alpha|^2}\cdot\left\{\left(1+e^{\lambda}\right)e^{e^{\lambda}|\alpha|^2}\left[e^{\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}-e^{\lambda\alpha\frac{\partial}{\partial\alpha}}\right]\right\}\xi_q\left(\alpha,\alpha^*;t\right)$$

(5.67b)

respectively.

Eqns. (5.67) represent the quantum Liouville equations for the q-deformed quantum harmonic oscillator in the α-representation. The probability distribution function $\xi_q \;\; \alpha,\alpha^*;t$ in eqn. (5.67a) is equal to $Q_q(\alpha,\alpha^*;t)$ since eqn. (5.67a) is the same as eqn. (5.54).

5.2 The Quantum Liouville Equation of the q-Deformed 1-D Quantum Harmonic Oscillator in the α_q-Representation

A quantum Liouville equation for the q-deformed quantum harmonic oscillator can be also obtained for the q-deformation defined by $f \;\; \hat{N} = \sqrt{\frac{\hat{N} \;\; q}{\hat{N}}}$ where $\left[\hat{N}\right]_q = \frac{q^{\hat{N}} - 1}{q-1}$ in the α_q-representation. The aim of this section is to show that this is possible by using the q-deformed P-representation based on the q-deformed coherent states that have been introduced by Arik and Coon [9].

5.2.1 The q-Analog of Glauber-Sudarshan P-Representation

The q-deformed boson operators $\hat{a}_q$ and $\hat{a}_q^\dagger$ in this case obey the q-commutation relation [9]:

$$\left[\hat{a}_q, \hat{a}_q^\dagger\right]_q = \hat{a}_q \hat{a}_q^\dagger - q\,\hat{a}_q^\dagger \hat{a}_q = 1 \tag{5.68}$$

where $\left[\hat{a}_q, \hat{a}_q^\dagger\right]_q$ represents the q-commutator.

The unnormalized q-deformed coherent state $\left\|\alpha_q\right\rangle$ is defined as the eigenfunction of the q-deformed annihilation boson operator $\hat{a}_q$, or [9]:

$$\hat{a}_q \left\| \alpha_q \right\rangle = \alpha_q \left\| \alpha_q \right\rangle \tag{5.69}$$

where,

$$\left\| \alpha_q \right\rangle = \sum_{n=0}^{\infty} \frac{\alpha_q^n}{[n]_q!} \left\| n \right\rangle_q \tag{5.70}$$

and the q-deformed Hilbert space bases are given by [9]:

$$\left\| n \right\rangle_q = \left(\hat{a}_q^\dagger\right)^n \left\| 0 \right\rangle_q \tag{5.71}$$

such that

$$\hat{a}_q \left\| 0 \right\rangle_q = 0 \tag{5.72}$$

$${}_q\left\langle 0 \right\| 0 \right\rangle_q = 1 \tag{5.73}$$

It can also be shown that [9]:

$$\hat{a}_q^\dagger \left\| n \right\rangle_q = \left\| n+1 \right\rangle_q \tag{5.74}$$

$$\hat{a}_q \left\| n \right\rangle_q = [n]_q \left\| n-1 \right\rangle_q \tag{5.75}$$

where $[n]_q$ is as defined by eqn. (3.5).

Therefore, the scalar product ${}_q\left\langle m \right\| n \right\rangle_q$, using eqns. (5.71), (5.74) and (5.75), becomes:

$${}_q\left\langle m \right\| n \right\rangle_q = {}_q\left\langle 0 \right\| \left(\hat{a}_q\right)^m \left(\hat{a}_q^\dagger\right)^n \left\| 0 \right\rangle_q = [n]_q!\ \delta_{nm} \tag{5.76}$$

where $[\mathrm{n}]_q!$ is as defined by eqn. (3.12).

The action of $\hat{a}_q^{\dagger}$ on $\|\alpha_q\rangle$ can also be written as (see Appendix-G)

$$\hat{a}_q^{\dagger}\|\alpha_q\rangle = \frac{D}{D\alpha_q}\|\alpha_q\rangle \tag{5.77}$$

where $\frac{D}{D\alpha_q}$ represents the q-differential operator defined as [9]:

$$\frac{D}{D\alpha_q} f\left(\alpha_q\right) = \frac{f\left(\alpha_q\right) - f\left(q\alpha_q\right)}{\alpha_q - q\alpha_q} \tag{5.78}$$

The operators $\hat{a}_q$ and $\hat{a}_q^{\dagger}$ are Hermitian adjoints of each other and the Hilbert space $\left\{\|\mathrm{n}\rangle_q\right\}$ is the Hilbert space adjoint of the Hilbert Space $\left\{{}_q\langle\mathrm{n}\|\right\}$. Therefore,

$$\left(\hat{a}_q\|\alpha_q\rangle\right)^{\dagger} = \langle\alpha_q\|\,\alpha_q^{*} \tag{5.79}$$

and,

$$\langle\alpha_q\|\hat{a}_q = \left(\hat{a}_q^{\dagger}\|\alpha_q\rangle\right)^{\dagger} = \langle\alpha_q\|\,\frac{D}{D\alpha_q^{*}} \tag{5.80}$$

Arik and Coon [9] have also proved that for the states $\left\{\|\alpha_q\rangle\right\}$ there is also a resolution of identity in the form:

$$\hat{\mathrm{I}} = \frac{1}{\pi}\,\mathbb{S}\,\frac{D^2\alpha_q}{E_q(\sigma)}\|\alpha_q\rangle\langle\alpha_q\| \tag{5.81}$$

where,

$$\sigma = q\left|\alpha_q\right|^2 \tag{5.82}$$

$$D^2\alpha_q = D\alpha_q\, D\alpha_q^* \tag{5.83}$$

and D is the q-deformed differential associated with the q-deformed differential operator $\frac{D}{D\alpha_q}$, $\mathbb{S}$ represents the basic integral, which is for a function F of real variable x defined as [9]:

$$\mathbb{S}_0^{\mathrm{b}} F(x)Dx \equiv (1-q)\mathrm{b}\sum_{\ell=0}^{\infty} q^{\ell} F\left(q^{\ell}\mathrm{b}\right) \tag{5.84}$$

and the q-deformed exponential $E_q(\sigma)$ is defined as [9]:

$$E_q(\sigma) = \frac{1}{G\left(q(1-q)\left|\alpha_q\right|^2\right)} \tag{5.85}$$

where, $G\left(q(1-q)\left|\alpha_q\right|^2\right)$ is an entire function and $\lim_{q\to 1} E_q(\sigma) = e^{|\alpha|^2}$.

Then, by using the resolution of identity as in eqn. (5.81), one can write the $\mathbb{P}_q$-representation of the density operator $\hat{\rho}_q$ in the α_q-representation as:

$$\hat{\rho}_q = \frac{1}{\pi}\mathbb{S}\,\frac{D^2\alpha_q}{E_q(\sigma)}\mathbb{P}_q\left(\alpha_q,\alpha_q^*\right)\left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \tag{5.86}$$

where, $\hat{\rho}_q$ and $\mathbb{P}_q\left(\alpha_q,\alpha_q^*\right)$ represent the q-analogs of the density operator $\hat{\rho}$ and the weight function $P\left(\alpha,\alpha^*\right)$ for the Glauber-Sudarshan P-representation respectively (see eqn. (2.59).

Using these results, one can write the following relations (see Appendix-H, eqns. (H.17), (H.18), (H.19) and (H.20))

$$\hat{a}_q \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \alpha_q \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \tag{5.87}$$

$$\hat{\rho}_q \hat{a}_q^\dagger = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \alpha_q^* \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \tag{5.88}$$

$$\hat{a}_q^\dagger \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \tag{5.89}$$

and,

$$\hat{\rho}_q \hat{a}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q - \frac{D}{D\alpha_q^*}\right] \mathbb{P}_q\left(\alpha_q, q^{-1}\alpha_q^*\right) \tag{5.90}$$

Hence, eqns. of correspondence which are the analogs of eqns. (2.65), (2.66), (2.67) and (2.68) in α-representation can be written in α_q-representation in the form:

$$\hat{a}_q \hat{\rho}_q \to \alpha_q \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \tag{5.91}$$

$$\hat{\rho}_q \hat{a}_q^\dagger \to \alpha_q^* \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \tag{5.92}$$

$$\hat{a}_q^\dagger \hat{\rho}_q \to \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \tag{5.93}$$

and,

$$\hat{\rho}_q \hat{a}_q \to \left[\alpha_q - \frac{D}{D\alpha_q^*}\right] \mathbb{P}_q\left(\alpha_q, q^{-1}\alpha_q^*\right) \tag{5.94}$$

These equations permit the derivation of q-analog equations corresponding to eqns. (2.61) - (2.64) in the α_q -representation. Thus, by multiplying both sides of eqn. (5.89) from the left by $\hat{a}_q$, one obtains:

$$\hat{a}_q \hat{a}_q^{\dagger} \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \hat{a}_q \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \quad (5.95)$$

The substitution of eqn. (5.69) into eqn. (5.95) gives:

$$\hat{a}_q \hat{a}_q^{\dagger} \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \alpha_q \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \quad (5.96)$$

But since $\hat{a}_q$ and $\hat{a}_q^{\dagger}$ are both Hermitian adjoints of each other and $\hat{\rho}_q$ is self-adjoint, then the conjugate of eqn. (5.96) becomes;

$$\hat{\rho}_q \hat{a}_q \hat{a}_q^{\dagger} = \left(\hat{a}_q \hat{a}_q^{\dagger} \hat{\rho}\right)^{\dagger} = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\|$$

$$\cdot\ \alpha_q^* \left[\alpha_q - \frac{D}{D\alpha_q^*}\right] \mathbb{P}_q\left(\alpha_q, q^{-1}\alpha_q^*\right) \quad (5.97)$$

Also, one can show that (see Appendix-H, eqns. (H.34) and (H.35))

$$\hat{a}_q^{\dagger} \hat{a}_q \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q^* - \frac{D}{D\alpha_q}\right]\left(q^{-1}\alpha_q\right) \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right)$$

$$(5.98)$$

and,

$$\hat{\rho}_q \hat{a}_q^{\dagger} \hat{a}_q = \left(\hat{a}_q^{\dagger} \hat{a}_q \hat{\rho}\right)^{\dagger} = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\|$$

$$\cdot \left[\alpha_q - \frac{D}{D\alpha_q^*}\right]\left(q^{-1}\alpha_q^*\right) \mathbb{P}_q\left(\alpha_q, q^{-1}\alpha_q^*\right) \quad (5.99)$$

Thus, the correspondence equations in the α_q-representation, which are the analogs of eqns. (D.27), (D.28), (D.29) and (D.30) in the α-representation given in Appendix-D, become (see Appendix-H, eqns. (H.36) and (H.37)):

$$\hat{a}_q \hat{a}_q^{\dagger} \hat{\rho}_q \rightarrow \alpha_q \left[\alpha_q^* - \frac{D}{D\alpha_q} \right] \mathbb{P}_q \left(q^{-1}\alpha_q, \alpha_q^* \right) \tag{5.100}$$

$$\hat{\rho}_q \hat{a}_q \hat{a}_q^{\dagger} \rightarrow \alpha_q^* \left[\alpha_q - \frac{D}{D\alpha_q^*} \right] \mathbb{P}_q \left(\alpha_q, q^{-1}\alpha_q^* \right) \tag{5.101}$$

$$\hat{a}_q^{\dagger} \hat{a}_q \hat{\rho}_q \rightarrow \left[\alpha_q^* - \frac{D}{D\alpha_q} \right] \left(q^{-1}\alpha_q \right) \mathbb{P}_q \left(q^{-1}\alpha_q, \alpha_q^* \right) \tag{5.102}$$

and,

$$\hat{\rho}_q \hat{a}_q^{\dagger} \hat{a}_q \rightarrow \left[\alpha_q - \frac{D}{D\alpha_q^*} \right] \left(q^{-1}\alpha_q^* \right) \mathbb{P}_q \left(\alpha_q, q^{-1}\alpha_q^* \right) \tag{5.103}$$

These correspondence equations represent novel results and will play an important role in deriving the quantum Liouville equation for the q-deformed quantum harmonic oscillator in the α_q-representation.

5.2.2 The Quantum Liouville Equation in Terms of q-Derivatives

The quantum Liouville equation in terms of q-derivative can be derived by substituting the Hamiltonian of the q-deformed quantum harmonic oscillator as given by eqn. (3.30a) in the Heisenberg equation of motion (i.e., eqn. (2.57)) to yield:

$$\frac{\partial \hat{\rho}_q}{\partial t} = -\left(\frac{i\omega}{2} \right) \left\{ \hat{a}_q \hat{a}_q^{\dagger} \hat{\rho}_q + \hat{a}_q^{\dagger} \hat{a}_q \hat{\rho}_q - \hat{\rho}_q \hat{a}_q \hat{a}_q^{\dagger} - \hat{\rho}_q \hat{a}_q^{\dagger} \hat{a}_q \right\} \tag{5.104}$$

Using the correspondence of eqns. (5.100) - (5.103), eqn. (5.104) for the density operator can be transformed to the $\mathbb{P}_q$ -representation in a manner similar to what has been done in the α -space. This gives:

$$\frac{\partial \mathbb{P}_q(\alpha_q,\alpha_q^*;t)}{\partial t} = -\left(\frac{i\omega}{2}\right)\left\{\alpha_q\left[\alpha_q^* - \frac{D}{D\alpha_q}\right]\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t)\right.$$
$$+\left[\alpha_q^* - \frac{D}{D\alpha_q}\right]\left(q^{-1}\alpha_q\right)\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t)$$
$$-\ \alpha_q^*\left[\alpha_q - \frac{D}{D\alpha_q^*}\right]\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t)$$
$$\left.-\left[\alpha_q - \frac{D}{D\alpha_q^*}\right](q^{-1}\alpha_q^*)\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t)\right\}$$

(5.105)

Eqn. (5.105) can be simplified to give:

$$\frac{\partial \mathbb{P}_q(\alpha_q,\alpha_q^*;t)}{\partial t} =$$
$$-\left(\frac{i\omega}{2}\right)\left\{\left[\left|\alpha_q\right|^2\right]\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t) - \alpha_q\frac{D}{D\alpha_q}\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t)\right.$$
$$+\left[q^{-1}\left|\alpha_q\right|^2\right]\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t) - \frac{D}{D\alpha_q}\left[\left(q^{-1}\alpha_q\right)\mathbb{P}_q(q^{-1}\alpha_q,\alpha_q^*;t)\right]$$
$$-\left[\left|\alpha_q\right|^2\right]\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t) + \left(\alpha_q^*\right)\frac{D}{D\alpha_q^*}\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t)$$
$$\left.-\left[q^{-1}\left|\alpha_q\right|^2\right]\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t) + \frac{D}{D\alpha_q^*}\left[(q^{-1}\alpha_q^*)\mathbb{P}_q(\alpha_q,q^{-1}\alpha_q^*;t)\right]\right\}$$

(5.106)

Applying the product rule of q-differentiation eqn. (3.25) to $\frac{D}{D\alpha_q}\left[\left(q^{-1}\alpha_q\right)\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)\right]$ with $F(x)=q^{-1}\alpha_q$ and $G(x)=\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)$ gives:

$$\frac{D}{D\alpha_q}\left[\left(q^{-1}\alpha_q\right)\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)\right]=$$

$$q^{-1}\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)+\alpha_q\frac{D}{D\alpha_q}\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right) \tag{5.107}$$

And, similarly for $\frac{D}{D\alpha_q^*}\left[\left(q^{-1}\alpha_q^*\right)\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)\right]$, but with $F(x)=q^{-1}\alpha_q^*$ and $G(x)=\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)$, one obtains:

$$\frac{D}{D\alpha_q^*}\left[\left(q^{-1}\alpha_q^*\right)\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)\right]=$$

$$q^{-1}\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)+\left(\alpha_q^*\right)\frac{D}{D\alpha_q^*}\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right) \tag{5.108}$$

Substituting eqns. (5.107) and (5.108) into eqn. (5.106) and simplifying the result, then:

$$\begin{aligned}\frac{\partial\mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)}{\partial t}&=-\left(\frac{i\omega}{2}\right)\\ &\cdot\left\{\left[2\,\alpha_q^*\frac{D}{D\alpha_q^*}\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)-2\,\alpha_q\frac{D}{D\alpha_q}\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)\right]\right.\\ &\quad+\left(1+q^{-1}\right)\left|\alpha_q\right|^2\left[\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)-\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)\right]\\ &\quad\left.+q^{-1}\left[\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*;t\right)-\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*;t\right)\right]\right\}\end{aligned} \tag{5.109}$$

The Jackson derivative $\mathcal{D}_z^q$ is defined as [115-117]:

$$\mathcal{D}_z^q = \frac{D}{Dz} \tag{5.110}$$

Then, using $q^{\pm 1} = e^{\pm \lambda}$ and eqn.(5.110) in eqn. (5.109), this equation becomes:

$$\begin{aligned}
\frac{\partial \mathbb{P}_q\left(\alpha_q, \alpha_q^*; t\right)}{\partial t} &= -\left(\frac{i\omega}{2}\right) \\
&\cdot \left\{ \left[2\,\alpha_q^*\,\mathcal{D}_{\alpha_q^*}^q\; \mathbb{P}_q\left(\alpha_q, e^{-\lambda}\,\alpha_q^*; t\right) - 2\,\alpha_q\,\mathcal{D}_{\alpha_q}^q\; \mathbb{P}_q\left(e^{-\lambda}\,\alpha_q, \alpha_q^*; t\right) \right] \right. \\
&+ \left(1 + e^{-\lambda}\right) \left|\alpha_q\right|^2 \left[\mathbb{P}_q\left(e^{-\lambda}\,\alpha_q, \alpha_q^*; t\right) - \mathbb{P}_q\left(\alpha_q, e^{-\lambda}\,\alpha_q^*; t\right) \right] \\
&\left. + e^{-\lambda} \left[\mathbb{P}_q\left(\alpha_q, e^{-\lambda}\,\alpha_q^*; t\right) - \mathbb{P}_q\left(e^{-\lambda}\,\alpha_q, \alpha_q^*; t\right) \right] \right\}
\end{aligned} \tag{5.111}$$

Using the analogs of eqns. (5.42) and (5.43), in the α_q -representation, one can write:

$$e^{-\lambda \alpha_q \frac{\partial}{\partial \alpha_q}}\; \mathbb{P}_q\left(\alpha_q, \alpha_q^*; t\right) = \mathbb{P}_q\left(e^{-\lambda}\,\alpha_q, \alpha_q^*; t\right) \tag{5.112}$$

$$e^{-\lambda \alpha_q^* \frac{\partial}{\partial \alpha_q^*}}\; \mathbb{P}_q\left(\alpha_q, \alpha_q^*; t\right) = \mathbb{P}_q\left(\alpha_q, e^{-\lambda}\,\alpha_q^*; t\right) \tag{5.113}$$

Substituting eqns. (5.112) and (5.113) into eqn. (5.111), the final result is:

$$\frac{\partial \mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)}{\partial t} = -\left(\frac{i\omega}{2}\right)$$

$$\cdot\left\{\left[2\,\alpha_q^*\,\mathcal{D}_{\alpha_q^*}^{q}\;e^{-\lambda\,\alpha_q^*\frac{\partial}{\partial\alpha_q^*}} - 2\,\alpha_q\,\mathcal{D}_{\alpha_q}^{q}\;e^{-\lambda\,\alpha_q\frac{\partial}{\partial\alpha_q}}\right]\right.$$

$$+\left(1+e^{-\lambda}\right)\left|\alpha_q\right|^2\left[e^{-\lambda\,\alpha_q\frac{\partial}{\partial\alpha_q}} - e^{-\lambda\,\alpha_q^*\frac{\partial}{\partial\alpha_q^*}}\right]$$

$$\left.+\,e^{-\lambda}\left[e^{-\lambda\,\alpha_q^*\frac{\partial}{\partial\alpha_q^*}} - e^{-\lambda\,\alpha_q\frac{\partial}{\partial\alpha_q}}\right]\right\}\mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)$$

(5.114)

Eqn. (5.114) represents the quantum Liouville equation for the probability distribution function $\mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)$ of the q-deformed quantum harmonic oscillator in the α_q -representation.

5.3 The Classical Limit of the q-Deformed Quantum Harmonic Oscillator

The present section is devoted to the investigation of the classical limit of the q-deformed 1-D quantum harmonic oscillator. This investigation is performed on the basis of the quantum Liouville equations corresponding to this oscillator that have been derived in Secs. (5.1) and (5.2).

5.3.1 The Approach to the Classical Limit

The classical limit for the q-deformed 1-D quantum harmonic oscillator can be approached in a way similar to what has been done by Ghosh et al [76] for the undeformed 1-D quantum oscillator with some necessary modifications.

In the case of the undeformed 1-D quantum oscillator, the following conditions are used to approach the classical limit [76]:

$$\left.\begin{array}{l}\hbar \to 0, |\alpha|^2 \to \infty \\ \text{such that} \\ \hbar|\alpha|^2 \to \text{finite}\end{array}\right\} \tag{5.115}$$

These conditions are necessary to obtain the energy as the classical limit of the expectation value of the Hamiltonian for the case of the non-deformed oscillator where it is required that this expectation value (i.e., eqn. (2.51)) remains finite as $\hbar \to 0$.

For the case of the q-deformed 1-D quantum harmonic oscillator, the expectation value of the Hamiltonian in a coherent state $|\alpha\rangle$ can be obtained as:

$$\langle\alpha|\hat{\mathbb{H}}_q|\alpha\rangle = \left(\frac{\hbar\omega}{4}\right)\left(\frac{1}{\sinh(\lambda)}\right)\left\{\left(1+e^{\lambda}\right) e^{\lambda|\alpha|^2} - \left(1+e^{-\lambda}\right) e^{-\lambda|\alpha|^2}\right\} \tag{5.116}$$

where the Hamiltonian $\hat{\mathbb{H}}_q$ is given by eqn. (3.34), and the q-number operators are given as $\hat{N}_q = \frac{\sinh \lambda \hat{N}}{\sinh \lambda}$ and $\hat{N}+1_q = \frac{\sinh \lambda[\hat{N}+1]}{\sinh \lambda}$.

Again, the classical limit of this expectation value could be taken to correspond to the classical energy of the q-deformed oscillator.

For this expectation value to remain finite as $\hbar \to 0$, a modification to the conditions given in eqns. (5.115) is necessary. This is due to the introduction of q-deformation which is related to the non-linearity parameter λ. The modified conditions can be obtained by expanding all the exponential functions appearing in the expression for the expectation value of eqn. (5.116) as power series in λ to obtain:

$$\langle\alpha|\hat{\mathbb{H}}_q|\alpha\rangle = \left(\frac{\hbar\omega}{4}\right)\left\{\frac{1}{\lambda\left[1+\frac{\lambda^2}{3!}+\frac{\lambda^4}{5!}+\cdots\right]}\right\}$$
$$\cdot\left\{\left[2+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right]\left(1+\lambda|\alpha|^2+\frac{1}{2!}\lambda^2|\alpha|^4+\frac{1}{3!}\lambda^3|\alpha|^6+\cdots\right)\right.$$
$$\left.-\left[2-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\cdots\right]\left(1-\lambda|\alpha|^2+\frac{1}{2!}\lambda^2|\alpha|^4-\frac{1}{3!}\lambda^3|\alpha|^6+\cdots\right)\right\}$$

(5.117)

Simplifying eqn. (5.117) then, after some lengthy mathematical manipulations, one obtains:

$$\langle\alpha|\hat{\mathbb{H}}_q|\alpha\rangle = \left(\frac{\hbar\omega}{4}\right)\left\{\frac{1}{\left[1+\frac{\lambda^2}{3!}+\frac{\lambda^4}{5!}+\cdots\right]}\right\}\left\{4|\alpha|^2+\frac{2\lambda^2}{3}|\alpha|^6+\cdots+2+\lambda^2|\alpha|^4+\cdots\right.$$
$$\left.+\lambda^2|\alpha|^2+\frac{\lambda^4}{6}|\alpha|^6+\cdots+\frac{\lambda^2}{3}+\frac{\lambda^4}{6}|\alpha|^4+\cdots\right\}$$

(5.118)

Letting $\lambda\to 0$ (i.e., $q\to 1$) in eqn. (5.118), this equation will reduce, as expected, to the energy equation for the undeformed quantum harmonic oscillator of eqn. (2.51). Therefore, to apply the condition for approaching the classical limit, $\hbar\to 0$ one should take into consideration the fact that $\lambda\neq 0$ in the present q-deformed case.

The main idea again is to let the energy as represented by the expectation value of eqn. (5.118) remain finite as $\hbar\to 0$. It appears from eqn. (5.118) that as

$\hbar \to 0$ this expectation value can remain finite only if some additional conditions on λ are applied. Table (5.5) shows the different cases that can arise.

Table (5.5)

Different cases that arise in taking the limit $\hbar \to 0$ for the q-deformed oscillator

	limiting condition	existence of classical limit
case-1	$\hbar \to 0$, $\lvert\alpha\rvert^2 \to \infty$ such that $\hbar\lvert\alpha\rvert^2 \to \text{finite}$, and keeping λ fixed.	No
case-2	$\hbar \to 0$, $\lvert\alpha\rvert^2 \to \infty$ such that $\hbar \to 0$ faster than $\lambda \to 0$, i.e., $\lambda = (const.) \cdot \hbar^{\delta}$ where $\delta = 1-\varepsilon$ and $0 < \varepsilon < 1$, hence, $\lambda\lvert\alpha\rvert^2 \to \text{undefined}$.	No
case-3	$\hbar \to 0$, $\lvert\alpha\rvert^2 \to \infty$ such that $\lambda \to 0$ as fast as $\hbar \to 0$, i.e. , $\lambda = (const.) \cdot \hbar$, hence $\lambda\lvert\alpha\rvert^2 \to \text{finite}$	Yes
case-4	same as case-3 but $\lambda \to 0$ faster than $\hbar \to 0$, i.e., $\lambda = (const.) \cdot \hbar^{\delta}$ where $\delta = 1+\varepsilon$ and $\varepsilon > 1$, hence, $\lambda\lvert\alpha\rvert^2 \to 0$	Yes, but the q-deformation must vanish

There are different reasons for the non-existence of classical limits in some of the cases shown in Table (5.5). For example, when applying the condition $\hbar \to 0$, $|\alpha|^2 \to \infty$ such that $\hbar|\alpha|^2 \to$ finite and keeping λ fixed (case-1), the terms in eqn. (5.118) containing $\lambda^{2m}|\alpha|^{2m+2}$, $\lambda^{2m}|\alpha|^{2m+4}$ and $\lambda^{2j}|\alpha|^{2j}$ (i.e., $m=1,2,\cdots$ and $j=2,3,\cdots$) will blow-up (i.e., go to ∞). This is interpreted to mean that under such a condition (i.e., keeping λ fixed), a classical limit does not exist [97]. While when applying the condition $\hbar \to 0$ faster than $\lambda \to 0$, $|\alpha|^2 \to \infty$ and $\lambda = (const.)\cdot\hbar\cdot\hbar^{-\varepsilon}$ such that $\lambda|\alpha|^2 \to$undefined (i.e., case 2 in Table (5.5)), eqn. (5.118) does not produce a finite value for the energy. This is also interpreted to mean that under such condition (i.e., $\hbar \to 0$ faster than $\lambda \to 0$) the classical limit does not exist.

Applying the other limiting condition $\lambda \to 0$ faster than $\hbar \to 0$, $|\alpha|^2 \to \infty$ and $\lambda = (const.)\cdot\hbar\cdot\hbar^{\varepsilon}$ such that $\lambda|\alpha|^2 \to 0$, (i.e., case 4 in Table (5.5)) then, eqn. (5.118) produces a finite value for the energy of the undeformed quantum oscillator (i.e. ,$\langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega|\alpha|^2$). Again, this means that under such a condition (i.e., $\lambda \to 0$ faster than $\hbar \to 0$), a classical limit exists but the q-deformation must vanish. This case is similar to the case of taking $q \to 1$ (i.e., $\lambda \to 0$) in eqn. (5.118) as mentioned before. However, applying the limiting condition, $\lambda \to 0$ as fast as $\hbar \to 0$, $|\alpha|^2 \to \infty$ and $\lambda = (const.)\cdot\hbar$ such that $\lambda|\alpha|^2 \to$finite (i.e., case 3 in Table (5.5)), then all terms in eqn. (5.118) containing $\lambda^m|\alpha|^{2m+2}$ $(m=1,2,\cdots)$ remain finite with other terms vanishing. Therefore, eqn. (5.118) reduces to the energy of the q-deformed classical harmonic oscillator as:

$$\langle\alpha|\hat{\mathbb{H}}_q|\alpha\rangle = \left(\frac{\hbar\omega}{4}\right)\left\{4|\alpha|^2 + \frac{2\lambda^2}{3}|\alpha|^6 + \cdots\right\} \tag{5.119}$$

This result means that under the condition, $\lambda \to 0$ as fast as $\hbar \to 0$ (i.e., case 3 in Table (5.5)), a classical limit exists. Therefore, this limiting condition (case 3) will be adopted in this thesis to approach the classical limit for the q-deformed quantum oscillator on the basis of its various representations of the Liouville equation.

5.3.2 The Classical Limit of the Liouville Equation for Q_q in the α-Representation

Expanding all functions appearing in eqn. (5.54) as power series in λ, and simplifying the result, one obtains:

$$\frac{\partial Q_q(\alpha,\alpha^*;t)}{\partial t} = -\left(\frac{i\omega}{4}\right)\left[\frac{1}{[1+S_1(\lambda)]}\right]$$
$$\cdot\left\{[2+S_2(\lambda)]\, e^{\left[\lambda|\alpha|^2+\frac{\lambda^2}{2!}|\alpha|^2+\cdots\right]}\right.$$
$$\cdot\left[\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)+\frac{\lambda}{2!}\left\{\left(\alpha^*\frac{\partial}{\partial\alpha^*}\right)^2-\left(\alpha\frac{\partial}{\partial\alpha}\right)^2\right\}+\cdots\right]$$
$$+[2+S_3(\lambda)]\, e^{\left[-\lambda|\alpha|^2+\frac{\lambda^2}{2!}|\alpha|^2-\cdots\right]}$$
$$\left.\cdot\left[\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)-\frac{\lambda}{2!}\left\{\left(\alpha^*\frac{\partial}{\partial\alpha^*}\right)^2-\left(\alpha\frac{\partial}{\partial\alpha}\right)^2\right\}+\cdots\right]\right\}$$
$$\cdot Q_q(\alpha,\alpha^*;t) \qquad (5.120)$$

where,

$$\left.\begin{aligned} S_1(\lambda) &= \frac{\lambda^2}{3!} + \frac{\lambda^4}{5!} + \frac{\lambda^6}{7!} + \cdots \\ S_2(\lambda) &= \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \\ S_3(\lambda) &= -\lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots \end{aligned}\right\} \tag{5.121}$$

Applying the adopted condition for the classical limit, i.e., $\hbar \to 0$, $|\alpha|^2 \to \infty$ such that $\lambda|\alpha|^2 \to \text{finite}$, to eqn. (5.120), where $\lambda = (const.)\cdot\hbar$, and letting $Q_q(\alpha,\alpha^*;t) \to \mathcal{P}_{CL}^q(\alpha,\alpha^*;t)$ in this limit, then eqn. (5.120) reduces to:

$$\frac{\partial \mathcal{P}_{CL}^q(\alpha,\alpha^*;t)}{\partial t} = -\left(\frac{i\omega}{4}\right)\left\{2e^{\lambda|\alpha|^2}\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\right.$$
$$\left. + 2\,e^{-\lambda|\alpha|^2}\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\right\}\mathcal{P}_{CL}^q(\alpha,\alpha^*;t)$$

(5.122)

where the fact that in this limit $S_1(\lambda) = S_2(\lambda) = S_3(\lambda) = 0$ has been used. Re-arranging the terms in eqn. (5.122), this equation becomes:

$$\frac{\partial \mathcal{P}_{CL}^q(\alpha,\alpha^*;t)}{\partial t} = -i\,\omega_q^{(1)}\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\mathcal{P}_{CL}^q(\alpha,\alpha^*;t) \tag{5.123}$$

where,

$$\omega_q^{(1)} = \omega\cosh\left(\lambda|\alpha|^2\right) \tag{5.124}$$

Eqn. (5.123) represents a classical Liouville equation for a classical harmonic oscillator having frequency $\omega_q^{(1)}$. By expanding the frequency of this oscillator $\omega_q^{(1)}$ up to λ^2, eqn. (5.123) becomes:

$$\frac{\partial \mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)}{\partial t}=$$
$$-i\omega\left\{\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)+\frac{\lambda^2}{2!}|\alpha|^4\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)\right\}\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right) \tag{5.125}$$

Eqn. (5.125) can be interpreted as a classical Liouville equation for a classical harmonic oscillator with frequency:

$$\omega_q^{(1)}=\omega\left(1+\frac{\lambda^2}{2!}|\alpha|^4\right) \tag{5.126}$$

It is apparent that the 1st term on the right hand side of eqn. (5.125) represents the classical Liouville equation for a simple harmonic oscillator with frequency ω. Thus, eqn. (5.125) agrees with that obtained by Ghosh [76] for the undeformed case but with an additional term $\omega\frac{\lambda^2}{2!}|\alpha|^4\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)$ resulting from the q-deformation.

Also, the comparison between eqn. (5.125) and the corresponding eqn. (4.70) shows that, when $\omega_q^{(1)}$ (i.e., eqn. (4.42a)), is expanded up to first order of λ^2, the result agrees with that of eqn. (5.126).

5.3.3 The Classical Limit of the Liouville Equation for P_q in the α-Representation

Using a technique similar to that of Sec. (5.3.2), eqn. (5.55) results in

$$\frac{\partial P_q(\alpha,\alpha^*;t)}{\partial t} = -\left(\frac{i\omega}{4}\right)\left[\frac{1}{\left(1+S_1(\lambda)\right)}\right]$$
$$\cdot\left\{\left(2+S_3(\lambda)\right)\left(1+\lambda+\frac{\lambda^2}{2!}+\cdots\right)e^{-\left[\lambda|\alpha|^2+\frac{\lambda^2}{2!}|\alpha|^2+\cdots\right]}\right.$$
$$\cdot\left[\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)+\frac{\lambda}{2!}\left(\left(\alpha^*\frac{\partial}{\partial\alpha^*}\right)^2-\left(\alpha\frac{\partial}{\partial\alpha}\right)^2\right)+\cdots\right]$$
$$-\left(2+S_2(\lambda)\right)\left(1-\lambda+\frac{\lambda^2}{2!}-\cdots\right)e^{-\left[-\lambda|\alpha|^2+\frac{\lambda^2}{2!}|\alpha|^2-\cdots\right]}$$
$$\left.\cdot\left[-\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)+\frac{\lambda}{2!}\left(\left(\alpha^*\frac{\partial}{\partial\alpha^*}\right)^2-\left(\alpha\frac{\partial}{\partial\alpha}\right)^2\right)-\cdots\right]\right\}$$
$$\cdot P_q(\alpha,\alpha^*;t)$$

(5.127)

Applying the limiting condition, $\hbar\to 0$, $|\alpha|^2\to\infty$ and $\lambda\to 0$ as` fast as $\hbar\to 0$, i.e., $\lambda=(const.)\cdot\hbar$, hence $\lambda|\alpha|^2\to\text{finite}$, to eqn. (5.127) and letting $P_q(\alpha,\alpha^*;t)\to\mathcal{P}_{CL}^{q}(\alpha,\alpha^*;t)$, one obtains the same result as in eqn. (5.122). Also, applying the same limiting condition to eqn. (5.67a) (i.e., the Liouville equation obtained from Zaslavskii's method) and letting $\xi_q\ \alpha,\alpha^*;t\ \to\mathcal{P}_{CL}^{q}\ \alpha,\alpha^*;t$, produce the same result as in eqn. (5.122).

Similarly, applying the same previously mentioned limiting conditions (i.e., case-3) to eqn. (5.67b), leads to eqn. (5.123) but with q-deformed frequency given as:

$$\omega_q^{(2)} = \omega\, e^{\lambda|\alpha|^2} \tag{5.128}$$

As in the case of the classical limit of the Liouville equation for Q_q, eqn. (5.123) can be compared with the corresponding eqn. (4.70) by expanding eqn. (5.128) up to first order in λ. The result is:

$$\omega_q^{(2)} = \omega\left(1+\lambda|\alpha|^2\right) \tag{5.129}$$

Similarly, the expansion of eqn. (4.42b) leads to the same result given by eqn. (5.129) and, hence, eqn. (5.123) agrees with the corresponding eqn. (4.70).

Again, substituting eqn. (5.129) into eqn. (5.123), one obtains a classical Liouville equation consisting of two terms. The 1st term represents the classical Liouville equation for a simple harmonic oscillator with frequency ω. This result is the same as that obtained by Ghosh et al. [76]. The 2nd term is $\omega\lambda\,|\alpha|^2\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)$ resulting from the effect of q-deformation.

5.3.4 The Classical Limit of the Liouville Equation for $\mathbb{P}_q$ in the α_q-Representation

Using the same technique of expanding the exponentials functions as power series in λ, and after some lengthy manipulations, eqn. (5.114) becomes:

$$\frac{\partial \mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)}{\partial t}=$$

$$-\left(\frac{i\omega}{2}\right)\left\{\left[2\,\alpha_q^*\,\mathcal{D}_{\alpha_q^*}^{q}\left[1-\lambda\,\alpha_q^*\frac{\partial}{\partial\alpha_q^*}+\frac{\lambda^2}{2!}\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}\right)^2-\cdots\right]\right.\right.$$

$$\left.-2\,\alpha_q\,\mathcal{D}_{\alpha_q}^{q}\left[1-\lambda\,\alpha_q\frac{\partial}{\partial\alpha_q}+\frac{\lambda^2}{2!}\left(\alpha_q\frac{\partial}{\partial\alpha_q}\right)^2-\cdots\right]\right]$$

$$+\left(2+S_3(\lambda)\right)\lambda\left|\alpha_q\right|^2\left[\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right)\right.$$

$$\left.-\left(\frac{\lambda}{2!}\right)\left[\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}\right)^2-\left(\alpha_q\frac{\partial}{\partial\alpha_q}\right)^2\right]+\cdots\right]$$

$$+\left(1+S_4(\lambda)\right)\lambda\left[-\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right)\right.$$

$$\left.\left.+\left(\frac{\lambda}{2!}\right)\left[\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}\right)^2-\left(\alpha_q\frac{\partial}{\partial\alpha_q}\right)^2\right]-\cdots\right]\right\}\mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)$$

(5.130)

where,

$$S_4(\lambda)=-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\cdots \tag{5.131}$$

Letting $\lambda\to 0$ (i.e., $q\to 1$) in eqn. (5.130) leads to

$$\frac{\partial P_{\mathrm{CL}}\left(\alpha,\alpha^*;t\right)}{\partial t}=-i\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right)P_{\mathrm{CL}}\left(\alpha,\alpha^*;t\right) \tag{5.132}$$

where, in this limit the relations [115-117]

$$\left.\begin{aligned}\lim_{\lambda\to 0}\mathcal{D}^{q}_{\alpha^*_q} &\to \frac{\partial}{\partial\alpha^*}\\ \lim_{\lambda\to 0}\mathcal{D}^{q}_{\alpha_q} &\to \frac{\partial}{\partial\alpha}\end{aligned}\right\} \tag{5.133}$$

have been used.

Eqn. (5.130) is the same as that obtained by Ghosh et al. [76] for the undeformed case as expected.

Now, applying the limiting conditions $\hbar\to 0$, $\left|\alpha_q\right|^2\to\infty$ and $\lambda\to 0$ as fast as $\hbar\to 0$, i.e., $\lambda=(const.)\cdot\hbar$ such that $\lambda\left|\alpha_q\right|^2\to\text{finite}$, to eqn. (5.130), with $\mathbb{P}_q\left(\alpha_q,\alpha_q^*;t\right)\to P^q_{CL}\left(\alpha_q,\alpha_q^*;t\right)$, the result is:

$$\frac{\partial P^q_{CL}\left(\alpha_q,\alpha_q^*;t\right)}{\partial t}=$$
$$-i\omega\left\{\left[\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right]+\lambda\left|\alpha_q\right|^2\left[\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right]\right\}P^q_{CL}\left(\alpha_q,\alpha_q^*;t\right) \tag{5.134}$$

Eqn. (5.134) can be re-written in the form:

$$\frac{\partial P^q_{CL}\left(\alpha_q,\alpha_q^*;t\right)}{\partial t}=-i\omega_q^{(3)}\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right)P^q_{CL}\left(\alpha_q,\alpha_q^*;t\right) \tag{5.135}$$

where,

$$\omega_q^{(3)}=\omega\left(1+\lambda\left|\alpha_q\right|^2\right) \tag{5.136}$$

Eqn. (5.135) is the classical Liouville equation for harmonic oscillator with frequency $\omega_q^{(3)}$. This result for $\omega_q^{(3)}$ could be shown to agree with the result obtained by Shabanov [89] using the path integral method to approach the classical limit for the q-deformed oscillator with q-deformation possessing the box function defined in eqn. (3.6).

The comparison between eqn. (5.135) and the corresponding eqn. (4.79) has been performed by using the same procedure presented in Sec. (5.3.2) and Sec. (5.3.3). The result shows agreement between these equations when the expansion of eqn. (4.57b) is accomplished up to first order in λ, thus, eqn. (5.135) agrees with eqn. (4.79).

Also, it is noticed that the classical Liouville equation for the simple harmonic oscillator corresponds to the first term in eqn. (5.135). The additional term in this equation $\omega\,\lambda\left|\alpha_q\right|^2\left(\alpha_q^*\frac{\partial}{\partial\alpha_q^*}-\alpha_q\frac{\partial}{\partial\alpha_q}\right)$ results from the effect of q-deformation.

5.4 Computer Visualizations

It can be noticed that the classical Liouville equations, obtained from applying the classical limiting procedure in this chapter, are similar to those derived in Chapter 4 where the frequency of the system is also a function of $|\alpha|^2$. Therefore, one can use the same solution procedure introduced in the previous chapter to solve these equations. Then, computer visualizations similar to those used in that chapter can be employed to study the behavior of the q-deformed oscillator in its different classical limits. Performing this procedure, the results depicted in Figs. (5.1) - (5.3) are obtained for the 2-dimensional time-evolution contours of the probability distribution functions $\mathcal{P}_{CL}^{q}\ \alpha,\alpha^*;t$ and $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ in phase space. These probability distributions exhibit whorl shapes, and it can be seen that these whorl shapes become finer as $t\to\infty$. Again, these whorl shapes can be compared with those obtained by Milburn [77] for the anharmonic oscillator shown in Fig. (4.1).

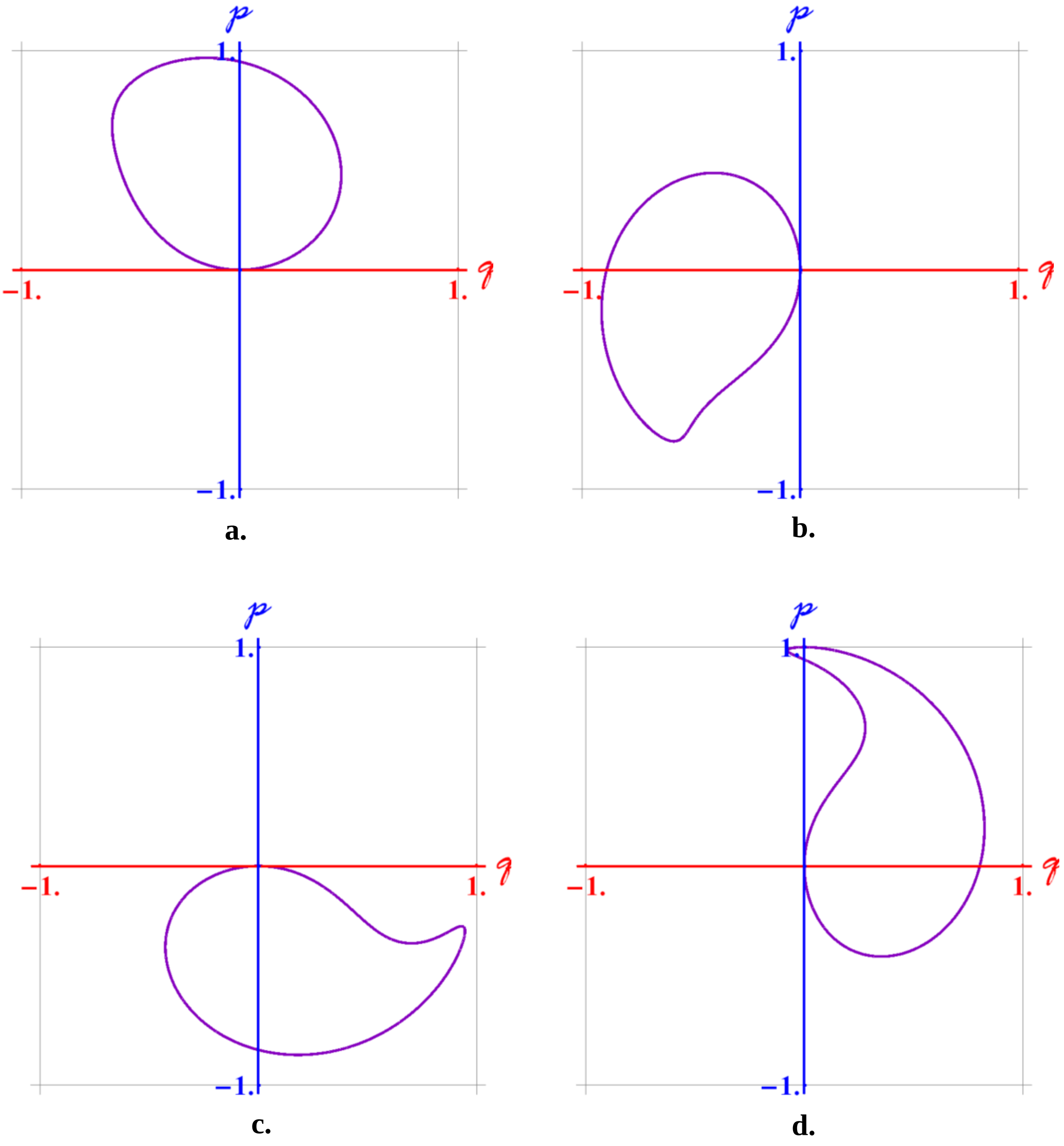

Fig. (5.1): The 2-D time-evolution contours of the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_{q}^{(1)}$ given by eqn. (5.124) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

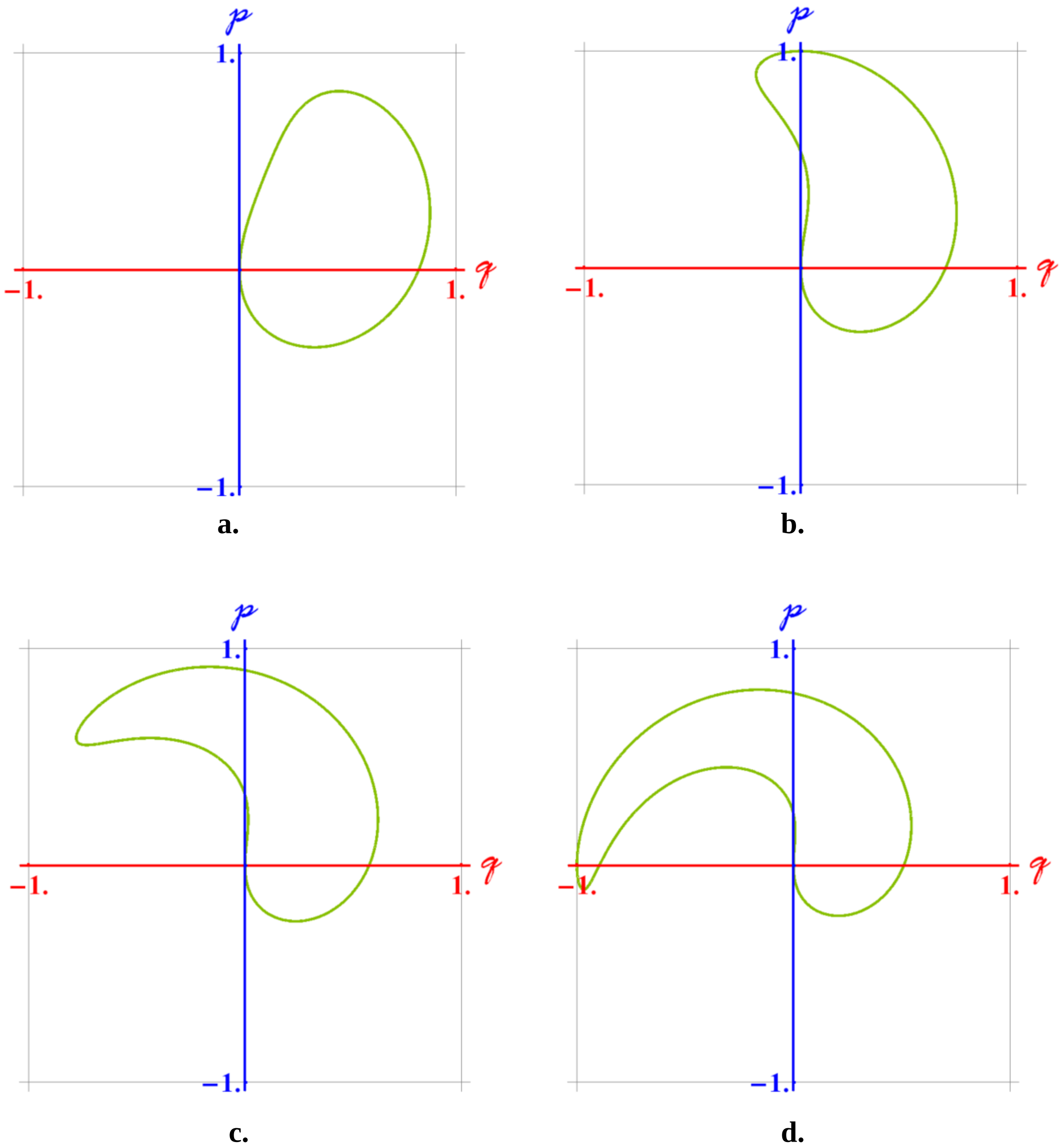

Fig. (5.2): The 2-D time-evolution contours of the classical probability distribution function $\mathcal{P}_{CL}^{q}(\alpha,\alpha^{*};t)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(2)}$ given by eqn. (5.128) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

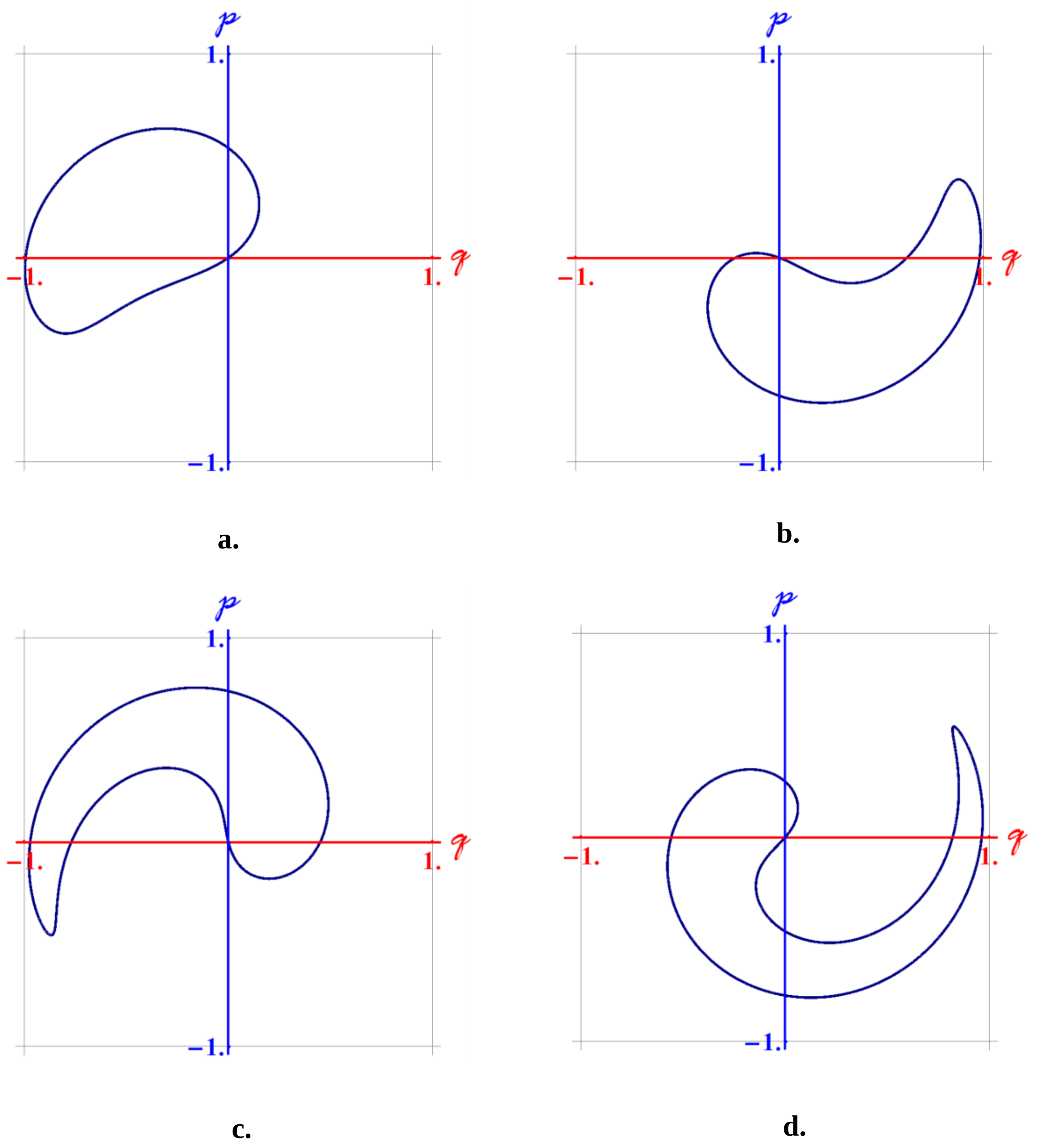

Fig. (5.3): The 2-D time-evolution contours of the classical probability distribution function $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(3)}$ given by eqn. (5.136) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

It should be noticed that Fig. (5.3) has been obtained by substituting the definition of $\left|\alpha_q\right|^2$ into eqn. (5.136), using eqns. (4.1), (4.2) and (4.3b).

From all the previous figures, it can be observed that the behavior of the classical probability distribution functions $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$ and $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ is more or less similar to that exhibited in Chapter 4, i.e., Figs. (4.2) and (4.3).

It is also observed that Fig. (5.3) is not similar to Fig. (5.2). This dissimilarity is basically the result of the non-commutative nature of quantum mechanics. That is, applying the classical limiting procedure in the present case leads to different expressions for $\omega_q^{(\mu)}$in the α - and α_q - representations (see eqns. (5.128) and (5.136)).

Also, in Figs. (5.4) - (5.6), the results of 3-D time-evolution of the classical probability distribution functions $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$ and $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ are depicted in phase space. It is clear from these figures that the peaks of the q-deformed Gaussians for the classical probability distributions $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^*;t\right)$ and $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ do not change with time and are equal to the maximum value (i.e., 1). These peaks follow the classical trajectories that are shown in Figs. (5.1) - (5.3) for the probability distribution functions. Another observation is the nature of the Gaussian shapes of these distributions which become more convoluted around themselves as $t\to\infty$, as is apparent in Figs. (5.4) - (5.6).
become more convoluted around themselves as $t\to\infty$, as is apparent in Fig. (5.4)-Fig. (5.6).

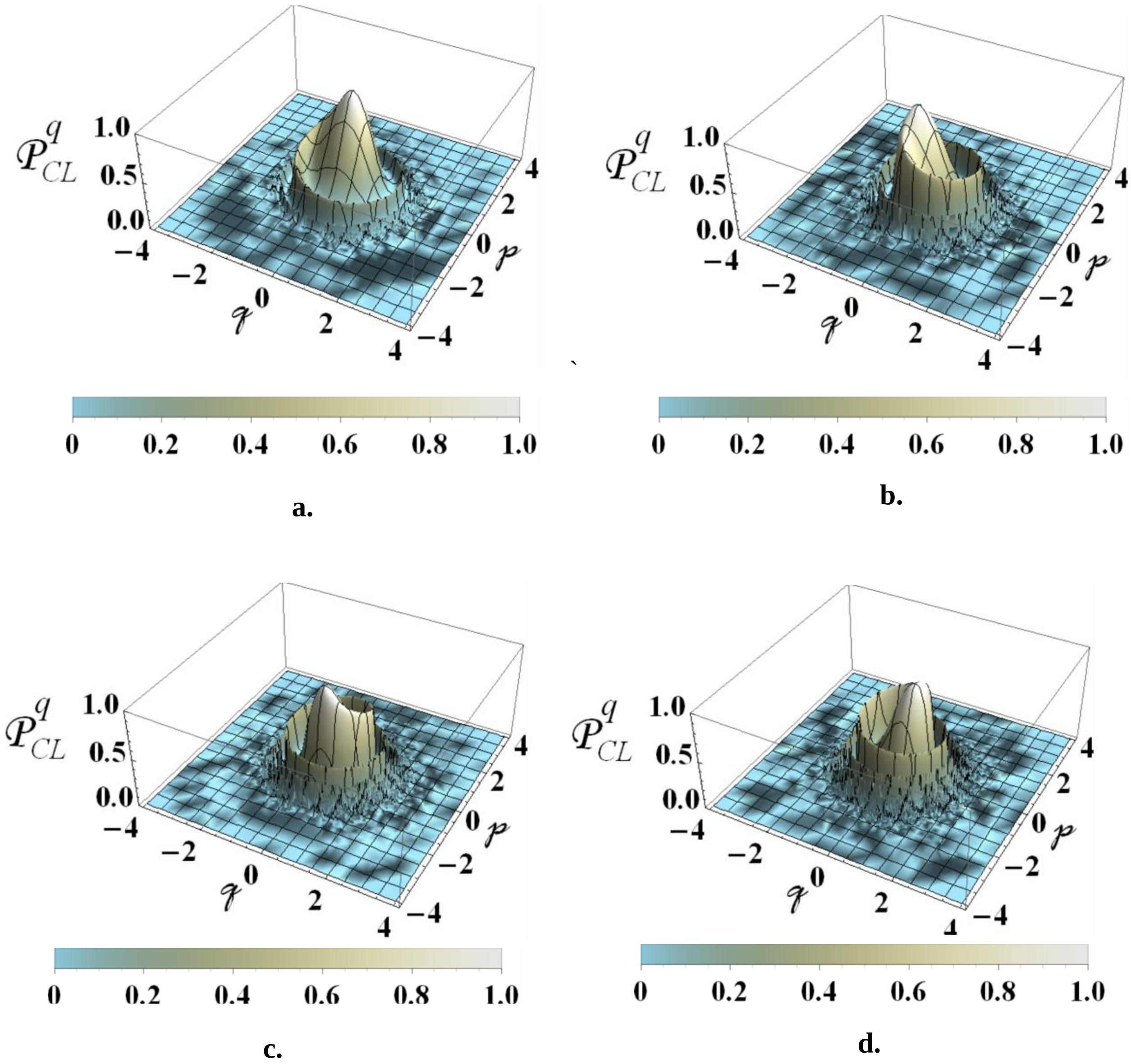

Fig. (5.4): The 3-D time-evolution of the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(1)}$ given by eqn. (5.124) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

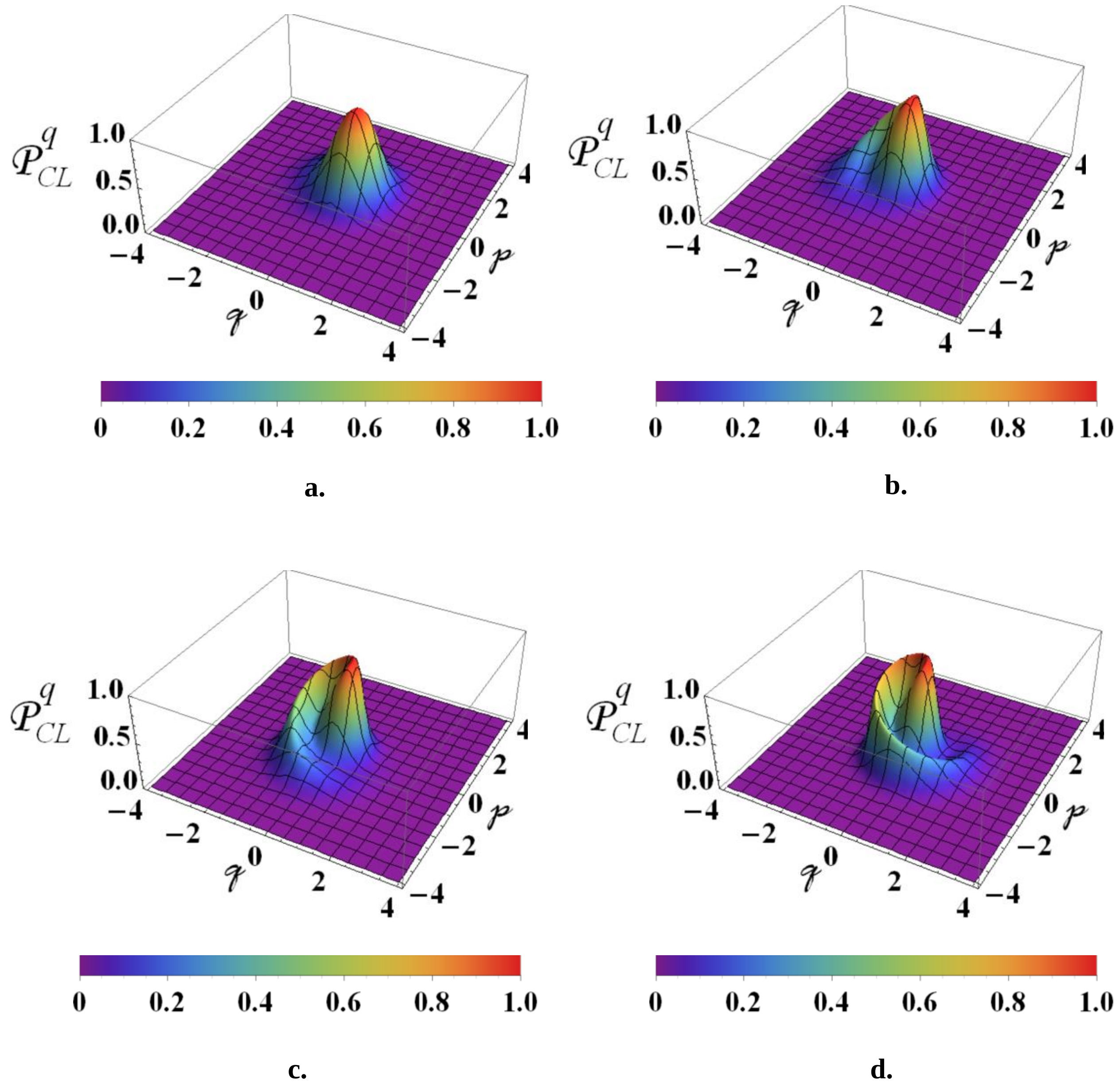

Fig. (5.5): The 3-D time-evolution of the classical probability distribution function $\mathcal{P}_{CL}^{q}\left(\alpha,\alpha^{*};t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_{q}^{(2)}$ given by eqn. (5.128) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

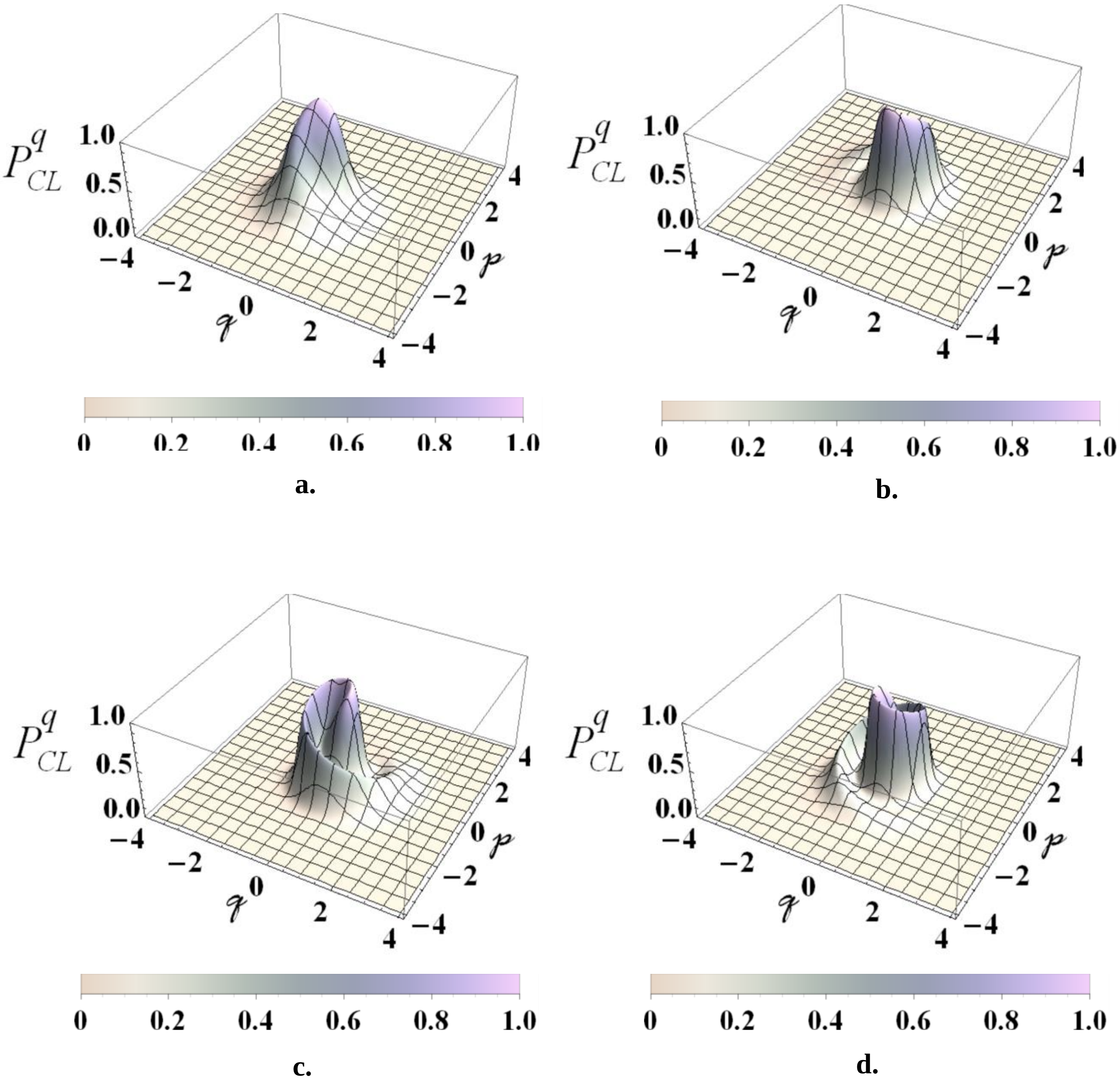

Fig. (5.6): The 3-D time-evolution of the classical probability distribution function $P_{CL}^{q}\left(\alpha_q,\alpha_q^*;t\right)$ for the q-deformed harmonic oscillator with frequency $\omega_q^{(3)}$ given by eqn. (5.136) and $q=0.5$ in phase space, for different values of time (τ): (a) $\tau=\pi/2$, (b) $\tau=\pi$, (c) $\tau=3\pi/2$, and (d) $\tau=2\pi$.

Chapter Six

Conclusions and Suggestions for Future Work

Conclusions and Suggestions for Future Work

6.2 Conclusions

1. The investigation of the classical limit of the 1-D q-deformed quantum harmonic oscillator leads to the conclusion that this limit is statistical in nature. This is clear from eqns. (5.125) and (5.135) where the classical Liouville equations are obtained for the 1-D q-deformed classical harmonic oscillator in α- and α_q-representations respectively. This is in conformity with Ghosh et al.'s work [76], where the classical Liouville equation was obtained for the 1-D classical simple harmonic oscillator, by applying the classical limiting conditions $\hbar \to 0, |\alpha|^2 \to \infty$, such that $\hbar|\alpha|^2 \to \text{finite}$. It is also concluded that this interpretation for the q-deformed quantum harmonic oscillator is more accurate than that introduced by Batouli and El Baz [95], because they interpreted this oscillator as a driven harmonic oscillator with the driving force and deformed frequency $\omega_q^2 = (\omega^2/4)(qt)^2$, both dependent on the deformation of this oscillator. Batouli and El Baz's interpretation [95] is based on using the undeformed Heisenberg equation of motion to calculate the q-deformed time dependent expectation values for the position and momentum. This methodology has intrinsic limitations compared with our methodology which is based on obtaining the Liouville classical limit that uncovers more details due to its phase space nature.

2. The q-deformed 1-D quantum harmonic oscillator can be interpreted as a nonlinear quantum oscillator where the nonlinearity parameter λ depends on $\hbar$ such that $\lambda = (const.) \cdot \hbar$. This dependence is as required for the classical limit to exist (see Table (5.1)). Based on the more detailed approach to the classical limit adopted in this work, this interpretation seems to be more accurate than that introduced by Man'ko [91] because this oscillator was interpreted as a nonlinear quantum oscillator with special type of nonlinearity where the frequency is energy dependent.

3. The behavior of the classical limit of the quantum Liouville equations for the q-deformed 1-D quantum harmonic oscillator in phase space shows whorl shapes evolving with time as in Figs. (5.1) - (5.3). These figures are similar to those obtained by Milburn [77] for the 1-D classical anharmonic oscillator as in Fig. (4.1). This similarity results from the fact that the anharmonicity itself represents a kind of deformation.
   This leads to the conclusion that the whorl shapes in phase space can be considered as a generalized phenomenon connected with q-deformation; the anharmonic oscillator being a special case.

4. It has been noted in Sec. (5.3.4) that the classical limit obtained in the present work using Arik and Coon's [9] coherent states for the q-deformed oscillator agrees with that obtained by Shabanov [89] based on path integrals for the same q-deformation type. This can be taken as a strong confirmation of the correctness of the results obtained in the present work for the classical limit based on Arik and Coon's work for the q-deformed oscillator.

5. It can also be concluded that the q-deformation of the 1-D quantum harmonic oscillator induces a non-commutative effect in the geometry. This can be understood in the light of Vitiello's work [130], where the q-deformation of the coherent states was studied to find that the fractal self-similarity, obtained by defining a fractal operator $q^{\alpha\frac{d}{d\alpha}}$, leads to a non-commutative geometry. The expression for this fractal operator is similar to the structure of the dilatation (shift) operators $e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}$, $e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}$, $e^{\pm\lambda\alpha_q\frac{\partial}{\partial\alpha_q}}$ and $e^{\pm\lambda\alpha_q^*\frac{\partial}{\partial\alpha_q^*}}$ appearing in the present work. These dilatation (shift) operators are inherent to the q-deformation and arise naturally in the quantum Liouville equations given in eqns. (5.54), (5.55), (5.67a), (5.67b) and (5.114) for the q-deformed 1-D quantum harmonic oscillator in the α- and α_q-representations respectively.

6.2 Suggestions for Future Work

One can suggest the following proposals to develop the present work in the future:

1. Studying the possibility of finding analytical or numerical solutions for the quantum Liouville equations of the 1-D quantum harmonic oscillator given by eqns. (5.54), (5.55), (5.67a), (5.67b) and (5.114). Such a study is essential to enable the visualization of the behavior of the q-deformed quantum oscillator in the quantum phase space.

2. The path integral approach introduced by Ajanapon [131] for the undeformed quantum harmonic oscillator can be used to investigate the classical limit of the q-deformed 1-D quantum harmonic oscillator. The results that emerge

from such an approach can then be compared with the results of this thesis. In this context, Shabanov's work [89], where the quantum and classical mechanics of the q-deformed quantum harmonic oscillator were also investigated using path integrals, can also be cited.

3. The use of other potentials, such as the Morse and Pöschl-Teller potentials, and the investigation of the interpretation of the q-deformation of such potentials along the same lines used in the present thesis are suggested. The importance of such an extension of the present work stems from the fact that such potentials are very useful in many fields of physics such as molecular, solid state and nuclear physics.

4. Treatment of the classical limit of the q-deformed 1-D quantum harmonic oscillator in the light of Lavagno's work [96] can be suggested. One may first construct the q-deformed quantum Liouville equation by using the q-deformed Heisenberg equation of motion. The construction of the q-deformed classical system may then be attempted by using the q-deformed Poisson bracket to derive the q-deformed 1-D classical Liouville equation. Then, one may investigate the classical limit of the resulting quantum Liouville equation and compare the result with that obtained from the q-deformed classical Poisson bracket approach.

Appendices

Appendix A

Evaluation of the Poisson Bracket $\left\{\alpha_f, \alpha_f^*\right\}$

The Poisson bracket $\left\{\alpha_f, \alpha_f^*\right\}$ is defined as [100]:

$$\left\{\alpha_f, \alpha_f^*\right\} = \left(\frac{\partial \alpha_f}{\partial q}\right)_p \left(\frac{\partial \alpha_f^*}{\partial p}\right)_q - \left(\frac{\partial \alpha_f}{\partial p}\right)_q \left(\frac{\partial \alpha_f^*}{\partial q}\right)_p \qquad \text{(A.1)}$$

But since,

$$\left(\frac{\partial \alpha_f}{\partial q}\right)_p = \left(\frac{\partial \alpha_f}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial q}\right)_p + \left(\frac{\partial \alpha_f}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial q}\right)_p \qquad \text{(A.2)}$$

$$\left(\frac{\partial \alpha_f}{\partial p}\right)_q = \left(\frac{\partial \alpha_f}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial p}\right)_q + \left(\frac{\partial \alpha_f}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial p}\right)_q \qquad \text{(A.3)}$$

and their conjugates are,

$$\left(\frac{\partial \alpha_f^*}{\partial q}\right)_p = \left(\frac{\partial \alpha_f^*}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial q}\right)_p + \left(\frac{\partial \alpha_f^*}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial q}\right)_p \qquad \text{(A.4)}$$

$$\left(\frac{\partial \alpha_f^*}{\partial p}\right)_q = \left(\frac{\partial \alpha_f^*}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial p}\right)_q + \left(\frac{\partial \alpha_f^*}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial p}\right)_q \qquad \text{(A.5)}$$

then, substituting eqns. (A.2) – (A.5) into eqn. (A.1), and after some mathematical manipulations, one obtains:

$$\left\{\alpha_f,\alpha_f^*\right\} = \left(\frac{\partial\alpha_f}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial\alpha_f^*}{\partial\alpha^*}\right)_{\alpha}\left\{\left(\frac{\partial\alpha}{\partial q}\right)_p\left(\frac{\partial\alpha^*}{\partial p}\right)_q - \left(\frac{\partial\alpha}{\partial p}\right)_q\left(\frac{\partial\alpha^*}{\partial q}\right)_p\right\}$$
$$+\left(\frac{\partial\alpha_f}{\partial\alpha^*}\right)_{\alpha}\left(\frac{\partial\alpha_f^*}{\partial\alpha}\right)_{\alpha^*}\left\{\left(\frac{\partial\alpha^*}{\partial q}\right)_p\left(\frac{\partial\alpha}{\partial p}\right)_q - \left(\frac{\partial\alpha^*}{\partial p}\right)_q\left(\frac{\partial\alpha}{\partial q}\right)_p\right\} \tag{A.6}$$

Now, since

$$\left\{\alpha,\alpha^*\right\} = \left(\frac{\partial\alpha}{\partial q}\right)_p\left(\frac{\partial\alpha^*}{\partial p}\right)_q - \left(\frac{\partial\alpha}{\partial p}\right)_q\left(\frac{\partial\alpha^*}{\partial q}\right)_p \tag{A.7}$$

and

$$\left\{\alpha,\alpha^*\right\} = -\left\{\alpha^*,\alpha\right\} \tag{A.8}$$

then, substituting eqns. (A-7) and (A-8) into eqn. (A-6), the result is:

$$\left\{\alpha_f,\alpha_f^*\right\} = \left\{\alpha,\alpha^*\right\}\cdot\left\{\left(\frac{\partial\alpha_f}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial\alpha_f^*}{\partial\alpha^*}\right)_{\alpha} - \left(\frac{\partial\alpha_f}{\partial\alpha^*}\right)_{\alpha}\left(\frac{\partial\alpha_f^*}{\partial\alpha}\right)_{\alpha^*}\right\} \tag{A.9}$$

But,

$$\left\{\alpha_f,\alpha_f^*\right\}_{\alpha,\alpha^*} = \left(\frac{\partial\alpha_f}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial\alpha_f^*}{\partial\alpha^*}\right)_{\alpha} - \left(\frac{\partial\alpha_f}{\partial\alpha^*}\right)_{\alpha}\left(\frac{\partial\alpha_f^*}{\partial\alpha}\right)_{\alpha^*} \tag{A.10}$$

then, substituting eqn. (A.10) into eqn. (A-9), one obtains:

$$\left\{\alpha_f,\alpha_f^*\right\} = \left\{\alpha,\alpha^*\right\}\cdot\left\{\alpha_f,\alpha_f^*\right\}_{\alpha,\alpha^*} \tag{A.11}$$

Appendix B

Evaluation of the Poisson Brackets $\left\{\alpha, \mathbb{H}_f(\alpha,\alpha^*)\right\}$ and $\left\{\alpha_f, \mathcal{H}_f(\alpha_f,\alpha_f^*)\right\}$

According to the definition of the Poisson bracket [100]:

$$\left\{\alpha, \mathbb{H}_f(\alpha,\alpha^*)\right\} = \left(\frac{\partial \alpha}{\partial q}\right)_p \left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial p}\right)_q - \left(\frac{\partial \alpha}{\partial p}\right)_q \left(\frac{\partial \mathbb{H}_f(\alpha,\alpha^*)}{\partial q}\right)_p \tag{B.1}$$

Considering $\mathbb{H}_f$ as a function of the two independent variables α and α^*, one can write:

$$\left(\frac{\partial \mathbb{H}_f}{\partial p}\right)_q = \left(\frac{\partial \mathbb{H}_f}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial p}\right)_q + \left(\frac{\partial \mathbb{H}_f}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial p}\right)_q \tag{B.2}$$

and,

$$\left(\frac{\partial \mathbb{H}_f}{\partial q}\right)_p = \left(\frac{\partial \mathbb{H}_f}{\partial \alpha}\right)_{\alpha^*} \left(\frac{\partial \alpha}{\partial q}\right)_p + \left(\frac{\partial \mathbb{H}_f}{\partial \alpha^*}\right)_{\alpha} \left(\frac{\partial \alpha^*}{\partial q}\right)_p \tag{B.3}$$

Then, substituting eqns. (B.2) and (B.3) into eqn. (B.1), the result is:

$$\left\{\alpha,\mathbb{H}_f(\alpha,\alpha^*)\right\} = \left(\frac{\partial\alpha}{\partial q}\right)_p \left\{\left(\frac{\partial\mathbb{H}_f}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial\alpha}{\partial p}\right)_q + \left(\frac{\partial\mathbb{H}_f}{\partial\alpha^*}\right)_\alpha\left(\frac{\partial\alpha^*}{\partial p}\right)_q\right\}$$
$$- \left(\frac{\partial\alpha}{\partial p}\right)_q \left\{\left(\frac{\partial\mathbb{H}_f}{\partial\alpha}\right)_{\alpha^*}\left(\frac{\partial\alpha}{\partial q}\right)_p + \left(\frac{\partial\mathbb{H}_f}{\partial\alpha^*}\right)_\alpha\left(\frac{\partial\alpha^*}{\partial q}\right)_p\right\} \tag{B.4}$$

Eqn. (B.4) can be simplified to obtain:

$$\left\{\alpha,\mathbb{H}_f(\alpha,\alpha^*)\right\} = \left\{\left(\frac{\partial\alpha}{\partial q}\right)_p\left(\frac{\partial\alpha^*}{\partial p}\right)_q - \left(\frac{\partial\alpha}{\partial p}\right)_q\left(\frac{\partial\alpha^*}{\partial q}\right)_p\right\}\left(\frac{\partial\mathbb{H}_f}{\partial\alpha^*}\right)_\alpha \tag{B.5}$$

Now, since

$$\left\{\alpha,\mathbb{H}_f(\alpha,\alpha^*)\right\}_{\alpha,\alpha^*} = \left(\frac{\partial\mathbb{H}_f}{\partial\alpha^*}\right)_\alpha \tag{B.6}$$

then, substituting eqns. (A.7) and (B.6) into eqn. (B.5), one obtains:

$$\left\{\alpha,\mathbb{H}_f(\alpha,\alpha^*)\right\} = \left\{\alpha,\alpha^*\right\}\cdot\left\{\alpha,\mathbb{H}_f(\alpha,\alpha^*)\right\}_{\alpha,\alpha^*} \tag{B.7}$$

Similarly, one can prove that

$$\left\{\alpha_f,\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right)\right\} = \left\{\alpha_f,\alpha_f^*\right\}\cdot\left\{\alpha_f,\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right)\right\}_{\alpha_f,\alpha_f^*} \tag{B.8}$$

Appendix C

Evaluation of the Poisson Brackets $\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}$ and $\left\{\mathbb{H}_f\left(\alpha,\alpha^*\right),\mathcal{P}_{CL}^f\left(\alpha,\alpha^*;t\right)\right\}$

The Poisson bracket $\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}$ is defined as [100]:

$$\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}=\left(\frac{\partial\mathcal{H}_f}{\partial q}\right)_p\left(\frac{\partial P_{CL}^f}{\partial p}\right)_q-\left(\frac{\partial\mathcal{H}_f}{\partial p}\right)_q\left(\frac{\partial P_{CL}^f}{\partial q}\right)_p \tag{C.1}$$

Considering $\mathcal{H}_f$ as a function of the two independent variables α_f and α_f^*, one can write:

$$\left(\frac{\partial\mathcal{H}_f}{\partial q}\right)_p=\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f}\right)_{\alpha_f^*}\left(\frac{\partial\alpha_f}{\partial q}\right)_p+\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f^*}\right)_{\alpha_f}\left(\frac{\partial\alpha_f^*}{\partial q}\right)_p \tag{C.2}$$

and,

$$\left(\frac{\partial\mathcal{H}_f}{\partial p}\right)_q=\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f}\right)_{\alpha_f^*}\left(\frac{\partial\alpha_f}{\partial p}\right)_q+\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f^*}\right)_{\alpha_f}\left(\frac{\partial\alpha_f^*}{\partial p}\right)_q \tag{C.3}$$

Also,

$$\left(\frac{\partial P_{CL}^{f}}{\partial q}\right)_{p}=\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial q}\right)_{p}+\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial q}\right)_{p} \tag{C.4}$$

$$\left(\frac{\partial P_{CL}^{f}}{\partial p}\right)_{q}=\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial p}\right)_{q}+\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial p}\right)_{q} \tag{C.5}$$

Substituting eqns. (C.2) – (C.5) into eqn. (C.1), the result becomes:

$$\begin{aligned}
&\left\{\mathcal{H}_{f}(\alpha_{f},\alpha_{f}^{*}),P_{CL}^{f}(\alpha_{f},\alpha_{f}^{*};t)\right\}= \\
&\quad\left\{\left(\frac{\partial \mathcal{H}_{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial q}\right)_{p}+\left(\frac{\partial \mathcal{H}_{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial q}\right)_{p}\right\} \\
&\cdot\left\{\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial p}\right)_{q}+\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial p}\right)_{q}\right\} \\
&-\left\{\left(\frac{\partial \mathcal{H}_{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial p}\right)_{q}+\left(\frac{\partial \mathcal{H}_{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial p}\right)_{q}\right\} \\
&\cdot\left\{\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}}\right)_{\alpha_{f}^{*}}\left(\frac{\partial \alpha_{f}}{\partial q}\right)_{p}+\left(\frac{\partial P_{CL}^{f}}{\partial \alpha_{f}^{*}}\right)_{\alpha_{f}}\left(\frac{\partial \alpha_{f}^{*}}{\partial q}\right)_{p}\right\}
\end{aligned} \tag{C.6}$$

After some mathematical manipulations, this equation can be simplified to yield:

$$\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}=$$

$$\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f^*}\right)_{\alpha_f}\left(\frac{\partial P_{CL}^f}{\partial\alpha_f}\right)_{\alpha_f^*}\left\{\left(\frac{\partial\alpha_f^*}{\partial q}\right)_p\left(\frac{\partial\alpha_f}{\partial p}\right)_q-\left(\frac{\partial\alpha_f^*}{\partial p}\right)_q\left(\frac{\partial\alpha_f}{\partial q}\right)_p\right\}$$

$$+\left(\frac{\partial\mathcal{H}_f}{\partial\alpha_f}\right)_{\alpha_f^*}\left(\frac{\partial P_{CL}^f}{\partial\alpha_f^*}\right)_{\alpha_f}\left\{\left(\frac{\partial\alpha_f}{\partial q}\right)_p\left(\frac{\partial\alpha_f^*}{\partial p}\right)_q-\left(\frac{\partial\alpha_f}{\partial p}\right)_q\left(\frac{\partial\alpha_f^*}{\partial q}\right)_p\right\}$$

(C.7)

Then, substituting eqn. (A-1) into eqn. (C-7) one gets:

$$\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}=$$

$$\left\{\alpha_f,\alpha_f^*\right\}\cdot\left\{\mathcal{H}_f\left(\alpha_f,\alpha_f^*\right),P_{CL}^f\left(\alpha_f,\alpha_f^*;t\right)\right\}_{\alpha_f,\alpha_f^*}$$

(C.8)

Similarly, one can prove that

$$\left\{\mathbb{H}_f(\alpha,\alpha^*),\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}=\left\{\alpha,\alpha^*\right\}\cdot\left\{\mathbb{H}_f(\alpha,\alpha^*),\mathcal{P}_{CL}^f(\alpha,\alpha^*;t)\right\}_{\alpha,\alpha^*}$$

(C.9)

Appendix D

The Correspondence Relations for $\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\hat{\rho}$ and $\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)$

Multiplying the Glauber-Sudarshan P-representation for the relations $\hat{a}\,\hat{\rho}$ and $\hat{\rho}\,\hat{a}^{\dagger}$ given in eqns. (2.61) and (2.62) by $\hat{a}^{\dagger}$ from the left and $\hat{a}$ from the right respectively, results in:

$$\hat{a}^{\dagger}\hat{a}\ \hat{\rho} = \int d^2\alpha\ \hat{a}^{\dagger}\ |\alpha\rangle\langle\alpha|\left\{\alpha P(\alpha,\alpha^*)\right\} \tag{D.1}$$

$$\hat{\rho}\,\hat{a}^{\dagger}\hat{a} = \int d^2\alpha\ |\alpha\rangle\langle\alpha|\ \hat{a}\left\{\alpha^* P(\alpha,\alpha^*)\right\} \tag{D.2}$$

The expression for $\hat{a}^{\dagger}|\alpha\rangle\langle\alpha|$ and its conjugate are given in refs. [108,109] as:

$$\hat{a}^{\dagger}|\alpha\rangle\langle\alpha| = \left[\alpha^* + \frac{\partial}{\partial\alpha}\right]|\alpha\rangle\langle\alpha| \tag{D.3}$$

$$|\alpha\rangle\langle\alpha|\ \hat{a} = \left[\alpha + \frac{\partial}{\partial\alpha^*}\right]|\alpha\rangle\langle\alpha| \tag{D.4}$$

Substituting eqns. (D.3) and (D.4) into eqns. (D.1) and (D.2) respectively, gives

$$\hat{a}^{\dagger}\hat{a}\ \hat{\rho} = \int d^2\alpha\left[\alpha^* + \frac{\partial}{\partial\alpha}\right]|\alpha\rangle\langle\alpha|\left\{\alpha P(\alpha,\alpha^*)\right\} \tag{D.5}$$

$$\hat{\rho}\,\hat{a}^{\dagger}\hat{a} = \int d^2\alpha\left[\alpha + \frac{\partial}{\partial\alpha^*}\right]|\alpha\rangle\langle\alpha|\left\{\alpha^* P(\alpha,\alpha^*)\right\} \tag{D.6}$$

Performing integration by parts in eqns. (D.5) and (D.6), one obtains [111,112]:

$$\hat{a}^{\dagger}\hat{a}\ \hat{\rho} = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha^* - \frac{\partial}{\partial\alpha} \right] (\alpha) P(\alpha,\alpha^*) \right\} \tag{D.7}$$

$$\hat{\rho}\hat{a}^{\dagger}\hat{a} = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha - \frac{\partial}{\partial\alpha^*} \right] (\alpha^*)\ P(\alpha,\alpha^*) \right\} \tag{D.8}$$

Now, starting from eqns. (2.61) and (2.62) to obtain $(\hat{a})^2\hat{\rho}$ and $\hat{\rho}(\hat{a}^{\dagger})^2$, then multiplying the results by $(\hat{a}^{\dagger})^2$ from the left and $(\hat{a})^2$ from the right, and using the same technique used previously to obtain eqns. (D.7) and (D.8), the results become:

$$(\hat{a}^{\dagger})^2(\hat{a})^2\ \hat{\rho} = \int d^2\alpha \left[\alpha^* + \frac{\partial}{\partial\alpha} \right]^2 |\alpha\rangle\langle\alpha| \left\{ (\alpha)^2 P(\alpha,\alpha^*) \right\} \tag{D.9}$$

$$\hat{\rho}(\hat{a}^{\dagger})^2(\hat{a})^2 = \int d^2\alpha \left[\alpha + \frac{\partial}{\partial\alpha^*} \right]^2 |\alpha\rangle\langle\alpha| \left\{ (\alpha^*)^2 P(\alpha,\alpha^*) \right\} \tag{D.10}$$

Applying again integration by parts to eqns. (D.9) and (D.10), gives [108,109]:

$$(\hat{a}^{\dagger})^2(\hat{a})^2\ \hat{\rho} = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha^* - \frac{\partial}{\partial\alpha} \right]^2 (\alpha)^2 P(\alpha,\alpha^*) \right\} \tag{D.11}$$

$$\hat{\rho}(\hat{a}^{\dagger})^2(\hat{a})^2 = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha - \frac{\partial}{\partial\alpha^*} \right]^2 (\alpha^*)^2 P(\alpha,\alpha^*) \right\} \tag{D.12}$$

Then, using the mathematical induction method, and following the same previous procedure used to derive eqns. (D.11) and (D.12), one obtains:

$$(\hat{a}^{\dagger})^m(\hat{a})^m\ \hat{\rho} = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha^* - \frac{\partial}{\partial\alpha} \right]^m (\alpha)^m P(\alpha,\alpha^*) \right\} \tag{D.13}$$

and,

$$\hat{\rho}(\hat{a}^{\dagger})^m(\hat{a})^m = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ \left[\alpha - \frac{\partial}{\partial\alpha^*} \right]^m (\alpha^*)^m P(\alpha,\alpha^*) \right\} \tag{D.14}$$

Now, letting $f(\hat{a}^{\dagger}\hat{a}) = (\hat{a}^{\dagger})^m(\hat{a})^m$, and substituting $f(\hat{a}^{\dagger}\hat{a})$ into eqns. (D.13) and (D.14), results in:

$$f\left(\hat{a}^{\dagger}\hat{a}\right)\hat{\rho} = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ f\left(\left[\alpha^{*} - \frac{\partial}{\partial\alpha}\right]^{m} (\alpha)^{m}\right) P(\alpha,\alpha^{*}) \right\} \tag{D.15}$$

and,

$$\hat{\rho}\, f\left(\hat{a}^{\dagger}\hat{a}\right) = \int d^2\alpha\ |\alpha\rangle\langle\alpha| \left\{ f\left(\left[\alpha - \frac{\partial}{\partial\alpha^{*}}\right]^{m} \left(\alpha^{*}\right)^{m}\right) P(\alpha,\alpha^{*}) \right\} \tag{D.16}$$

The function of the number operator, $f\left(\hat{a}^{\dagger}\hat{a}\right)$, represents a special case of the function $f\left(\hat{a},\hat{a}^{\dagger}\right)$, where $f\left(\hat{a},\hat{a}^{\dagger}\right)$ represents an arbitrary operator ordered function. This permits the generalization of eqns. (D.15) and (D.16) using $f\left(\hat{a}^{\dagger}\hat{a}\right)\hat{\rho} \rightarrow f\left(\hat{a},\hat{a}^{\dagger}\right)\hat{\rho}$ and $\hat{\rho}\, f\left(\hat{a}^{\dagger}\hat{a}\right) \rightarrow \hat{\rho}\, f\left(\hat{a},\hat{a}^{\dagger}\right)$ (see Sec. (5.1)). This generalization process can be satisfied if:

$$f\left(\hat{a},\hat{a}^{\dagger}\right)\hat{\rho} = f\left(\hat{a}^{\dagger}\hat{a}\right)\hat{\rho} = f\left(\hat{a}\hat{a}^{\dagger}\right)\hat{\rho} \tag{D.17}$$

and,

$$\hat{\rho}\, f\left(\hat{a},\hat{a}^{\dagger}\right) = \hat{\rho}\, f\left(\hat{a}^{\dagger}\hat{a}\right) = \hat{\rho}\, f\left(\hat{a}\hat{a}^{\dagger}\right) \tag{D.18}$$

where $f\left(\hat{a}\hat{a}^{\dagger}\right)$ represents the anti-normal ordered operator function. Both functions $f\left(\hat{a}^{\dagger}\hat{a}\right)$ and $f\left(\hat{a}\hat{a}^{\dagger}\right)$ can be obtained from the function $f\left(\hat{a},\hat{a}^{\dagger}\right)$ by using the commutator $\left[\hat{a},\hat{a}^{\dagger}\right]$.

The derivation of eqn. (D.17) is as follows. Supposing any expression for arbitrary operator ordered function $f\left(\hat{a},\hat{a}^{\dagger}\right)$ then, letting this function act on the density operator $\hat{\rho}$ from the left, and using the one-to-one correspondence relations given by eqns. (2.65) and (2.67), one can obtain the expression for $f\left(\hat{a},\hat{a}^{\dagger}\right)\hat{\rho}$. This expression is compared with the expressions obtained for the $f\left(\hat{a}^{\dagger}\hat{a}\right)\hat{\rho}$ and $f\left(\hat{a}\hat{a}^{\dagger}\right)\hat{\rho}$. Similarly, one can derive eqn. (D.18) by using the same expression for the operator ordered function $f\left(\hat{a},\hat{a}^{\dagger}\right)$ that was used to prove eqn. (D.17), but with the one-to-one correspondence relations that are defined in eqns. (2.66) and (2.68). Thus, by using the mathematical induction method for any arbitrary operator ordered function $f(\hat{a},\hat{a}^{\dagger})$, and following the same previously mentioned procedure (see Sec. (5.1)), one can prove that the eqns. (D.17) and (D.18) are still satisfied.

Hence, eqns. (D.15) and (D.16) can be re-written as:

$$f(\hat{a},\hat{a}^{\dagger})\,\hat{\rho}=\int d^2\alpha\;|\alpha\rangle\langle\alpha|\left\{f\left(\left[\alpha^{*}-\frac{\partial}{\partial\alpha}\right]^{m},\,(\alpha)^{m}\right)P(\alpha,\alpha^{*})\right\} \qquad \text{(D.19)}$$

$$\hat{\rho}\,f(\hat{a},\hat{a}^{\dagger})=\int d^2\alpha\;|\alpha\rangle\langle\alpha|\left\{f\left(\left[\alpha-\frac{\partial}{\partial\alpha^{*}}\right]^{m},\,\left(\alpha^{*}\right)^{m}\right)P(\alpha,\alpha^{*})\right\} \qquad \text{(D.20)}$$

Eqns. (D.19) and (D.20), together with eqns. (D.17) and (D.18), permit the generalization of eqns. (D.7) and (D.8) by using $\hat{a}^{\dagger}\hat{a}\,\hat{\rho}=\left[\hat{a}^{\dagger}\hat{a}\right]^{m}\left(f(\hat{a},\hat{a}^{\dagger})\right)\hat{\rho}$ and $\hat{\rho}\,\hat{a}^{\dagger}\hat{a}=\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]^{m}\left(f(\hat{a},\hat{a}^{\dagger})\right)$.Therefore, eqns. (D.19) and (D.20) become:

$$\left[\hat{a}^{\dagger}\hat{a}\right]^{m}\left(f^{2}(\hat{a},\hat{a}^{\dagger})\right)\hat{\rho}=$$
$$\int d^2\alpha\;|\alpha\rangle\langle\alpha|\left\{\left[\alpha^{*}-\frac{\partial}{\partial\alpha}\right]^{m}(\alpha)^{m}\;f^{2}\left(\left[\alpha^{*}-\frac{\partial}{\partial\alpha}\right]^{m},\,(\alpha)^{m}\right)P(\alpha,\alpha^{*})\right\}$$

(D.21)

and,

$$\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]^{m}\left(f^{2}(\hat{a},\hat{a}^{\dagger})\right)=$$
$$\int d^2\alpha\;|\alpha\rangle\langle\alpha|\left\{f^{2}\left(\left[\alpha-\frac{\partial}{\partial\alpha^{*}}\right]^{m},\,\left(\alpha^{*}\right)^{m}\right)\left[\alpha-\frac{\partial}{\partial\alpha^{*}}\right]^{m}\left(\alpha^{*}\right)^{m}P(\alpha,\alpha^{*})\right\}$$

(D.22)

respectively, where the function f has been replaced by f^2 in these equations. Eqns. (D.21) and (D.22) can be re-written equivalently in terms of the quasiprobability distribution function $\varphi^{(s)}(\alpha,\alpha^*)$ given in Table (2.1) as:

$$\left[\hat{a}^\dagger\hat{a}\right]^m \left(f^2(\hat{a},\hat{a}^\dagger)\right)\hat{\rho}=$$
$$\int d^2\alpha\,|\alpha\rangle\langle\alpha|\left\{\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]^m\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]^m\right)\right.$$
$$\left.\cdot\, f^2\left(\left[\alpha^*+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]^m,\ \left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]^m\right)\varphi^{(s)}(\alpha,\alpha^*)\right\}$$

(D.23)

and,

$$\hat{\rho}\left[\hat{a}^\dagger\hat{a}\right]^m \left(f^2(\hat{a},\hat{a}^\dagger)\right)=$$
$$\int d^2\alpha\,|\alpha\rangle\langle\alpha|\left\{f^2\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]^m,\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]^m\right)\right.$$
$$\left.\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^*}\right]^m\left[\alpha^*+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]^m\right)\varphi^{(s)}(\alpha,\alpha^*)\right\}$$

(D.24)

respectively.

Eqns. (D.23) and (D.24) represent the general relations for $\left[\hat{a}^\dagger\hat{a}\right]\left(f^2(\hat{a}^\dagger,\hat{a})\right)\hat{\rho}$ and $\hat{\rho}\left[\hat{a}^\dagger\hat{a}\right]\left(f^2(\hat{a}^\dagger,\hat{a})\right)$.

A special case of the relations for $\left[\hat{a}^\dagger\hat{a}\right]^m\left(f^2(\hat{a},\hat{a}^\dagger)\right)\hat{\rho}$ and $\hat{\rho}\left[\hat{a}^\dagger\hat{a}\right]^m\left(f^2(\hat{a},\hat{a}^\dagger)\right)$ can be obtained by letting $m=1$ and $f^2\left(\hat{a}\,,\hat{a}^\dagger\right)=f^2\left(\hat{a}^\dagger\,\hat{a}\right)$ in eqns. (D.23) and (D.24), then these equations become:

$$\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)\hat{\rho} =$$
$$\int d^{2}\alpha\,|\alpha\rangle\langle\alpha|\left\{\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)\right.$$
$$\left.\cdot\, f^{2}\left(\left[\alpha^{*}+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha}\right]\left[\alpha+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*})\right\}$$

(D.25)

and,

$$\hat{\rho}\left[\hat{a}^{\dagger}\hat{a}\right]\left(f^{2}(\hat{a}^{\dagger}\hat{a})\right)=$$
$$\int d^{2}\alpha\,|\alpha\rangle\langle\alpha|\left\{f^{2}\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\right.$$
$$\left.\cdot\left(\left[\alpha+\left(\frac{s-1}{2}\right)\frac{\partial}{\partial\alpha^{*}}\right]\left[\alpha^{*}+\left(\frac{s+1}{2}\right)\frac{\partial}{\partial\alpha}\right]\right)\varphi^{(s)}(\alpha,\alpha^{*})\right\}$$

(D.26)

respectively.

It is clear that in the limit $f^{2}\rightarrow 1$, and using the ordering parameter $s=-1$ (i.e., the anti-normal ordered bosons operators , as given in Table (2.1)), then the s-ordered quasiprobability function $\varphi^{(-1)}(\alpha,\alpha^{*})\rightarrow P(\alpha,\alpha^{*})$ and, hence, eqns. (D.25) and (D.26), reduce to eqns. (D.7) and (D.8) respectively. The following one-to-one correspondence relations for the Glauber-Sudarshan P-representation in the α-representation, which are the same as those introduced by Walls and Milburn [111], can then be obtained:

$$\hat{a}\,\hat{a}^{\dagger}\,\hat{\rho} \rightarrow \alpha\left(\alpha^{*} - \frac{\partial}{\partial\alpha}\right)P(\alpha,\alpha^{*}) \tag{D.27}$$

$$\hat{\rho}\,\hat{a}\,\hat{a}^{\dagger} \rightarrow \alpha^{*}\left(\alpha - \frac{\partial}{\partial\alpha^{*}}\right)P(\alpha,\alpha^{*}) \tag{D.28}$$

$$\hat{a}^{\dagger}\hat{a}\,\hat{\rho} \rightarrow \left(\alpha^{*} - \frac{\partial}{\partial\alpha}\right)\alpha\,P(\alpha,\alpha^{*}) \tag{D.29}$$

$$\hat{\rho}\,\hat{a}^{\dagger}\hat{a} \rightarrow \left(\alpha - \frac{\partial}{\partial\alpha^{*}}\right)\alpha^{*}\,P(\alpha,\alpha^{*}) \tag{D.30}$$

Appendix E

Evaluation of the Commutators

$\left[\hat{\mathbf{A}}_{+},\hat{\mathbf{B}}_{1}\right], \left[\hat{\mathbf{A}}_{-},\hat{\mathbf{B}}_{2}\right], \left[\hat{\mathbf{A}}_{+},\left[\hat{\mathbf{A}}_{+},\hat{\mathbf{B}}_{1}\right]\right], \left[\hat{\mathbf{B}}_{1},\left[\hat{\mathbf{A}}_{+},\hat{\mathbf{B}}_{1}\right]\right], \left[\hat{\mathbf{A}}_{-},\left[\hat{\mathbf{A}}_{-},\hat{\mathbf{B}}_{2}\right]\right]$ and $\left[\hat{\mathbf{B}}_{2},\left[\hat{\mathbf{A}}_{-},\hat{\mathbf{B}}_{2}\right]\right]$

Using the definition of the commutator:

$$\left[\hat{A}_{+},\hat{B}_{1}\right]=\hat{A}_{+}\hat{B}_{1}-\hat{B}_{1}\hat{A}_{+} \tag{E.1}$$

and multiplying both sides of eqn. (E.1) form the right by an arbitrary function $g=g\left(\alpha,\alpha^{*}\right)$, then substituting the definitions of $\hat{A}_{+}$ and $\hat{B}_{1}$ given in eqns. (5.16) and (5.17) in these equations and simplifying the result , gives:

$$\begin{aligned}\left[\hat{A}_{+},\hat{B}_{1}\right]g= \\ &\left(\frac{s+1}{2}\right)\lambda^{2}\alpha\,\alpha^{*2}\frac{\partial g}{\partial\alpha^{*}}+\left(\frac{s-1}{2}\right)\lambda^{2}\alpha^{2}\,\alpha^{*}\frac{\partial g}{\partial\alpha} \\ &+\left[\frac{(s-1)(s+1)}{4}\right]\lambda^{2}\alpha\alpha^{*}\frac{\partial^{2}g}{\partial\alpha\,\partial\alpha^{*}}-\left(\frac{s+1}{2}\right)\lambda^{2}\alpha^{*}\frac{\partial}{\partial\alpha^{*}}\left(\alpha\alpha^{*}g\right) \\ &-\left(\frac{s-1}{2}\right)\lambda^{2}\alpha\frac{\partial}{\partial\alpha}\left(\alpha\alpha^{*}g\right)-\left[\frac{(s-1)(s+1)}{4}\right]\lambda^{2}\frac{\partial^{2}}{\partial\alpha\,\partial\alpha^{*}}\left(\alpha\alpha^{*}g\right)\end{aligned} \tag{E.2}$$

where,

$$\alpha\frac{\partial}{\partial\alpha}\left(\alpha\alpha^{*}g\right)=\alpha\alpha^{*}g+\alpha^{2}\alpha^{*}\frac{\partial g}{\partial\alpha} \tag{E.3}$$

$$\alpha^{*}\frac{\partial}{\partial\alpha^{*}}\left(\alpha\alpha^{*}g\right)=\alpha\alpha^{*}g+\alpha\,\alpha^{*2}\frac{\partial g}{\partial\alpha^{*}} \tag{E.4}$$

and,

$$\frac{\partial^2 g}{\partial\alpha\,\partial\alpha^*}\left(\alpha\alpha^* g\right) = \alpha\frac{\partial g}{\partial\alpha} + \alpha^*\frac{\partial g}{\partial\alpha^*} + \alpha\alpha^*\frac{\partial^2 g}{\partial\alpha\,\partial\alpha^*} + g \tag{E.5}$$

Substituting eqns. (E.3), (E.4) and (E.5) into eqn. (E.2) and simplifying the result, one obtains:

$$\left[\hat{A}_+,\hat{B}_1\right] = -\lambda^2\left\{s|\alpha|^2 + \left(\frac{s^2-1}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + 1\right]\right\} \tag{E.6}$$

Similarly, one can prove that:

$$\left[\hat{A}_-,\hat{B}_2\right] = \left[\hat{A}_+,\hat{B}_1\right] \tag{E.7}$$

Also, one can write,

$$\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right] = \hat{A}_+\left[\hat{A}_+,\hat{B}_1\right] - \left[\hat{A}_+,\hat{B}_1\right]\hat{A}_+ \tag{E.8}$$

Then, substituting the expressions for $\hat{A}_+$ from eqn. (5.16) and for the commutator $\left[\hat{A}_+,\hat{B}_1\right]$ from eqn. (E.6) into eqn. (E.8), applying the same technique used to obtain the commutator $\left[\hat{A}_+,\hat{B}_1\right]$ and simplifying the result, yields:

$$\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right] = \left(\frac{s^2-1}{2}\right)\lambda^3|\alpha|^2 = \left(\frac{s^2-1}{2}\right)\lambda^2\hat{A}_+ \tag{E.9}$$

Similarly, the commutator $\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right]$ can be evaluated by using the same technique used to evaluate $\left[\hat{A}_+,\left[\hat{A}_+,\hat{B}_1\right]\right]$, and after some lengthy mathematical manipulations, the result is:

$$\left[\hat{B}_1,\left[\hat{A}_+,\hat{B}_1\right]\right] =$$

$$-\lambda^3\left\{s^2|\alpha|^2 + \left(\frac{s\left(s^2-1\right)}{4}\right)\left[\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + 1\right] + \left[\frac{(s+1)^2(s-1)^2}{4}\right]\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right\} \tag{E.10}$$

The following relations and there complex conjugates, as well as eqns. (E.3) – (E.5), have been used to obtain eqn. (E.10):

$$\alpha \frac{\partial}{\partial \alpha}\left(\alpha \frac{\partial g}{\partial \alpha} \right) = \alpha \frac{\partial g}{\partial \alpha} + \alpha^2 \frac{\partial^2 g}{\partial \alpha^2} \tag{E.11}$$

$$\alpha \frac{\partial}{\partial \alpha}\left(\alpha^* \frac{\partial g}{\partial \alpha^*} \right) = \alpha \alpha^* \frac{\partial^2 g}{\partial \alpha \partial \alpha^*} \tag{E.12}$$

$$\frac{\partial^2 g}{\partial \alpha \partial \alpha^*}\left(\alpha \frac{\partial g}{\partial \alpha} \right) = \frac{\partial^2 g}{\partial \alpha \partial \alpha^*} + \alpha \frac{\partial^3 g}{\partial \alpha^2 \partial \alpha^*} \tag{E.13}$$

Using the same technique, the commutators $\left[\hat{A}_-, \left[\hat{A}_-, \hat{B}_2 \right] \right]$ and $\left[\hat{B}_2, \left[\hat{A}_-, \hat{B}_2 \right] \right]$ can be evaluated. The results are:

$$\left[\hat{A}_-, \left[\hat{A}_-, \hat{B}_2 \right] \right] = -\left[\hat{A}_+, \left[\hat{A}_+, \hat{B}_1 \right] \right] \tag{E.14}$$

and,

$$\left[\hat{B}_2, \left[\hat{A}_-, \hat{B}_2 \right] \right] = -\left[\hat{B}_1, \left[\hat{A}_+, \hat{B}_1 \right] \right] \tag{E.15}$$

Appendix F

The Action of the Dilatation (Shift) Operators $e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}$ and $e^{\pm\lambda\alpha^*\frac{\partial}{\partial\alpha^*}}$ on $F(\alpha,\alpha^*)\,G(\alpha,\alpha^*)$

Assume two functions $F(x)$ and $G(x)$ that have power series expansions of the form:

$$F(x) = \sum_{m=0}^{\infty} a_m\, x^m \tag{F.1}$$

$$G(x) = \sum_{m=0}^{\infty} b_m\, x^m \tag{F.2}$$

Using these expressions, the product $F(x)\,G(x)$ can be written as:

$$F(x)G(x) = \left(a_o + a_1x + a_2x^2 + a_3x^3 + \cdots\right) \cdot \left(b_o + b_1x + b_2x^2 + b_3x^3 + \cdots\right) \tag{F.3}$$

Multiplying both sides of eqn. (F.3) from left by the dilatation (shift) operator $e^{\beta x\frac{\partial}{\partial x}}$, where β represent any arbitrary constant, the result is:

$$e^{\beta x\frac{\partial}{\partial x}}F(x)G(x)=$$

$$e^{\beta x\frac{\partial}{\partial x}}\Big\{\left[a_ob_o+a_ob_1x+a_ob_2x^2+a_ob_3x^3+\cdots\right]$$

$$+\left[a_1b_ox+a_1b_1x^2+a_1b_2x^3+a_1b_3x^4+\cdots\right]$$

$$+\left[a_2b_ox^2+a_2b_1x^3+a_2b_2x^4+a_2b_3x^5+\cdots\right]$$

$$+\left[a_3b_ox^3+a_3b_1x^4+a_3b_2x^5+a_3b_3x^6+\cdots\right]\Big\}$$

(F.4)

Applying eqn. (5.41) to eqn. (F.4) and simplifying, the result becomes:

$$e^{\beta x\frac{\partial}{\partial x}}F(x)G(x)=$$

$$a_o\left\{b_o+b_1\left(e^{\beta}x\right)+b_2\left(e^{\beta}x\right)^2+b_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

$$+a_1\left(e^{\beta}x\right)\left\{b_o+b_1\left(e^{\beta}x\right)+b_2\left(e^{\beta}x\right)^2+b_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

$$+a_2\left(e^{\beta}x\right)^2\left\{b_o+b_1\left(e^{\beta}x\right)+b_2\left(e^{\beta}x\right)^2+b_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

$$+a_3\left(e^{\beta}x\right)^3\left\{b_o+b_1\left(e^{\beta}x\right)+b_2\left(e^{\beta}x\right)^2+b_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

(F.5)

Collecting similar terms, this gives:

$$e^{\beta x\frac{\partial}{\partial x}}F(x)G(x)=$$

$$\left\{a_o+a_1\left(e^{\beta}x\right)+a_2\left(e^{\beta}x\right)^2+a_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

$$\cdot\left\{b_o+b_1\left(e^{\beta}x\right)+b_2\left(e^{\beta}x\right)^2+b_3\left(e^{\beta}x\right)^3+\cdots\right\}$$

(F.6)

But since,

$$F\left(e^{\beta}x\right)=\sum_{\mathrm{m}=0}^{\infty}a_{\mathrm{m}}\left(e^{\beta}x\right)^{\mathrm{m}} \tag{F.7}$$

and

$$G\left(e^{\beta}x\right)=\sum_{\mathrm{m}=0}^{\infty}b_{\mathrm{m}}\left(e^{\beta}x\right)^{\mathrm{m}} \tag{F.8}$$

then, substituting eqns. (F.7) and (F.8) into eqn. (F.6), one obtains:

$$e^{\beta x\frac{\partial}{\partial x}}F(x)G(x)=F\left(e^{\beta}x\right)G\left(e^{\beta}x\right) \tag{F.9}$$

Using $F(x)=F(\alpha,\alpha^{*})$ and $G(x)=G(\alpha,\alpha^{*})$ in eqn. (F.9), then substituting $\beta=\pm\lambda$ and applying eqns. (5.42), (5.43) for $F(\alpha,\alpha^{*})$ and $G(\alpha,\alpha^{*})$ respectively to eqn. (F.9), the, results become:

$$e^{\pm\lambda\alpha^{*}\frac{\partial}{\partial\alpha^{*}}}F(\alpha,\alpha^{*})G(\alpha,\alpha^{*})=F\left(\alpha,e^{\pm\lambda}\alpha^{*}\right)e^{\pm\lambda\alpha^{*}\frac{\partial}{\partial\alpha^{*}}}G(\alpha,\alpha^{*}) \tag{F.10}$$

and,

$$e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}F(\alpha,\alpha^{*})G(\alpha,\alpha^{*})=F\left(e^{\pm\lambda}\alpha,\alpha^{*}\right)e^{\pm\lambda\alpha\frac{\partial}{\partial\alpha}}G(\alpha,\alpha^{*}) \tag{F.11}$$

Appendix G

The Expressions for $\hat{a}_q^{\dagger}\|\alpha_q\rangle$ and $\langle\alpha_q\|\hat{a}_q$ in Terms of q-Derivatives

The unormalized q-deformed coherent state $\|\alpha_q\rangle$ is given by eqn. (5.70). Then,

$$\hat{a}_q^{\dagger}\|\alpha_q\rangle=\sum_{n=0}^{\infty}\frac{\alpha_q^{n}}{[n]_q!}\hat{a}_q^{\dagger}\|n\rangle_q \qquad (G.1)$$

Substituting for $\hat{a}_q^{\dagger}\|n\rangle_q$ from eqn. (5.74) into the right hand side of eqn. (G.1), then multiplying both sides of the result by $[n+1]_q$ and taking $m=n+1$, the result becomes:

$$\hat{a}_q^{\dagger}\|\alpha_q\rangle=\sum_{m=1}^{\infty}\frac{[m]_q\,\alpha_q^{m-1}}{[m]_q\,[m-1]_q!}\|m\rangle_q \qquad (G.2)$$

But,

$$[m]_q!=[m]_q\,[m-1]_q! \qquad (G.3)$$

Then, substituting eqn. (G.3) into eqn. (G.2) yields:

$$\hat{a}_q^{\dagger}\|\alpha_q\rangle=\sum_{m=1}^{\infty}\frac{[m]_q\,\alpha_q^{m-1}}{[m]_q!}\|m\rangle_q \qquad (G.4)$$

Using the expression for the action of the q-differential operator (Jackson's derivative) and $f\left(\alpha_q\right)=\alpha_q^{\mathrm{m}}$ given in eqn. (5.78), leads to:

$$\hat{a}_q^{\dagger}\left\|\alpha_q\right\rangle=\frac{D}{D\alpha_q}\left\{\sum_{\mathrm{m}=1}^{\infty}\frac{\alpha_q^{\mathrm{m}}}{[\mathrm{m}]_q!}\left\|\mathrm{m}\right\rangle_q+\left\|0\right\rangle_q\right\} \tag{G.5}$$

where,

$$\frac{D}{D\alpha_q}\alpha_q^{\mathrm{m}}=[\mathrm{m}]_q\,\alpha_q^{\mathrm{m}-1} \tag{G.6}$$

and,

$$\frac{D}{D\alpha_q}\left\|0\right\rangle_q=0 \tag{G.7}$$

But since $\left\|\alpha_q\right\rangle$ is as defined in eqn. (5.70), then using this definition in eqn. (G.5), one obtains:

$$\hat{a}_q^{\dagger}\left\|\alpha_q\right\rangle=\frac{D}{D\alpha_q}\left\|\alpha_q\right\rangle \tag{G.8}$$

Similarly, it can be shown that:

$$\left\langle\alpha_q\right\|\hat{a}_q=\left(\hat{a}_q^{\dagger}\left\|\alpha_q\right\rangle\right)^{\dagger}=\left\langle\alpha_q\right\|\frac{D}{D\alpha_q^{*}} \tag{G.9}$$

Appendix H

The Correspondence Relations for $\hat{a}_q^\dagger \hat{\rho}_q$, $\hat{\rho}_q \hat{a}_q$, $\hat{a}_q^\dagger \hat{a}_q \hat{\rho}_q$ and $\hat{\rho}_q \hat{a}_q^\dagger \hat{a}_q$

The correspondence relations for $\hat{a}_q^\dagger \hat{\rho}_q$ and $\hat{\rho}_q \hat{a}_q$ in terms of the $\mathbb{P}_q$ -representation can be derived as follows.

Using the product rule of q-differentiation defined in eqn. (3.25) with $\mathcal{D}_x^q = \frac{D}{D\alpha_q}$, $F(x) = F(\alpha_q, \alpha_q^*)$ and $G(x) = G(\alpha_q, \alpha_q^*)$, then eqn. (3.25) becomes:

$$\frac{D}{D\alpha_q}\left\{F(\alpha_q,\alpha_q^*)G(\alpha_q,\alpha_q^*)\right\} = \left[\frac{D}{D\alpha_q}F(\alpha_q,\alpha_q^*)\right]G(\alpha_q,\alpha_q^*) + F(q\alpha_q,\alpha_q^*)\frac{D}{D\alpha_q}G(\alpha_q,\alpha_q^*) \tag{H.1}$$

Eqn. (H.1) is similar to the equation introduced by Arik and Coon [9], where $F(\alpha_q, \alpha_q^*)$ and $G(\alpha_q, \alpha_q^*)$ are two arbitrarily functions defined as [9]:

$$F(\alpha_q,\alpha_q^*) = G\left((1-q)\left|\alpha_q\right|^2\right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) \tag{H.2}$$

Therefore,

$$F\left(q\alpha_q,\alpha_q^*\right)=G\left(q(1-q)\left|\alpha_q\right|^2\right)f\left(\alpha_q,\alpha_q^*\right) \tag{H.3}$$

and,

$$G\left(\alpha_q,\alpha_q^*\right)=g\left(\alpha_q,\alpha_q^*\right) \tag{H.4}$$

where, $f\left(\alpha_q,\alpha_q^*\right)$ and $g\left(\alpha_q,\alpha_q^*\right)$ are two arbitrary functions.

Substituting eqns. (H.2) – (H.4) into eqn. (H.1) and re-arranging the result, gives:

$$\begin{aligned}G\left(q(1-q)\left|\alpha_q\right|^2\right)f\left(\alpha_q,\alpha_q^*\right)\frac{D}{D\alpha_q}g\left(\alpha_q,\alpha_q^*\right)=\\ \frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)g\left(\alpha_q,\alpha_q^*\right)\right\}\\ -\frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)\right\}g\left(\alpha_q,\alpha_q^*\right)\end{aligned} \tag{H.5}$$

Applying the product rule of q-differentiation (i.e., eqn. (H.1)) again to the expression $\frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)\right\}$ that appears in eqn. (H.5) with $F\left(\alpha_q,\alpha_q^*\right)=G\left((1-q)\left|\alpha_q\right|^2\right)$ and $G\left(\alpha_q,\alpha_q^*\right)=f\left(q^{-1}\alpha_q,\alpha_q^*\right)$, then using the definition of $E_q\left(\sigma/q\right)$ that have been introduced in [9] as:

$$E_q\left(\sigma/q\right)=\frac{1}{G\left((1-q)\left|\alpha_q\right|^2\right)} \tag{H.6}$$

or,

$$G\left(q(1-q)\left|\alpha_q\right|^2\right)=E_q\left(-q\left|\alpha_q\right|^2\right) \tag{H.7}$$

where σ was given in eqn. (5.82), yields:

$$\frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)\right\}=$$
$$-\frac{\alpha_q^* f\left(q^{-1}\alpha_q,\alpha_q^*\right)}{E_q\left(q\left|\alpha_q\right|^2\right)}+\left[\frac{1}{E_q\left(q\left|\alpha_q\right|^2\right)}\right]\left[\frac{D}{D\alpha_q}f\left(q^{-1}\alpha_q,\alpha_q^*\right)\right] \tag{H.8}$$

and the following equation has been used:

$$\frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)\right\}=\frac{D}{D\alpha_q}\left\{E_q\left(-\left|\alpha_q\right|^2\right)\right\}=-\alpha_q^* E_q\left(-q\left|\alpha_q\right|^2\right) \tag{H.9}$$

Substituting eqns. (H.7) and (H-8) into eqn. (H.5) and simplifying the result, produces

$$\left[\frac{1}{E_q\left(q\left|\alpha_q\right|^2\right)}\right]f\left(\alpha_q,\alpha_q^*\right)\frac{D}{D\alpha_q}g\left(\alpha_q,\alpha_q^*\right)=$$
$$\frac{D}{D\alpha_q}\left\{G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)g\left(\alpha_q,\alpha_q^*\right)\right\}$$
$$+\left[\frac{g\left(\alpha_q,\alpha_q^*\right)}{E_q\left(q\left|\alpha_q\right|^2\right)}\right]\left(\alpha_q^*-\frac{D}{D\alpha_q}\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right) \tag{H.10}$$

Using the basic integral $\mathbb{S}$ on both sides of eqn. (H.10), yields:

$$\mathbb{S}\left[\frac{D^2\alpha_q}{E_q\left(q\left|\alpha_q\right|^2\right)}\right] f\left(\alpha_q,\alpha_q^*\right)\frac{D}{D\alpha_q}g\left(\alpha_q,\alpha_q^*\right)=$$

$$\mathbb{S}\,D^2\alpha_q\left\{\frac{D}{D\alpha_q}\left[G\left((1-q)\left|\alpha_q\right|^2\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)g\left(\alpha_q,\alpha_q^*\right)\right]\right\}$$

$$+\,\mathbb{S}\left[\frac{D^2\alpha_q\ g\left(\alpha_q,\alpha_q^*\right)}{E_q\left(q\left|\alpha_q\right|^2\right)}\right]\left(\alpha_q^*-\frac{D}{D\alpha_q}\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)$$

(H.11)

Eqn. (H.11) is similar to the equation that was introduced by Arik and Coon [9] where the 1st term on the right hand side represents the boundary term [9] that vanishes at $\left|\alpha_q\right|^2=[0]_q$ and at $\left|\alpha_q\right|^2\rightarrow[\infty]_q$, where $[0]_q=0$ and $[\infty]_q=(1-q)^{-1}$ are as defined in [9].

Then, eqn. (H-11) becomes:

$$\mathbb{S}\left[\frac{D^2\alpha_q}{E_q\left(q\left|\alpha_q\right|^2\right)}\right] f\left(\alpha_q,\alpha_q^*\right)\frac{D}{D\alpha_q}g\left(\alpha_q,\alpha_q^*\right)=$$

$$\mathbb{S}\left[\frac{D^2\alpha_q\ g\left(\alpha_q,\alpha_q^*\right)}{E_q\left(q\left|\alpha_q\right|^2\right)}\right]\left(\alpha_q^*-\frac{D}{D\alpha_q}\right)f\left(q^{-1}\alpha_q,\alpha_q^*\right)$$

(H.12)

But since,

$$\hat{a}_q^\dagger \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \hat{a}_q^\dagger \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \tag{H.13}$$

then, by applying eqn. (H.12) to eqn. (H.13) and using eqn. (G.8), one can get:

$$f\left(\alpha_q, \alpha_q^*\right) \equiv \mathbb{P}_q\left(\alpha_q, \alpha_q^*\right) \tag{H.14}$$

$$g\left(\alpha_q, \alpha_q^*\right) \equiv \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \tag{H.15}$$

Hence,

$$\hat{a}_q^\dagger \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \equiv \frac{D}{D\alpha_q} g\left(\alpha_q, \alpha_q^*\right) \tag{H.16}$$

Therefore, one can deduce that eqn. (H.13) takes the form:

$$\hat{a}_q^\dagger \hat{\rho}_q = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \tag{H.17}$$

The adjoint of eqn. (H.13) is given as:

$$\hat{\rho}_q \hat{a}_q = \left(\hat{a}_q^\dagger \hat{\rho}_q\right)^\dagger = \frac{1}{\pi} \mathbb{S} \frac{D^2\alpha_q}{E_q(\sigma)} \left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \left[\alpha_q - \frac{D}{D\alpha_q^*}\right] \mathbb{P}_q\left(\alpha_q, q^{-1}\alpha_q^*\right) \tag{H.18}$$

Then, the one-to-one correspondence for eqns. (H.17) and (H.18) in terms of the $\mathbb{P}_q$-representation can be deduced as:

$$\hat{a}_q^\dagger \hat{\rho}_q \to \left[\alpha_q^* - \frac{D}{D\alpha_q}\right] \mathbb{P}_q\left(q^{-1}\alpha_q, \alpha_q^*\right) \tag{H.19}$$

and,

$$\hat{\rho}_q \hat{a}_q \rightarrow \left[\alpha_q - \frac{D}{D\alpha_q^*} \right] \mathbb{P}_q \left(\alpha_q, q^{-1} \alpha_q^* \right) \tag{H.20}$$

respectively.

Similarly, the correspondence relations for $\hat{a}_q^\dagger \hat{a}_q \hat{\rho}_q$ and $\hat{\rho}_q \hat{a}_q^\dagger \hat{a}_q$ can be derived by the same technique, but using:

$$F\left(\alpha_q, \alpha_q^*\right) = G\left((1-q) \left| \alpha_q \right|^2 \right) \left[f\left(q^{-1} \alpha_q, \alpha_q^* \right) \right] \alpha_q \tag{H.21}$$

instead of eqn. (H.2).

Therefore,

$$F\left(q\alpha_q, \alpha_q^*\right) = G\left(q\,(1-q) \left| \alpha_q \right|^2 \right) \left[f\left(\alpha_q, \alpha_q^* \right) \right] \left(q\alpha_q \right) \tag{H.22}$$

Then, after some lengthy mathematical manipulations, the result becomes:

$$\begin{aligned} &\mathbb{S} \left[\frac{D^2 \alpha_q}{E_q\left(q \left| \alpha_q \right|^2 \right)} \right] f\left(\alpha_q, \alpha_q^* \right) \left(q\alpha_q \right) \frac{D}{D\alpha_q} g\left(\alpha_q, \alpha_q^* \right) = \\ &\quad \mathbb{S}\, D^2 \alpha_q \left\{ \frac{D}{D\alpha_q} \left[G\left((1-q) \left| \alpha_q \right|^2 \right) f\left(q^{-1} \alpha_q, \alpha_q^* \right) \alpha_q \; g\left(\alpha_q, \alpha_q^* \right) \right] \right. \\ &\qquad\qquad \left. - G\left(q(1-q) \left| \alpha_q \right|^2 \right) g\left(\alpha_q, \alpha_q^* \right) f\left(\alpha_q, \alpha_q^* \right) \right\} \\ &\qquad\qquad + \mathbb{S} \left[\frac{D^2 \alpha_q \; g\left(\alpha_q, \alpha_q^* \right)}{E_q\left(q \left| \alpha_q \right|^2 \right)} \right] \left[\alpha_q^* - \frac{D}{D\alpha_q} \right] \alpha_q \; f\left(q^{-1} \alpha_q, \alpha_q^* \right) \end{aligned} \tag{H.23}$$

Similarly, the 1st term on the right hand side of eqn. (H.23) represents a boundary term [9] which also vanishes at $\left|\alpha_q\right|^2 = [0]_q$ and at $\left|\alpha_q\right|^2 \to [\infty]_q$.

This can be proved in a similar manner a done by Arik and Coon [9] and as follows.

Since $\mathbb{S}\, D^2\alpha_q$ consists of an ordinary integration over the argument ϕ of the complex variable $\alpha_q = \left|\alpha_q\right| e^{i\phi}$ and a basic integration over $\left|\alpha_q\right|^2$, then one can write [9]:

$$\mathbb{S}\, D^2\alpha_q F\left(\alpha_q,\alpha_q^*\right) \equiv \underset{[0]_q}{\overset{[\infty]_q}{\mathbb{S}}} D\left(\left|\alpha_q\right|^2\right) \int_0^{2\pi} d\phi F\left(\alpha_q,\alpha_q^*\right) \qquad \text{(H.24)}$$

Letting

$$d\phi = \left(\frac{i}{2}\right) \frac{d\alpha_q^*}{\alpha_q^*} \qquad \text{(H.25)}$$

$$F\left(\alpha_q,\alpha_q^*\right) = \frac{D}{D\alpha_q} \left\{ G\left((1-q)\left|\alpha_q\right|^2 \right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) \alpha_q \, g\left(\alpha_q,\alpha_q^*\right) \right\}$$
$$- G\left(q(1-q)\left|\alpha_q\right|^2 \right) f\left(\alpha_q,\alpha_q^*\right) g\left(\alpha_q,\alpha_q^*\right)$$
$$\text{(H.26)}$$

and using the fact that for $\frac{D}{D\alpha_q}\left(\left|\alpha_q\right|^2\right) = \alpha_q^*$, then

$$\frac{D}{D\alpha_q} = \alpha_q^* \frac{D}{D\left(\left|\alpha_q\right|^2\right)} \qquad \text{(H.27)}$$

After some mathematical manipulations, eqn. (H.24) becomes:

$$\mathbb{S}\, D^2\alpha_q \left\{ \frac{D}{D\alpha_q} \left[G\left((1-q)\left|\alpha_q\right|^2 \right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) \alpha_q\, g\left(\alpha_q,\alpha_q^*\right) \right. \right.$$
$$\left. - G\left(q(1-q)\left|\alpha_q\right|^2 \right)\, f\left(\alpha_q,\alpha_q^*\right) g\left(\alpha_q,\alpha_q^*\right) \right\} = \left[\frac{i}{2}\right] \oint d\alpha_q^* \overset{[\infty]_q}{\underset{[0]_q}{\mathbb{S}}} D$$
$$\cdot \left\{ \left[G\left((1-q)\left|\alpha_q\right|^2 \right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) - G\left(q\,(1-q)\left|\alpha_q\right|^2 \right) f\left(\alpha_q,\alpha_q^*\right) \right] \right.$$
$$\left. \cdot\, \alpha_q\, g\left(\alpha_q,\alpha_q^*\right) \right\}$$

(H.28)

But $\overset{b}{\underset{0}{\mathbb{S}}} DF\left(\alpha_q,\alpha_q^*\right)$ can be written as [9]:

$$\overset{[\infty]_q}{\underset{[0]_q}{\mathbb{S}}} DF\left(\alpha_q,\alpha_q^*\right) = F\left(\alpha_q,\alpha_q^*\right)\Bigg|_{\left|\alpha_q\right|^2 = [0]_q}^{\left|\alpha_q\right|^2 = [\infty]_q} \tag{H.29}$$

where,

$$F\left(\alpha_q,\alpha_q^*\right) =$$
$$\left[G\left((1-q)\left|\alpha_q\right|^2 \right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) - G\left(q\,(1-q)\left|\alpha_q\right|^2 \right) f\left(\alpha_q,\alpha_q^*\right) \right] \alpha_q\, g\left(\alpha_q,\alpha_q^*\right)$$

(H.30)

Then, applying eqn. (H.29) to the right hand side of eqn. (H.28), and simplifying the result, yields:

$$\mathbb{S}\, D^2\alpha_q \left\{ \frac{D}{D\alpha_q} \left[G\left((1-q)\left|\alpha_q\right|^2 \right) f\left(q^{-1}\alpha_q,\alpha_q^*\right) \alpha_q\, g\left(\alpha_q,\alpha_q^*\right) \right] \right.$$

$$\left. - G\left(q(1-q)\left|\alpha_q\right|^2 \right) g\left(\alpha_q,\alpha_q^*\right) f\left(\alpha_q,\alpha_q^*\right) \right\} =$$

$$\left[\frac{i}{2}\right] \oint d\alpha_q^* \left\{ F\left(\alpha_q,\alpha_q^*\right) \Bigg|_{\left|\alpha_q\right|^2 = [0]_q}^{\left|\alpha_q\right|^2 = [\infty]_q} \right\} = 0$$

(H.31)

The notation $\oint$ represents a closed contour integration on the circle $\left|\alpha_q\right|^2$.

The term on the right hand side of eqn. (H.31) vanishes for $\left|\alpha_q\right|^2 = [0]_q$ and for $\left|\alpha_q\right|^2 \rightarrow [\infty]_q$ as in ref. [9].

Therefore, eqn. (H.23) becomes:

$$\mathbb{S} \left[\frac{D^2\alpha_q}{E_q\left(q\left|\alpha_q\right|^2 \right)} \right] f\left(\alpha_q,\alpha_q^*\right) \left(\alpha_q\right) \frac{D}{D\alpha_q}\, g\left(\alpha_q,\alpha_q^*\right) =$$

$$\mathbb{S} \left[\frac{D^2\alpha_q\, g\left(\alpha_q,\alpha_q^*\right)}{E_q\left(q\left|\alpha_q\right|^2 \right)} \right] \left(\alpha_q^* - \frac{D}{D\alpha_q} \right) \left(q^{-1}\alpha_q\right) f\left(q^{-1}\alpha_q,\alpha_q^*\right)$$

(H.32)

But, since

$$\hat{a}_q^{\dagger}\,\hat{a}_q\,\hat{\rho}_q = \frac{1}{\pi}\,\mathbb{S}\,\frac{D^2\alpha_q}{E_q(\sigma)}\,\mathbb{P}_q\left(\alpha_q,\alpha_q^*\right)\left(\alpha_q\right)\hat{a}_q^{\dagger}\left\|\alpha_q\right\rangle\left\langle\alpha_q\right\| \qquad \text{(H.33)}$$

then, by applying eqn. (H.32) to eqn. (H.33), and using eqn. (G.8), one can deduce that eqn. (H.33) takes the form:

$$\hat{a}_q^{\dagger}\,\hat{a}_q\,\hat{\rho}_q = \frac{1}{\pi}\,\mathbb{S}\,\frac{D^2\alpha_q}{E_q(\sigma)}\left\|\alpha_q\right\rangle\left\langle\alpha_q\right\|\left[\alpha_q^* - \frac{D}{D\alpha_q}\right]\left(q^{-1}\alpha_q\right)\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*\right) \qquad \text{(H.34)}$$

The adjoint of eqn. (H.34) is:

$$\hat{\rho}_q\,\hat{a}_q^{\dagger}\,\hat{a}_q = \frac{1}{\pi}\,\mathbb{S}\,\frac{D^2\alpha_q}{E_q(\sigma)}\left\|\alpha_q\right\rangle\left\langle\alpha_q\right\|\left[\alpha_q - \frac{D}{D\alpha_q^*}\right]\left(q^{-1}\alpha_q^*\right)\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*\right) \qquad \text{(H.35)}$$

Then, the one-to-one correspondence relations for eqns. (H.34) and (H.35) in terms of the $\mathbb{P}_q$ -representation become:

$$\hat{a}_q^{\dagger}\,\hat{a}_q\,\hat{\rho}_q \to \left[\alpha_q^* - \frac{D}{D\alpha_q}\right]\left(q^{-1}\alpha_q\right)\mathbb{P}_q\left(q^{-1}\alpha_q,\alpha_q^*\right) \qquad \text{(H.36)}$$

and

$$\hat{\rho}_q\,\hat{a}_q^{\dagger}\,\hat{a}_q \to \left[\alpha_q - \frac{D}{D\alpha_q^*}\right]\left(q^{-1}\alpha_q^*\right)\mathbb{P}_q\left(\alpha_q,q^{-1}\alpha_q^*\right) \qquad \text{(H.37)}$$

respectively.

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جمهورية العراق
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الحالات الكمية المتشاكهة المشوهة وبعض تطبيقاتها

أطروحة

مقدمة الى كلية العلوم / جامعة النهرين

وهي جزء من متطلبات نيل درجة الدكتوراه في الفيزياء

من قبل

أحمد شاكر محمود ياس

بكالوريوس علوم فيزياء-كلية العلوم / جامعة النهرين (١٩٩٨)

ماجستيرعلوم فيزياء-كلية العلوم / جامعة النهرين (٢٠٠١)

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جمادى الأولى **شباط**

١٤٣٧هـ **٢٠١٦ م**

المُلخَص

لقد وَجَدَ مفهوم التشوه من النوع q طريقه الى العديد من التطبيقات في حقول الفيزياء المختلفة، مثل البصريات الكمية، الفيزياء الذرية، فيزياء الحالة الصلبة، الفيزياء النووية وعلم الكونيات، مما شجع على تعميم فكرة هذا النوع من التشوه لتشمل مفاهيم أخرى مثل الحالات المتشاكهة المعروفة في حقل البصريات الكمية. هذا من جانب ومن جانب أخر فأن تفسير المعنى الفيزيائي للتشوه من النوع q يبقى المشكلة الأبرز في جميع هذه التطبيقات.

إن العمل الحالي هو محاولة لتطبيق مفهوم الحالات المتشاكهة المشوهة من أجل أيجاد حل لمسألة تفسير المعنى الفيزيائي هذه، حيث تم أستخدام المتذبذب التوافقي الكمي ذي التشوه من النوع q ببعد الواحد كنموذج لتطبيق هذه المنهجية في أستخدام الحالات المتشاكهة المشوهة لحل هذه المسألة.

في العمل الحالي ، تم أبتداءاً أشتقاق معادلة ليوفل الكلاسيكية للمتذبذب التوافقي الكلاسيكي ذو التشوه من النوع q ببعد واحد في حالتي فضاء الطور المشوه وغير المشوه، حيث تم حل المعادلة آنفة الذكر بأستخدام "طريقة الخصائص" المعروفة في حل المعادلات التفاضلية الجزئية التي تعطي دالة توزيع الأحتمالية الكلاسيكية لهذا المتذبذب في فضاء الطور. ثم تم تحري سلوك هذه الدالة في فضاء الطور بأستخدام طريقة حاسوبية لتوضيح تفاصيل هذا السلوك من خلال بناء برنامج حاسوبي بأستخدام حزمة البرمجيات Mathematica®.

وعلى المستوى الكمي فقد تم في العمل الحالي إعادة صياغة معادلة هايزنبيرك للحركة لمؤثر الكثافة للمتذبذب التوافقي الكمي غير المشوه ببعد واحد بدلالة دوال توزيع شبه الأحتمالية المعروفة، لتمثيل المتذبذب ذو التشوه من النوع q في فضائي الطور المشوه وغير المشوه. ساعد هذا الأمر في الحصول على معادلة ليوفل الكمية للمتذبذب في هذين الفضائين. تم التوصل للغايات الكلاسيكية لمعادلات ليوفل الكمية بتعميم الطريقة للغايات التقليدية المستخدمة في الحالة غيرالمشوهة الى الحالة المشوهة. بالاضافة الى ذلك، فقد تم في العمل الحالي توظيف تقريب جديد لتحري الغاية الكلاسيكية لمعادلة ليوفل الكمية لهذا المتذبذب بأستخدام الحالات المتشاكهة المشوهة التي تمت صياغتها سابقا من قبل الباحثين (أريك وكوون).

أثبتت النتائج المترشحة عن الأشتقاقات الرياضية للغايات الكلاسيكية لمعادلات ليوفل الكمية لهذا المتذبذب بأن هذه الغايات ذات طبيعة أحصائية كما هو الحال للغاية الكلاسيكية للمتذبذب الكمي غير المشوه. كذلك، فإن هذه النتائج كشفت مع نتائج البرامجيات الحاسوبية المعدة بأستخدام Mathematica® عن معلومات أكثر تفصيلاً.

تخص المعنى الفيزيائي للتشوه من النوع q لهذا المتذبذب، وتضمن ملاحظة أن المتذبذب ذي التشوه من النوع q بالبعد الواحد يمكن إعتباره مكافئاً لمتذبذب لاخطي يتصف بمعامل لاخطية يعتمد على $\hbar$. لوحظ أيضاً بأن سلوك الغايات الكلاسيكية لمعادلة ليوفل الكمية لهذا المتذبذب يتصف بأشكال دوامية في فضاء الطورحيث با لأمكان مقارنتها مع نظيراتها الكلاسيكية لأظهار التفاوت في السلوك. ويمكن أعتبار هذا السلوك ذي الشكل الدوامي ظاهرة عامة مرتبطة بالتشوه من النوع q، حيث يكون المتذبذب اللاتوافقي حالة خاصة.

تم التوصل أيضاً من خلال نتائج العمل الحالي إلى وجود مؤشرات على طبيعة لاتبادلية لهندسة فضاء الطور للمتذبذب ذي التشوه من النوع q الذي تمت دراسته.