بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

وَمَا تَوْفِيقِيٓ إِلَّا بِٱللَّهِۚ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ ﴿٨٨﴾

صدق الله العظيم

هود-٨٨

وما أوتيتم من العلم إلا قليلا

AL-NAHRAIN UNIVERSITY

جامعة النهرين بغداد - العراق

المستخلص

الهدف من هذه الاطروحة هو أثبات وجود ووحدانية الحل العام (**Mild Solution**) لمسألة سيطرة شبه خطية ذات قيمة أبتدأية في فضاء باناخ مناسب وكذالك قابليتها على السيطرة. بعض النظريات التي تتعلق في قابلية السيطرة، وأثبات وجودية الحل العام (محلي ومطلق) وكذالك وحدانيته في فضاء باناخ مناسب بأستخدام نظرية النقطة الثابته لشويدرومنهج شبه الزمرة (شبه الزمره المتراصه). بأستخدام مبدأ التقلص لبناخ ومنهج شبه الزمرة (شبه الزمره التحليلى) في المسألة غير محدودة البعد في فضاءات غير محدودة البعد، لقد نوقشت و طورت في فضاء باناخ مناسب وجودية ووحدانية الحل العام (محلي) لمسألة سيطرة شبه خطية ذات قيمة أبتدأية. بعض الامثلة التوضيحيه ومدياتها التطبيقية للمسألة نوقشت وعرضت.

Abstract

The aim of this thesis is to prove the existence and uniqueness of the mild solutions of semilinear initial value control problems in a suitable Banach spaces as well as their controllability. Some theorems regarding controllability, local and global existence as well as uniqueness of the mild solution in infinite dimensional spaces have been developed in suitable Banach space using the Schauder fixed point theorem and the semigroup theory (compact semigroup). By using the Banach contraction principle and the semigroup theory (analytic semigroup) in infinite dimensional spaces, have been discussed and developed in suitable Banach spaces the local existence and uniqueness of the mild solution to the semilinear initial value control problem. Some illustrations and practical scopes of the problems have been discussed and present.

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Manaf Adnan

June, 2005.

Chapter One

Some Basic Concepts of Semigroup Theory

Chapter Three

Existence and Uniqueness of Mild Solution to the Semilinear Initial Value Control Problem Via "Banach Contraction Principle"

Chapter Two

Existence, Uniqueness and Controllability of Mild Solution to the Semilinear Initial Value Control Problem Via "Schauder Fixed Point Theorem"

APPENDICES

Appendix A:

The following remark which is useful here:

Remark:

(1) KerW= $\{w(t) \in O : Ww(t) = 0\}$.

(2) KerW is the vector space {see [Taylor, 58]}.

(3) KerW is the closed subspace of O. {see [Taylor, 58]}.

(4) Since O is a Banach space and Ker W is a subspace of O, we can define a Quotient space denoted by O/Ker W, define as follow:

$$O/\text{Ker } W = \{[w(t)] : w(t) \in O\},$$

Where $[w(t)] = \{w^*(t) \in O: w^*(t) - w(t) \in \text{KerW}\}$, {see [Taylor, 58]},

Where $[w(t)]$ is said to be an equivalent classes of $w(t)$.

(5) The Quotient space O/Ker W = $\{[w(t)]:w(t)\in O\}$, forms a vector space over the field of scalars by given the following definitions:

$$[\overline{w}(t)] \oplus [\overline{\overline{w}}(t)] = [\overline{w}(t) + \overline{\overline{w}}(t)], \forall \overline{w}(t), \overline{\overline{w}}(t) \in O$$

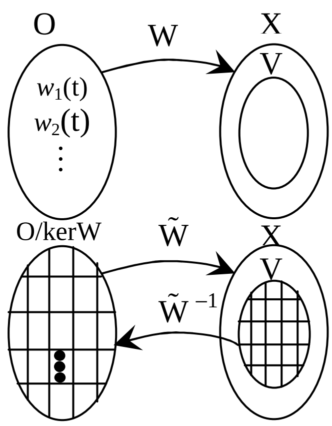
$$\alpha \odot [w(t)] = [\alpha w(t)], \text{for } \alpha \in R \text{ and } w(t) \in O.$$

(6) Since KerW is a closed subspace of a Banach space O then define a norm on O/Ker W, as follow: $\|[w(t)]\|_{O/\text{KerW}} = \inf_{w(t)\in[w(t)]} \|w(t)\|_O$.

, Moreover O is a Banach space then O/Ker W is also a Banach space. {See [Taylor, 58]}.

Remark (construction of $\tilde{\mathrm{W}}$ ***) {[Balachandran, 01], [Quinn, 85]}:***

Define a linear operator $\tilde{\mathrm{W}}$: O/Ker W $\longrightarrow$ X, by:



$\tilde{\mathrm{W}}\left[w(\mathrm{t})\right] = \mathrm{W}w(\mathrm{t}), w(\mathrm{t}) \in \left[w(\mathrm{t})\right]$

$\tilde{\mathrm{W}}$ Is one-to-one

{Since $\tilde{\mathrm{w}}\left[\bar{\bar{w}}(\mathrm{t})\right] = \tilde{\mathrm{w}}\left[\bar{w}(\mathrm{t})\right], \forall\left[\bar{\bar{w}}(\mathrm{t})\right], \left[\bar{w}(\mathrm{t})\right] \in \mathrm{O}/\mathrm{KerW}$

$\Rightarrow \mathrm{W}\bar{\bar{w}}(\mathrm{t}) = \mathrm{W}\bar{w}(\mathrm{t}), \forall \bar{\bar{w}}(\mathrm{t}) \in \left[\bar{\bar{w}}(\mathrm{t})\right], \bar{w}(\mathrm{t}) \in \left[\bar{w}(\mathrm{t})\right]$

$\Rightarrow \mathrm{W}\bar{\bar{w}}(\mathrm{t}) - \mathrm{W}\bar{w}(\mathrm{t}) = 0$

$\Rightarrow \mathrm{W}\left(\bar{\bar{w}}(\mathrm{t}) - \bar{w}(\mathrm{t})\right) = 0$

$\Rightarrow \bar{\bar{w}}(\mathrm{t}) - \bar{w}(\mathrm{t}) \in \mathrm{KerW}$

$\Rightarrow \bar{\bar{w}}(\mathrm{t}) \in \left[\bar{w}(\mathrm{t})\right]$ {Since $\left[\bar{w}(\mathrm{t})\right] = \{\bar{\bar{w}}(\mathrm{t}) \in \mathrm{O}\colon \bar{\bar{w}}(\mathrm{t}) - \bar{w}(\mathrm{t}) \in \mathrm{KerW}\}$

$\Rightarrow \left[\bar{\bar{w}}(\mathrm{t})\right] = \left[\bar{w}(\mathrm{t})\right]$

So, There exist $\tilde{\mathrm{W}}^{-1}$ define from V into O/Ker W.

To prove Range W=V is a Banach spaces via the norm define as follow:

$$\|\mathrm{v}\|_{\mathrm{V}} = \left\|\tilde{\mathrm{W}}^{-1}\mathrm{v}\right\|_{\mathrm{O}/\mathrm{KerW}}$$

Notice that:

$$\|\mathrm{W}w(\mathrm{t})\|_{\mathrm{V}} = \left\|\tilde{\mathrm{W}}^{-1}\mathrm{W}w(\mathrm{t})\right\|_{\mathrm{O}/\mathrm{KerW}} = \left\|\tilde{\mathrm{W}}^{-1}\mathrm{W}\left[w(\mathrm{t})\right]\right\|_{\mathrm{O}/\ker \mathrm{W}}, \forall w(\mathrm{t}) \in \left[w(\mathrm{t})\right]$$

$$= \left\|\left[w(\mathrm{t})\right]\right\|_{\mathrm{O}/\mathrm{KerW}} = \inf_{w(\mathrm{t}) \in [w(\mathrm{t})]} \|w(\mathrm{t})\|_{\mathrm{O}} \leq \|w(\mathrm{t})\|_{\mathrm{O}}, \forall w(\mathrm{t}) \in \mathrm{O},$$

So, W is a bounded linear operator for $0 \leq \mathrm{t} \leq \gamma$.

And $\left\|\tilde{\mathrm{W}}\left[w(\mathrm{t})\right]\right\|_{\mathrm{X}} = \|\mathrm{W}w(\mathrm{t})\|_{\mathrm{X}}, \forall w(\mathrm{t}) \in \left[w(\mathrm{t})\right]$

$$\Rightarrow \left\|\tilde{W}[w(t)]\right\|_X \leq \|W\|\|w(t)\|_O, \forall w(t) \in [w(t)]$$

$$\Rightarrow \left\|\tilde{W}[w(t)]\right\|_X \leq \|W\| \inf_{w(t)\in[w(t)]} \|w(t)\|_O = \|W\|\|[w(t)]\|_{O/\ker W}$$

$$\Rightarrow \left\|\tilde{W}[w(t)]\right\|_X \leq \|W\|\|[w(t)]\|_{O/\ker W}.$$

Since $\tilde{W}$ is bounded and D($\tilde{W}$) = O/KerW is closed which implies that $\tilde{W}^{-1}$ is closed {see Appendix D}.

Since $\tilde{W}^{-1}$ is closed operator and by the norm $\|v\|_V = \left\|\tilde{W}^{-1}v\right\|_{O/\mathrm{KerW}}$, which implies that V = Range W a Banach space. {[Balachandran, 01]}.

Since O is reflexive Banach space and KerW is weakly closed {see Appendix D}, So the infimum is actually attained, we can choose a control function $\underline{w}(t) \in [w(t)]$ such that $\underline{w}(t) = \tilde{W}^{-1}W\underline{w}(t)$.{see [Balachandran, 01], [Quinn, 85]}

$\Rightarrow \tilde{W}\underline{w}(t) = W\underline{w}(t)$, for $0 \leq t \leq \gamma$.

Remark:

Notice that:

Given $u_w(0) = u_0, u_w(\gamma) = v_0 \in V \subseteq X$, for arbitrary $w(.) \in L^p([0,r):O)$.

Notice that, we have a unique mild solution $u_w \in C([0,t_1]:X)$, given by:

$$u_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[Bw(s) + f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds$$

For $0 \leq t \leq t_1$ and for every arbitrary control function $w(.) \in L^p([0,t_1):O)$.

When $t = \gamma$ such that $0 < \gamma < t_1$

$$\Rightarrow u_w(\gamma)=T(\gamma)u_0+\int_{s=0}^{\gamma}T(\gamma-s)\left[Bw(s)+f(s,u_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds,$$

$\forall\ w(.)\in L^p([0,t_1):O)$.

Since $u_w(\gamma)=v_0$, we get:

$$v_0-T(\gamma)u_0-\int_{s=0}^{\gamma}T(\gamma-s)\left[f(s,u_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds=$$

$$\int_0^{\gamma}T(\gamma-s)Bw(s)ds,\forall w(.)\in L^p([0,r]:O).$$

Since $Gw(\gamma)=\int_0^{\gamma}T(\gamma-s)Bw(s)ds$, $\forall\ w(.)\in L^p([0,\gamma):O)$, we get:

$$v_0-T(\gamma)u_0-\int_{s=0}^{\gamma}T(\gamma-s)\left[f(s,u_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds=Gw(\gamma),$$

$\forall\ w(.)\in L^p([0,\gamma):O)$.

From {Remark (construction of $\tilde{W}$)}, there exist a control function $\underline{w}(t)\in[w(t)]$, for $0\le t\le\gamma$, i.e, $\underline{w}(t)\in O$ and $\underline{w}(t)-w(t)\in \mathrm{Ker}G$,

i.e., $\underline{w}\in L^p([0,\gamma]:O)$ and $\underline{w}(t)-w(t)\in O$, $W(\underline{w}-w)(t)=0$, such that:

$$\underline{w}(t)=\tilde{W}^{-1}W\underline{w}(t).$$

$\Rightarrow\ \tilde{W}\underline{w}(t)=W\underline{w}(t)$, for $0\le t\le\gamma$.

$$\Rightarrow v_0-T(\gamma)u_0-\int_{s=0}^{\gamma}T(\gamma-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds=G\underline{w}(\gamma)=\tilde{G}\underline{w}(\gamma)$$

$$\Rightarrow \tilde{G}\underline{w}(\gamma) = v_0 - T(\gamma)u_0 - \int_{s=0}^{\gamma} T(\gamma - s)\left[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau \right]ds,$$

For $\underline{w}(\gamma) \in [w(\gamma)]$.

Taking the inverse of $\tilde{G}$ ($\tilde{G}^{-1}$) of the both sides of the above equality, we get:

$$\underline{w}(\gamma) = \tilde{G}^{-1}\left[v_0 - T(\gamma)u_0 - \int_{s=0}^{\gamma} T(\gamma - s)\left[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau \right]ds \right]$$

$$\Rightarrow \underline{w}(t) = \tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(\gamma - s)\left[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau \right]ds \right]$$

Appendix B: "Some basic concepts of operator theory"

B.1 Linear operator[Taylor, 58]:

An operator T: X $\longrightarrow$ X, {X is a real or complex Banach space} is called a linear operator if it satisfies:

(i) $T(x + y) = Tx + Ty, \forall\ x, y \in X$.

(ii) $T(\alpha x) = \alpha x, \forall x \in X$ and $\alpha \in$ R or , Where (R is a real number, is a complex number).

B.2 Bounded Linear Operator [Taylor, 58]:

Let X, Y are the (real or complex) Banach spaces and T: X $\longrightarrow$ Y a linear operator. The operator T is said to be bounded if there is a real number c such that $\|Tx\| \le c\|x\|, \forall x \in X$.

The following are some useful examples:

B.2.1 Examples [Taylor, 58]:

(i) ***Identity operator:*** Let X is a Banach space, I: X $\rightarrow$ X is bounded.

(ii) ***Differentiation operator:*** Let X be the Banach space of all polynomials on J = [0,1], with norm given by $\|x\| = \max_{t \in J} |x(t)|$, A differentiation operator T is defined on X by:

$$Tx(t) = x'(t),$$

The operator is linear but not bounded.

(iii) ***Integral operator:***

We define an integral operator, T: C[0,1] $\longrightarrow$ C[0, 1], by:

$$Tx(t) = \int_0^1 K(t,\tau)x(\tau)d\tau,$$

Here K is a given function, which is called the keral of T and is assumed to be continuous on the closed square G = J×J in the tτ–plane, where J = [0, 1]. This operator is linear and bounded.

B.3 Compact Linear Operator:

Compact linear operator is very important in applications. For instance, they play the control role in the theory of the integral equations and in various problems of mathematical physics.

The following are needed later on:

B.3.1 Compact Set [Marsden,95]:

A subset M of a Banach space X is said to be compact if every open covering of M can be reduced to a finite open covering of M, i.e., if $M \subset \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ and $\Omega_\lambda \subset X$ is open for every λ in the set Λ, then there exists already finitely many, say $\Omega_{\lambda_1} \ldots\ldots \Omega_{\lambda_m}$,such that $M \subset \bigcup_{i=1}^{m} \Omega_{\lambda_i}$ ..

B.3.2 Relatively Compact [Marsden,95]:

A subset M of X is called relatively compact if $\bar{M}$ is compact.

B.3.3 Compact Linear Operator[Erwin,] :

Let X and Y are a (real or complex) Banach spaces. An operator $T: X \longrightarrow Y$ is said to be compact linear operator if for each bounded subset M of X, the image T(M) is relatively compact.

B.4 Precompact linear operator:

***B.4.1 Precompact set* [Taylor, 58] :**

Let X be a Banach space, a subset S of X is said to be precompact if for each ε > 0, there exists some finite set $S = \{x_1, \ldots, x_n\}$ in X such that S is contained in $\bigcup_{i=1}^{n} \beta(x_i, \varepsilon)$, where $\beta(x_i, \varepsilon) = \{y \in X : \|y - x_i\| < \varepsilon\}$.

B.4.2 Precompact linear operator[Harro,82]:

A linear operator T: X$\rightarrow$ Y, (X,Y are Banach spaces) is said to be precompact if for any bounded subset M of X, the image T(M) is precompact set in Y.

B.4.3 Theorem [Harro,82]:

Let X is a normed space and M is a subset of X, then the following properties hold:

(i) If M is relatively compact set then M is precompact set.

(ii) If M is precompact set in a complete space then M is relatively compact set.

(iii) If M is compact set then M is precompact set in X.

(iv) If M is precompact set then M is bounded set.

B.4.4 Remark [Harro,82]:

Let A and B are two precompact sets in Banach space X, then A + B is precompact set in X.

***B.4.5 Theorem* [Taylor, 58] :**

Let X, Y is normed spaces, then:

(i) Every compact linear operator T: X $\longrightarrow$ Y is bounded.

(ii) If dim X = ∞, the identity operator I : X $\longrightarrow$ X is not compact.

(iii) If dim X $<\infty$, then the identity operator I: X $\rightarrow$ X is compact.

Appendix C:

(1) ***Equicontinuous set [Marsden,95] :***

A subset S of C[a,b] is said to be equicontinuous, for each $\varepsilon > 0$, there is a $\delta > 0$, such that:

$$|x - y| < \delta \text{ And } u \in M \text{ imply } \|u(x) - u(y)\|_{C[a,b]} < \varepsilon$$

(2) ***Arzela-Ascoli's theorem [Dieudonne,60] :***

Suppose F is a Banach space and E is a compact metric space. In order that a subset H of the Banach space $\Im_F(E)$ be relatively compact, if and only if H be equicontinuous and that, for each x $\in$ E, the set H(x) = {f(x): f $\in$ H} be relatively compact in F.

(3) ***Schauder fixed point theorem [Zeidler,86]:***

Let M be a nonempty closed, bounded, convex subset of a Banach space X and the map T: M $\longrightarrow$ M is compact then T has a fixed point.

(4) ***Compact map [Zeidler,86]:***

Let S, M are two sets, a map T: S $\longrightarrow$ M is said to be compact if the following conditions are hold:

(i) T is continuous map.

(ii) For each bounded subsets of S, T(S) is relatively compact set in M.

(5) ***Strict contraction map [Klaus,85]:***

Suppose X is a Banach space X A mapping T: X$\longrightarrow$X is said to be strict contraction, with strict contraction constant L, if $\|Tx - Ty\|_X \le L\|x - y\|_X$, $\forall$ x, y $\in$ X, where 0 < L < 1.

(6) ***Closed set [Zeidler,86]:***

A subset S of the Banach space X is said to be closed, if for $x_n \in S$ such that $x_n \longrightarrow x$, then $x \in S$.

(7) ***Bounded set [Zeidler,86]:***

A subset S of the Banach space X is said to be bounded, if there exist $L > 0$ such that $\|s\|_S \le L$, $\forall s \in S$.

(8) ***Convex set [Zeidler,86] :***

A subset S of the Banach space X is said to be convex, if for each $s_1, s_2 \in S$, $\lambda s_1 + (1-\lambda)s_2 \in S$. Where $\lambda \in [0, 1]$.

(9) ***Banach contraction principle [Marsden,95]:***

Let M is a closed nonempty set in the Banach space X over k, where k are a scalar field and the operator T: $M \longrightarrow M$ is strict contraction operator then T has a unique fixed point.

(10) ***Locally Hölder continuous map [Paz,83] :***

Let I be an interval, A function f: $I \longrightarrow X$, where X is a Banach space is said to be Hölder continuous with exponent ϑ, $0 < \vartheta < 1$ on I, if there is a constant L such that $\|f(t) - f(s)\|_X \le L|t-s|^{\vartheta}$, for $s, t \in I$.

(11) ***Gronwall's inequality [Zeidler,86]:***

If the following conditions hold:

(1) $f(t) \ge 0$, $g(t) \ge 0$ and $h(t) \ge 0$ $\forall t \in [a,b]$.

(2) f, g and h are continuous function on (a, b).

(3) $f(t) \le h(t) + \int_a^t f(s)g(s)ds, \forall t \in [a,b]$

Then $f(t) \le h(t)e^{\int_a^t g(s)ds}$

Appendix D:

D.1 Corollary [Paz,83]:

Let A be the infinitesimal generator of an analytic semigroup T(t). If f is locally Hölder continuous map on (0, a], then for every $x \in X$ the initial value problem given by:

$$\frac{du(t)}{dt} = Au(t) + f(t)$$
$$u(0) = x$$

Has a unique mild solution given by:

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds$$

D.2 Theorem (Uniform convergence) [Erwin,78]:

Convergence $x_n \to x$ in the space C[a,b] is uniform convergence, that is (x_n) converges uniformly on [a,b] to x.

1.1 Introduction

The theory of one parameter semigroups of linear operators on Banach spaces started earlier, acquired its core in 1948 with the Hille-Yosida generation theorem, and attained its first apex with the 1957 edition of “semigroups and functional analysis” by E. Hille and R. S. Phillips. In the 1970's and 80's, the theory reached a certain state of perfection, which is well represented in the monographs by [Dav, 80], [Gol, 85], [Paz, 83] and others.

Today, the situation is characterized by manifold application of this theory not only to the traditional areas such as partial differential equations or stochastic processes. Semigroup has become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. The purpose of this chapter is to recall some definitions, basic concepts, propositions, theorems and some properties of the semigroup theory which are important for the discussion of our later results. This chapter consists of eight sections, in section one, we recall the elementary properties of the complex (matrix)−valued exponential function and introduce the definition of the semigroup on finite−dimensional space(matrix semigroup). In section two, we study the properties of the operator−valued exponential function and introduce the definition of the semigroup on infinite−dimensional space and give some typical examples of the semigroup and introduce the definition of the uniformly continuous semigroup with some examples of it. In section three,

We introduce the definition of the strongly continuous semigroup with some examples of it. In section four, we introduce the definition of the generator of the semigroup with some examples and study the properties of it and discuss the relation between the semigroup, its generator and the resolvent set. In section five, we study the fundamental theorem of the semigroup theory which is "Hille-Yosida Generation Theorems ". In section sex, we introduce the two important special classes of the semigroup which are "compact semigroup" and "analytic semigroup". In section seven, we study the fractional powers of certain unbounded linear operator and study some of their properties and introduce some remarks and theorem which display that the analytic semigroup play an important tool for define the fractional power of unbounded linear operator. In section eight, we study the solution of the homogenous abstract Cauchy problem and the inhomogeneous also we introduce two concepts of the solution to abstract Cauchy problem which are "classical solution" and "mild solution".

1.2 Finit-Dimensional "Matrix Semigroup"

In this section, we study the properties of the "scalar-valued exponential function" and pass to a more general case; we discuss the properties of the "Matrix-valued exponential function".

Consider the finite-dimensional vector space X = $\mathbb{C}$, where $\mathbb{C}$ is a set of complex numbers, in the following problem we will find all maps given by a "scalar-valued function",

T (.): $R^+ \longrightarrow \mathbb{C}$, which satisfy the conditions:

$$\left.\begin{aligned} &T(t+s) = T(t)T(s), \text{ for all } t,s \geq 0 \\ &T(0) = 1 \end{aligned}\right\} \tag{1.1}$$

1.2.1 Problem [Klaus,00]:

Find all maps T (.): $R^+ \longrightarrow \mathbb{C}$, satisfying equation (1.1).

Evidently, the exponential function $t \longrightarrow e^{ta}$ satisfies (1.1) for any $a \in \mathbb{C}$. We take a closer look at the exponential functions and study some its properties by giving the following propositions:

1.2.2 Proposition [Klaus,00]:

Let $T(t)=e^{ta}$, for some $a \in \mathbb{C}$ and all $t \geq 0$. Then the solution T(.) is differentiable and satisfies the initial value problem)

$$\left.\begin{aligned} &\frac{d}{dt}T(t) = aT(t), \text{ for } t \geq 0 \\ &T(0) = 1 \end{aligned}\right\} \tag{1.2}$$

1.2.3 Proposition [Klaus,00]:

Let T(.): $R^+ \longrightarrow \mathbb{C}$ be a continuous function satisfying equation (1.1). Then T(.) is differentiable and $\exists!\ a \in \mathbb{C}$, such that equation (1.2) holds.

1.2.4 Theorem [Klaus,00]:

Let T(.): $R^+ \longrightarrow \mathbb{C}$ be a continuous function satisfying equation(1.1) then $\exists!\ a \in \mathbb{C}$, such that $T(t) = e^{ta}$, $\forall\ t \geq 0$.

With this answer, we stop the discussion of this elementary situation and close this subsection with comment on problem (1.2.1).

1.2.5 Comment [Klaus,00] :

Once, as in the previous theorem, that a certain function T(.):$R^+ \longrightarrow \mathbb{C}$ is of the form $T(t) = e^{ta}$. It is clear that it can be extended to all $t \in R$, where R is the set of real numbers and even all $t \in \mathbb{C}$, still satisfying the equation (1.1), for all $t, s \in \mathbb{C}$.

Now, we pass to a more general case and consider finite-Dimensional vector spaces $X = \mathbb{C}^n$. The space $\ell(X)$ of all linear operators on X will then be identified with the space $M_n(\mathbb{C})$ of all complex $n \times n$ matrices, and in the following problem we will find all maps given by a “matrix valued function” $T(.) : R^+ \longrightarrow M_n(\mathbb{C})$ satisfying the conditions

$$\left.\begin{aligned} &T(t+s) = T(t)T(s),\ \forall\ t,s \geq 0 \\ &T(0) = I \end{aligned}\right\} \qquad (1.3)$$

Where I stands the identity matrix.

1.2.6 Problem [Klaus,00] :

Find all maps $T(.) : R^+ \longrightarrow M_n(\mathbb{C})$ satisfying the equation (1.3).

As before, the solutions of equation (1.1) are the scalar-valued "exponential functions". We see later that the solutions of equation (1.3) for this problem are the matrix-valued "exponential functions".

1.2.7 Definition [Paz, 83] :

For any $A \in M_n(\mathbb{C})$ and $t \in R$, the matrix – valued exponential function e^{tA} is defined by:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

1.2.8 Definition [Klaus,00] :

We call $\left\{e^{tA}\right\}_{t\geq 0}$ the (one-parameter) semigroup generated by the matrix $A \in M_n(\mathbb{C})$.

The family $\left\{e^{tA}\right\}_{t\geq 0}$ form (one – parameter) semigroup for any $A \in M_n(\mathbb{C})$ and the answer to problem (1.2.6) are given by the following proposition.

1.2.9 Proposition [Klaus,00]:

For any $A \in M_n(\mathbb{C})$, the map $R^+ \ni t \longrightarrow e^{tA} \in M_n(\mathbb{C})$, is continuous and satisfies:

$$e^{(t+s)A} = e^{tA}e^{sA}, \forall\, t,s \geq 0$$
$$e^{0A} = I$$

The properties of the matrix – valued exponential function $T(t)=e^{tA}$, given by the following proposition and theorem.

1.2.10 Proposition [Klaus,00]:

Let $T(t)=e^{tA}$,for some $A\in M_n(\mathbb{C})$.Then the function $T(.):R^{+} \longrightarrow M_n(\mathbb{C})$ is differentiable an satisfy the initial value problem:

$$\left.\begin{aligned} &\frac{d}{dt}T(t) = AT(t), \text{ for } t \geq 0 \\ &T(0) = I \end{aligned}\right\} \qquad (1.4)$$

Conversely, every differential function $T(.) : R^{+} \longrightarrow M_n(\mathbb{C})$ satisfy equation (1.4) is already of the form $T(t) = e^{tA}$, for some $A\in M_n(\mathbb{C})$, finally we observed that $A = \dot{T}(0)$.

1.2.11 Theorem [Klaus,00]:

Let $T(.): R^{+}\longrightarrow M_n(\mathbb{C})$ be a continuous function satisfying the equation (1.3). Then there exist $A\in M_n(\mathbb{C})$, such that $T(t) = e^{tA}$, $\forall\ t \geq 0$.

1.3 Uniformly Continuous Semigroup

In this section a more general case is discussed to a semigroup on infinite-dimensional spaces. From now on, we take X to be a real Banach space with norm ||.||. We denoted $\ell(X)$ the Banach algebra of all bounded linear operators on X.

1.3.1 Problem [Klaus,00]:

Find all maps $T(.) : R^{+} \longrightarrow \mathcal{L}(X)$ satisfying the conditions:

$$\left.\begin{array}{l} T(t+s) = T(t)T(s),\ \forall\ t,s \geq 0 \\ T(0) = I \end{array}\right\} \qquad (1.5)$$

Where I stands the identity operator.

In analogy to section (1.2), the solutions of equation (1.1) and (1.3) are scalar-valued " exponential functions " and the matrix – valued " exponential functions " respectively, the answer for this problem will be much more complex than as before. We see later that the solutions of equation (1.5) are the operator– valued "exponential functions ".

1.3.2 Definition [Klaus,00]:

A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a (one-parameter) semigroup on X if it satisfies the equation (1.5).

As in the matrix case (see definition 1.2.7), we can define an operator-valued "exponential function" by the following terminology:

1.3.3 Definition [Paz,83]:

For any $A \in \mathcal{L}(X)$ and $t \in R$, the operator-valued "exponential function" e^{tA} is defined by:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!},\ A \in \mathcal{L}(X)$$

The family $\left\{e^{tA}\right\}_{t\geq 0}$ form (one – parameter) semigroup for any $A\in \mathcal{L}(X)$ and the answer to problem (1.3.1) are given by the following proposition.

1.3.4 Proposition [Klaus,00]:

For any $A\in \mathcal{L}(X)$, the map $R^+ \ni t \longrightarrow e^{tA} \in \mathcal{L}(X)$ is continuous and satisfies:

$$e^{(t+s)A} = e^{tA}e^{sA},\ \forall\ t,s \geq 0$$
$$e^{0A} = I$$

The properties of the operator – valued exponential function T(t)= e^{tA} , given by the following proposition:

1.3.5 Proposition [Klaus,00]:

Let $T(t) = e^{tA}$, for some $A\in \mathcal{L}(X)$. Then the function $T(.) : R^+ \longrightarrow \mathcal{L}(X)$ is differentiable and satisfy the initial value problem:

$$\left.\begin{array}{l} \frac{d}{dt}T(t) = AT(t),\ \text{for}\ t \geq 0 \\ T(0) = I \end{array}\right\} \qquad (1.6)$$

Conversely, every differential function $T(.) : R^+ \longrightarrow \mathcal{L}(X)$ satisfy equation (1.6) is already of the form T(t)=e^{tA}, for some $A \in \mathcal{L}(X)$, finally we observed that $A = \dot{T}(0)$.

As seen before the ‘typical example’ of one-parameter semigroup of operators on a Banach space X is $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$, for any $A \in \ell(X)$, also we introduce another “ typical example” of one-parameter semigroup of operators on a Banach space X, before this we define some basic concepts for bounded linear operator A, we define

$\rho(A) = \{\lambda \in C \mid \lambda - A : D(A) \longrightarrow X \text{ is bijective}\}$, it is called resolvent set,

$\sigma(A) = \rho(A)^c$, i.e., $\sigma(A) = C \backslash \rho(A)$, it is called spectrum set, and,

$R(\lambda; A) = (\lambda - A)^{-1}$ at $\lambda \in \rho(A)$, it is called resolvent operator.

Consider now for each $t \geq 0$ the function $\lambda \mapsto e^{t\lambda}$, which is analytic for all λ belongs to□ . Therefore, one can define {see [DS, 58] or [TL, 80]} the exponential of A through the operator-valued version of Cauchy’s integral formula.

1.3.6 Definition [Klaus,00]:

Let $A \in \ell(X)$ and $t \in R$, the operator – valued “exponential function”

e^{tA} Is defining by:

$$e^{tA} = \frac{1}{2\pi i} \int_{+\partial U} e^{t\lambda} R(\lambda, A)\, d\lambda, \ \forall\ \lambda \geq 0 \qquad (1.7)$$

Where $+\partial U$ is a smooth positively oriented boundary.

The following proposition display that the family $\left\{e^{tA}\right\}_{t \geq 0}$ define by equation (1.7) form a (one – parameter) semigroup for any $A \in \ell(X)$.

1.3.7 Proposition [Klaus,00]:

For any $A \in \mathcal{L}(X)$, the map $R^+ \ni t \longrightarrow e^{tA} \in \mathcal{L}(X)$ is continuous and satisfies:

$$e^{(t+s)A} = e^{tA}e^{sA}, \ \forall\ t,s \geq 0$$
$$e^{0A} = I$$

Next, we introduce the following terminology:

1.3.8 Definition [Paz,83] :

A one-parameter semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X is called uniformly continuous or (norm continuous), if:

The map $R^+ \ni t \longrightarrow T(t) \in \mathcal{L}(X)$, satisfies the following conditions:

1. $T(t + s) = T(t)T(s),\ \forall\ t, s \in R^+$.
2. $T(0) = I$.
3. $\lim_{t\downarrow 0} \| T(t) - I \| = 0$.

To illustrate this definition, see the following examples:

1.3.9 Examples [Klaus,00]:

(i) Let $A \in \mathcal{L}(X)$, where X is a Banach space and set:

$T(t)=e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$, it is clearly that the family $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup.

Before discuss the next example, we introduce remark, definition and proposition which is useful here.

Remark:

Let $X = C_0(\Omega)$ (space of all continuous, complex-valued functions on compact set Ω that vanish at infinity)

This space is the Banach space (with sup-norm).

Definition:

The multiplication operator M_q induced on $C_0(\Omega)$ by some continuous function $q : \Omega \longrightarrow C$, is defined by:

$$M_q f = q.f, \text{ for all f in the domain}$$

Where $D(M_q) = \{f \in C_0(\Omega) : q.f \in C_0(\Omega)\}$

Proposition:

Let M_q with domain $D(M_q)$ be the multiplication operator induced on $C_0(\Omega)$ by some continuous function q. The operator M_q is bounded if and only if the function q is bounded.

(ii) Set $T_q(t) = e^{tM_q}$, it is easy to verify that $\{T_q(t)\}_{t\geq 0}$ is a uniformly continuous semigroup.

1.3.10 Remark [Klaus,00]:

The family $\{T_q(t)\}_{t\geq 0}$ is said to be "Multiplication semigroup".

Now, we introduce the fundamental theorem:

1.3.11 Theorem [Paz, 83]:

A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded.

1.4 Strongly Continuous Semigroup

In this section, we introduce definition and some examples of a strongly continuous semigroup and proposition which is give some properties of it.

1.4.1 Definition [Balakrishnan,78]:

A semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X is called strongly continuous semigroup of bounded linear operators if

The map $R^+ \ni t \longrightarrow T(t) \in \ell(X)$, satisfies the following conditions:

1. $T(t + s) = T(t)T(s), \forall\, t, s \in R^+$.
2. $T(0) = I$.
3. $\lim_{t \downarrow 0} \| T(t)x - x \| = 0$, for every $x \in X$.

1.4.2 Remark [Paz,83]:

A strongly continuous semigroup of bounded linear operators on X will be denoted by C_0 semigroup.

To illustrate this definition, see the following examples:

1.4.3 Examples :

(i) In example (1.3.9.i), $T(t) = e^{tA}$.

It is clear that the family $\left\{e^{tA}\right\}_{t\geq 0}$ is a strongly continuous semigroup generated by a bounded operator A.

(ii) In example (1.3.9.ii), $T_q(t) = e^{tM_q}$.

It is clear that the family $\{e^{tM_q}\}_{t\geq 0}$ is a strongly continuous semigroup generated by a bounded operator M_q.

(iii) Let X be the Banach space of continuous, bounded functions on $[0,\infty)$ with the sup-norm, and consider the translation operator:

$$(T(t)x)(z) = x(z + t), \quad x \in X, \quad z \geq 0 \quad \text{and} \quad t \geq 0$$

Clearly $T(t+s) = T(t)T(s)$, $\forall$ t, s ≥ 0 and $T(0) = I$ and we give

$$\|T(t)x - x\| = \sup_{z\geq 0} |x(z + t) - x(z)| \longrightarrow 0, \text{ as } t \longrightarrow 0^+, \forall\, x \in X.$$

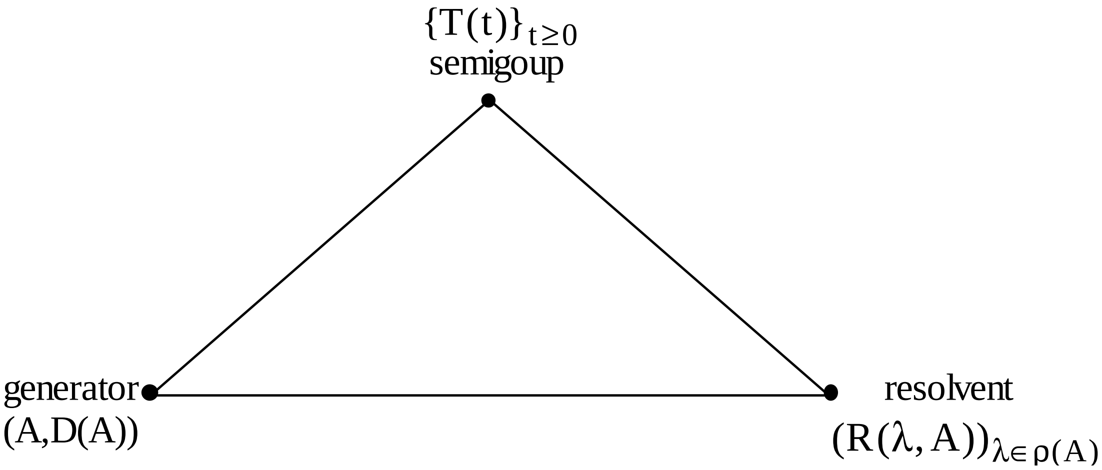
1.4.4 Proposition [Paz 83]:

For every strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ there exist $w \in R$ and $M \geq 1$, such that $\|T(t)\| \leq M e^{wt}$, $\forall$ $t \geq 0$.

1.5 Generators of Semigroups and their Resolvents

In this section, we introduce definition, examples and some properties of the generator A of the semigroup $\{T(t)\}_{t\geq 0}$. It will be a linear, but generally unbounded linear operator defined only on a dense subspace D(A) of the

Banach space X. In order to retrieve the semigroup $\{T(t)\}_{t\geq 0}$ from its generator (A, D(A)), we will need a third object, namely the resolvent operator of A, defined by $R(\lambda, A) = (\lambda - A)^{-1} \in \mathcal{L}(X)$, which is defined for all complex numbers in the resolvent set $\rho(A)$. To find and discuss the various relations between these objects, which can be illustrated by the following triangle:



1.5.1 Definition [Balakrishnan,78]:

The operator $A : D(A) \subset X \longrightarrow X$ of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X is define by $Ax = \lim\limits_{t\downarrow 0} \frac{T(t)x - x}{t}$, for every x in its domain D(A), where $D(A) = \{x \in X : \lim\limits_{t\downarrow 0} \frac{T(t)x - x}{t}$ exists$\}$, which is called "The infinitesimal generator" of a semigroup $\{T(t)\}_{t\geq 0}$.

To illustrate this definition, see the following examples:

1.5.2 Example:

(i) The infinitesimal generator of the example (1.4.3.i) is $A \in \mathcal{L}(X)$.

(ii) The infinitesimal generator of the example (1.4.3.ii) is M_q.

(iii) The infinitesimal generator of the example (1.4.3.iii) is A= $\frac{d}{dx}$, with

D(A)= $\{x : \frac{dx}{dz} \in X\}$.

The following proposition and theorem give some properties of the generator A and its domain.

1.5.3 Proposition [Klaus,00]:

For the generator (A, D(A)) of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, the following properties hold:

(i) A: $D(A) \subseteq X \longrightarrow X$ is a linear operator.

(ii) If $x \in D(A)$, then $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$, for all $t \geq 0$.

(iii) For every $t \geq 0$ and $x \in X$, one has $\int_0^T T(s)x\,ds \in D(A)$.

(iv) For every $t \geq 0$, one has:

$$T(t)x - x = A\int_{s=0}^{T} T(s)x\,ds, \qquad \text{if } x \in X$$

$$= \int_{s=0}^{T} T(s)Ax\,ds, \qquad \text{if } x \in D(A)$$

1.5.4 Theorem [Klaus,00]:

The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.

1.5.5 Remark [Klaus,00]:

The above proposition and theorem shows that the semigroup plays an important role of determining the solution of an abstract evolution equation $\dot{x}=Ax$, $x(0)=x_0$, where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$. In particular, we know $x(t) = T(t)x_0$ is the solution if A generates a strongly continuous semigroup and $x_0 \in D(A)$. so it is important to obtain a characterization of the operator which generate a strongly continuous semigroup.

1.5.6 Proposition [Klaus,00]:

For a strongly semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X with generator (A, D(A)), the following assertions are equivalent:

(a) The generator A is bounded.

(b) The domain of A {D(A)} is all of X.

(c) The domain of A {D(A)} is closed in X.

(d) The semigroup $\{T(t)\}_{t\geq 0}$ is uniformly continuous.

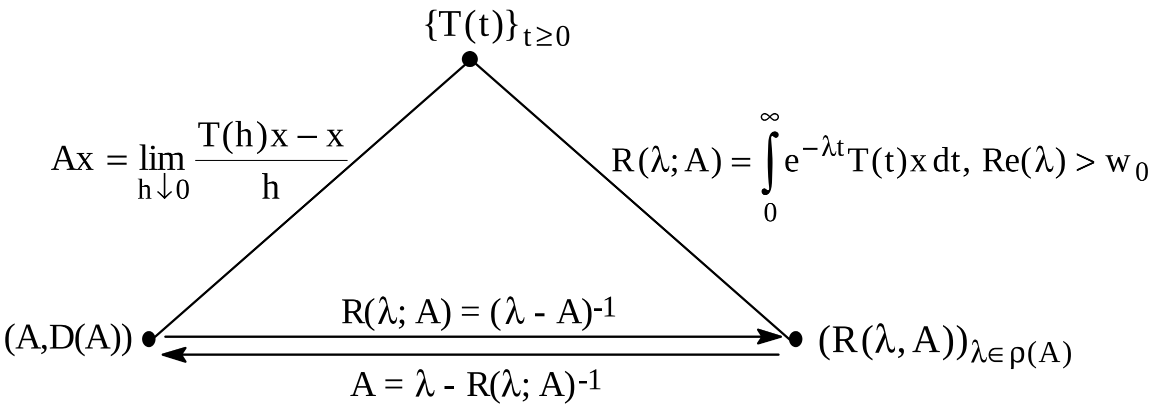
1.5.7 Remark [Balakrishnan,76]:

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X, then for every $\lambda \in C$, define a linear bounded operator:

$$R(\lambda;A)x = \int_{0}^{\infty} e^{-\lambda t} T(t)x\,dt \text{, for all } x \in X.$$

1.5.8 Diagram [Klaus,00]:

To close this section, we collect in a diagram the information obtained so far on the relations between a semigroup, its generator and its resolvent.



1.6 Hille-Yasida Generation Theorems

In this section, we discuss the fundamental theorem of the semigroup theory; this theorem shows essentially that the "exponential function" for unbounded linear operator is well defined. "Yosida's idea" was to approximate the unbounded linear operator A by a sequence $(A_n)_{n\in N}$ of bounded operators and show that:

$$e^{tA} = \lim_{n\to\infty} e^{tA_n}$$

Exists and is a strongly continuous semigroup.

1.6.1 Remark [Paz 83]:

A semigroup $\{T(t)\}_{t\geq 0}$ is said to be contractions if $\|T(t)\| \leq 1$.

We need some lemmas which is useful here.

1.6.2 Lemma [Paz 83]:

Let A be satisfying the following conditions:

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set ρ(A) of A contains R^+ and for every $\lambda > 0$,

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda} \tag{1.8}$$

and let $R(\lambda; A) = (\lambda I - A)^{-1}$. Then $\lim_{\lambda \to \infty} \lambda R(\lambda;A)x = x$, for $x \in X$

Proof:

Suppose first that $x \in D(A)$, then:

$\|\lambda R(\lambda; A)x - x\| = \|AR(\lambda; A)x\| = \|R(\lambda, A)|Ax\| \leq \frac{1}{\lambda}\|Ax\| \longrightarrow 0$, as $\lambda \longrightarrow \infty$

But D(A) is dense in X and $\|\lambda R(\lambda; A)\| \leq 1$. Therefore:

$\lambda R(\lambda; A)x \longrightarrow x$, as $\lambda \longrightarrow \infty$, for every $x \in X$.

Remark 1.6.3[Paz 83]:

We define for every $\lambda > 0$, the "Yosida approximation" of A by:

$$A_\lambda = \lambda AR(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I \tag{1.9}$$

Where A_λ is an approximation of A in the following sense:

Lemma 1.6.4 [Paz 83]:

Let A satisfy the above conditions (i) and (ii), if A_λ is the "Yosida approximation" of A, then:

$$\lim_{\lambda\to\infty} A_\lambda x = Ax, \text{ for } x \in D(A).$$

Proof:

For $x \in D(A)$, we have by the above lemma and the definition of A_λ that $\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda R(\lambda; A)Ax = Ax$.

Lemma 1.6.5 [Paz 83]:

Let A satisfy the above conditions (i) and (ii). If A_λ is the "Yosida approximation" of A, then A_λ is the infinitesimal generator of a uniformly continuous semigroup of continuous e^{tA_λ}, furthermore, for every $x \in X$, $\lambda, \mu > 0$, we have:

$$\| e^{tA_\lambda} x - e^{tA_\mu} x\| \le t\|A_\lambda x - A_\mu x\|$$

Now, we introduce an important theorem:

1.6.6 Theorem "Generation theorem (Hill-Yosida, 1948)" [Paz,83]:

A linear unbounded operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \ge 0$ if and only if:

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains R^+ and for every $\lambda > 0$,

$$\|R(\lambda; A)\| \le \frac{1}{\lambda}$$

Proof

(⇒)

If A is the infinitesimal generator of a C_0 semigroup of contraction, then it is closed and $\overline{D(A)} = X$ {by theorem (1.5.4)},

For $\lambda > 0$ and $x \in X$,

Since $R(\lambda;A)x = \int_{t=0}^{\infty} e^{-\lambda t} T(t)x \, dt$ {by Remark (1.5.7)}

Satisfying:

$$\|R(\lambda;A)x\| \le \int_{t=0}^{\infty} e^{-\lambda t} \| T(t)x \| \, dt \le \frac{1}{\lambda}\|x\|$$

Furthermore, for h > 0

$$\frac{T(h)-I}{h} R(\lambda;A)x = \frac{1}{h}\int_{0}^{\infty} e^{-\lambda t}\left[T(t+h)x - T(t)x\right] dt$$

$$= \frac{e^{\lambda h}-1}{h}\int_{h}^{\infty} e^{-\lambda t} T(t)x \, dt - \frac{1}{h}\int_{0}^{h} e^{-\lambda t} T(t)x \, dt \ldots\ldots\ldots \text{(1.10)}$$

As $h \downarrow 0$, the right hand side of equation (1.10) converges to $\lambda R(\lambda;A)x - x$

This implies that for every $x \in X$ and $\lambda > 0$, $R(\lambda;A)x \in D(A)$ and $AR(\lambda;A) = \lambda R(\lambda) - I$, or

$$(\lambda I - A)R(\lambda;A) = I \qquad \text{(1.11)}$$

For $x \in D(A)$, we have:

$$R(\lambda;A)Ax = \int_{0}^{\infty} e^{-\lambda t} T(t)Ax \, dt$$

$$= \int_0^\infty e^{-\lambda t} AT(t)x \, dt \quad \{\text{By proposition (1.6.3.ii)}\}$$

$$= A\left(\int_0^\infty e^{-\lambda t} T(t)x \, dt\right) = AR(\lambda;A)x$$

This implies that:

$$R(\lambda;A)Ax = AR(\lambda;A)x \qquad (1.12)$$

From (1.11) and (1.12), it follows that:

$R(\lambda;A)(\lambda I - A)x = x$, for $x \in D(A)$

Thus, $R(\lambda;A)$ is the inverse of $(\lambda I - A)$, it is exists for all $\lambda > 0$ and satisfies equation (1.8) therefore the necessary condition is satisfied.

$(\Leftarrow)$

If A satisfy the conditions (i) and (ii), to prove A is the infinitesimal generator of a C_0 semigroup of contraction.

We anticipate that the following properties hold.

(i) $e^{tA}x = \lim_{\lambda\to\infty} e^{tA_\lambda} x$ exist for each $x \in X$.

(ii) $\{\lim_{\lambda\to\infty} e^{tA_\lambda} x\}$ is a strongly continuous semigroup of contraction on X.

(iii) This semigroup has generator (A, D(A)).

By establishing these statements, we will complete the proof.

(i) let $x \in D(A)$, then :

$$\| e^{tA_\lambda} x - e^{tA_\mu} x \|_X \le t \| A_\lambda x - A_\mu x \|_X \text{ (by lemma (1.6.5))} \qquad (1.13)$$

But $\|A_\lambda x - A_\mu x\| \longrightarrow 0$ as $\lambda,\ \mu \longrightarrow \infty$ (1.14)

Since $A_\lambda x \longrightarrow Ax$, as $\lambda \longrightarrow \infty$ {by lemma (1.6.4)}

From equation (1.13) and equation (1.14), it follows that for $x \in D(A)$,

e^{tA_λ} Converges as $\lambda \longrightarrow \infty$. And the converges is uniform on bounded interval. Since D(A) is dense in X and $\|e^{tA_\lambda}\| \leq 1$, it follows that :

$$e^{tA}x = \lim_{\lambda \to \infty} e^{tA_\lambda} x,\ \forall\ x \in X \tag{1.15}$$

(ii) from equation (1.15), it follows readily that $\{T(t)\}_{t\geq 0}$ satisfies the semigroup properties :

(a) T(0)=1

(b) $\lim\limits_{\lambda \to \infty} e^{(t_1+t_2)A_\lambda} = \lim\limits_{\lambda \to \infty} e^{t_1 A}\ e^{t_2 A} = \lim\limits_{\lambda \to \infty} e^{t_1 A}\ \lim\limits_{\lambda \to \infty} e^{t_2 A}$

(c) $\|\lim\limits_{\lambda \to \infty} e^{tA_\lambda}\| \leq \lim\limits_{\lambda \to \infty} \|e^{tA_\lambda}\| \leq \lim\limits_{\lambda \to \infty} 1 = 1$

$\Rightarrow \|\lim\limits_{\lambda \to \infty} e^{tA_\lambda}\| \leq 1,\ \forall\, t \geq 0.$

(d) From the uniform convergence, it follows that $\{T(t)\}_{t\geq 0}$ is strongly continuous.

To conclude the proof we will show that A is the infinitesimal generator of $T(t)= \lim\limits_{\lambda \to \infty} e^{tA_\lambda}$

Let A' denoted the generator of the semigroup $\{\lim\limits_{\lambda \to \infty} e^{tA_\lambda}\}_{t\geq 0}$, for $\mu > 0$,

$$R(\mu; A')x = \int_{t=0}^{\infty} e^{-\mu t} T(t)x\,dt = \int_{t=0}^{\infty} e^{-\mu t} \lim_{\lambda \to \infty} e^{tA_\lambda} x\,dt = \lim_{\lambda \to \infty} \int_{t=0}^{\infty} e^{-\mu t} e^{tA_\lambda} x\,dt$$

$$= \lim_{\lambda \to \infty} R(\mu, A_\lambda)x$$

$\Rightarrow$

$$R(\mu; A')x = \lim_{\lambda\to\infty} R(\mu, A_\lambda)x$$

Next, $\mu R(\mu, A_\lambda)x - R(\mu, A_\lambda)A_\lambda x = x$ (1.16)

And hence for $x \in D(A)$, take limits as λ goes to infinity to the (1.16), we get:

$$\mu \lim_{\lambda\to\infty} R(\mu, A_\lambda)x - \lim_{\lambda\to\infty} R(\mu, A_\lambda)A_\lambda x = x$$

$\Rightarrow$

$$\mu R(\mu, A')x - R(\mu, A')Ax = x$$

or $R(\mu, A')(\mu x - Ax) = x$.

Let $y \in X$, then $x = R(\mu, A)y$ is in the domain of A, and hence

$$R(\mu, A')\left[\mu R(\mu, A)y - AR(\mu, A)y\right] = R(\mu, A)y$$

$\Rightarrow$

$$R(\mu, A')\left[\left((\mu - A)R(\mu, A)\right)y\right] = R(\mu, A)y$$

$\Rightarrow$

$R(\mu, A')y = R(\mu, A)y$, $y \in X$.

Hence, $D(A) = D(A')$, and from

$$\mu R(\mu, A)x - AR(\mu, A)x = x \qquad (1.17)$$

$$\mu R(\mu, A')x - A'R(\mu, A')x = x \qquad (1.18)$$

From (1.17) and (1.18),

It follows that $(A - A')R(\mu - A')x = 0$, and hence

$Ax = A'x$ for $x \in D(A) = D(A')$.

1.7 Special Classes of Semigroups

This section consists of two parts, first part is "compact semigroup" and the second part is "Analytic semigroup" which is considering the important two special classes of semigroup theory.

1.7.1 Compact Semigroups:

In what follows, some basic definitions, remarks and a theorem will be presented which are necessary to understand the work in chapter two.

1.7.1.1 Definition [Balakrishnan,76]:

A semigroup $\{T(t)\}_{t\geq 0}$ is said to be compact if T(t) is a compact operator for each $t > 0$.

1.7.1.2 Remark [Paz,83]:

The identity operator T(0) is not compact operator on infinite dimensional space.

1.7.1.3 Remark [Paz,83]:

A semigroup $\{T(t)\}_{t\geq 0}$ is called a continuous in the uniform operator topology if:

(1) $\|T(t + \Delta)x - T(t)x\| \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall\ x \in X$.

(2) $\|T(t)x - T(t - \Delta)x\| \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall\ x \in X$.

1.7.1.4 Theorem [Paz, 83]:

Let $T(t)$ be a C_0 semigroup. If $T(t)$ is compact for $t > t_0$, then $T(t)$ is a continuous in the uniform operator topology for $t > t_0$.

1.7.2 Analytic Semigroup:

In what follows, some basic definitions, remarks and theorems will be presented which are necessary to understand the work in chapter three. As before we dealt with semigroup whose domain was the nonnegative real axis. We will now consider the possibility of extending the domain of the parameter to regions in the complex plane that include the nonnegative real axis. However, we will restrict ourselves to very special domains, namely, angles around the positive real axis.

We introduce a general definition of the analytic semigroup:

1.7.2.1 Definition [Paz,83]:

Let $\Delta_\phi = \{z\in \mathbb{C} : |\arg z| < \phi, \phi>0 \}$ and for $z \in \Delta_\phi$, let $T(z)$ be a bounded linear operator. The family $\{T(z)\}_{z\in\Delta}$ is said to be an analytic semigroup in Δ_ϕ if the following conditions are satisfied :

(i) $z \longrightarrow T(z)$ is analytic in Δ_ϕ.

(ii) $T(0) = I$ and $\lim\limits_{\substack{z\to 0 \\ z\in\Delta_\phi}} T(z)x = x, \forall\, x \in X$.

(iii) $T(z_1 + z_2) = T(z_1)T(z_2), \forall\, z_1, z_2 \in \Delta_\phi$.

1.7.2.2 Definition [Paz,83]:

A family $\{T(t)\}_{t\geq 0}$ is said to be an analytic semigroup if the following conditions are satisfy :

(i) $t \longrightarrow T(t)$ is analytic in some sector Δ, where Δ is a sector containing the nonnegative real axis.

(ii) $T(0) = I$ and $\lim_{t\downarrow 0} \| T(t)x - x \| = 0$, for every $x \in X$

(iii) $T(t_1 + t_2) = T(t_1)T(t_2)$, $\forall\, t_1, t_2 \in R^+$.

1.7.2.3 Remark:

It is clear that the analytic semigroup $\{T(t)\}_{t\geq 0}$ is a C_0 semigroup.

1.8 Fractional Power of Closed Operators

In this section, we define fractional powers of certain unbounded linear operators and study some of their properties. We concentrate mainly on fractional powers of operators A for which $-A$ is the infinitesimal generator of an analytic semigroup. The results of this section will be used in the study the existence of solution of semilinear initial value problem.

The following theorem which is useful here:

Theorem 1.8.1 [Paz,83]:

Let T(t) be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of T(t) and assume $0 \in \rho(A)$. The following statements are equivalent:

(a) T(t) can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \in C: |\arg z| < \delta\}$.

(b) There exist a constant μ, such that for every $\sigma > 0$, $\tau \neq 0$,

$$\|R(\sigma + i\tau : A)\| \leq \frac{\mu}{|\tau|}$$

(c) There exist $0 < \delta < \pi/2$ and $M > 0$, such that:

$$\rho(A) \supset \sum_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$$

and $$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|}, \text{ for } \lambda \in \Sigma, \lambda \neq 0.$$

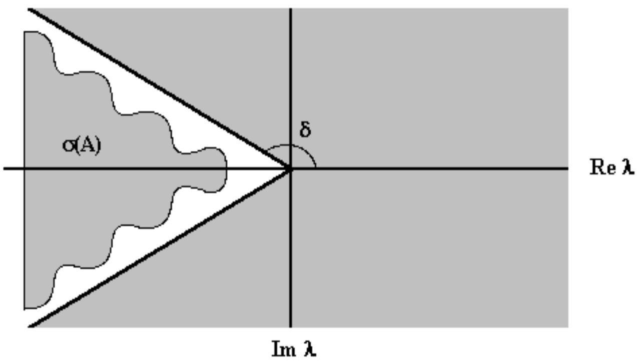
(d) T(t) is differentiable for $t > 0$ and there is a constant C such that:

$$\|AT(t)\| \leq \frac{\mu}{t}, \text{ for } t > 0.$$

Remark(1.8.2)

In theorem (1.8.1.c), the set:

$\rho(A) \supset \sum_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$, can be display by the following figure:



We introduce the following definition:

Definition 1.8.3 [Paz,83]:

For a closed densely defined linear operator A satisfy the following conditions:

(i) $\rho(A) \supset \Sigma^{+} = \{\lambda \in \mathbb{C} : 0 < w < |\arg\lambda| \leq \pi\} \cup \{0\}$

(ii) $\|R(\lambda; A)\| \leq \frac{M}{1+|\lambda|}$, for $\lambda \in \Sigma^{+}$, $M>0$

Define a bounded linear operator $A^{-\alpha} : X \rightarrow X_{\alpha}$,

$$A^{-\alpha} = \frac{1}{2\pi i}\int_{\Gamma'} \lambda^{-\alpha}(A-\lambda I)^{-1}\, d\lambda, \text{ for } \alpha \geq 0 \qquad (1.19)$$

where the path Γ' is a smooth curve in Σ^{+}.

Where X is a Banach space and X_{α} is a Banach space being dense in X, define as follow:

$$X_{\alpha} = \left\{ x \in X : \lim_{t\downarrow 0} \left\| t^{1-\alpha} AT(t)x \right\| = 0 \right\}, \text{ for } 0 \leq \alpha \leq 1$$

With norm which is equivalent to the graph norm of $A^{-\alpha}$

$\|x\|_{\alpha} = \|A^{\alpha}x\|_{X}$. Where A^{α} is the inverse of $A^{-\alpha}$, which can be display the construction of the operator A^{α} by the following definition and theorem.

Lemma(1.8.4)[Klaus,00]:

$A^{-\alpha}$ define by (1.18) is injective.

Definition(1.8.5 [Paz,83]:

Let $\alpha > 0$.the operator A^{α} define as the inverse of $A^{-\alpha}$ with domain $D(A^{\alpha}) = R(A^{-\alpha})$ is called the α-power of A, which denoted X_{α} the domain of A^{α}, i.e. $X_{\alpha} = D(A^{\alpha})$.

Theorem1.8.6 [Paz,83]:

Let A^{α} be defined as the inverse of $A^{-\alpha}$.Then:

(a) A^{α} is closed linear operator with domain $D(A^{\alpha}) = R(A^{-\alpha})$ = the range $A^{-\alpha}$.

(b) $\overline{D(A^{\alpha})} = X$, for every $\alpha \geq 0$.

(c) If α, β are real, then $A^{\alpha+\beta}x = A^{\alpha}A^{\beta}x$, for every $x \in D(A^{\gamma})$, where $\gamma = \max(\alpha, \beta, \alpha + \beta)$.

Theorem1.8.7 [Paz,83]:

Let $-A$ be the infinitesimal generator of an analytic semigroup T(t) if: $0 \in \rho(A)$, then:

(a) $T(x): X \longrightarrow D(A^{\alpha})$, for every $t > 0$ and $\alpha \geq 0$.

(b) For every $x \in D(A^{\alpha})$, we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.

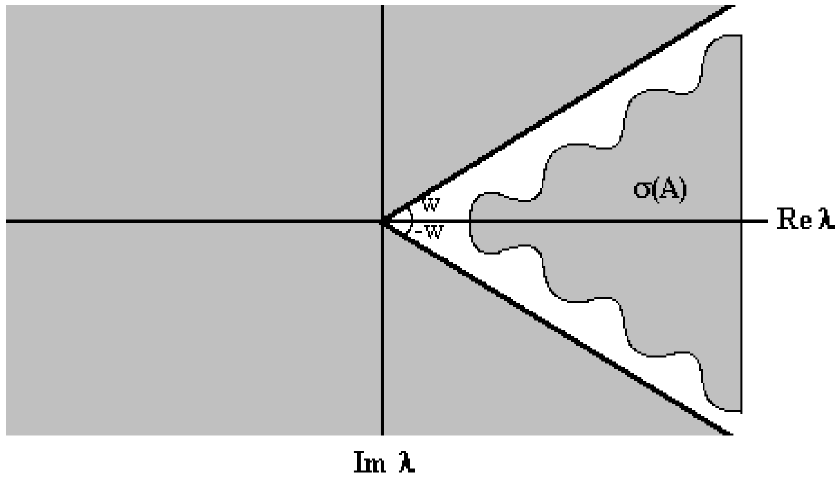
(c) For every $t \geq 0$, the operator $A^{\alpha}T(t)$ is bounded and $\|A^{\alpha}T(t)\| \leq M_{\alpha}t^{-\alpha}$.

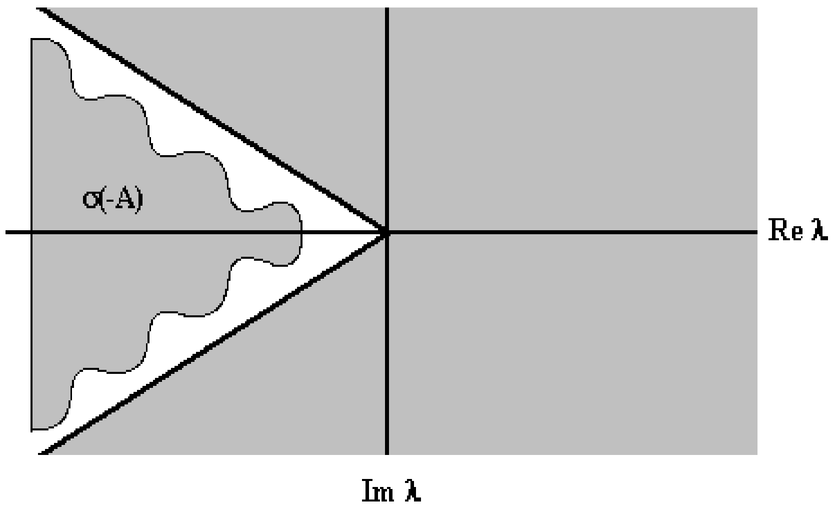
(d) Let $0<\alpha\leq 1$ and $x \in D(A^{\alpha})$ then $\|T(t)x - x\| \leq C_{\alpha}t^{\alpha}\|A^{\alpha}x\|_X$

where C_{α} is the positive constant depend onα.

Remark 1.8.8 [Paz,83]:

From the definition (1.8.3), if w <π/2, then −A is the infinitesimal generator of an analytic semigroup {see theorem (1.8.1)}, which can be display by the following figures:





Conclusion 1.8.9:

If −A is the infinitesimal generator of bounded analytic semigroup

Then the fractional power $A^{-\alpha}$ exist for $\alpha>0$.

1.9 The Abstract Cauchy Problem

In this section, our aim is to solve the homogeneous initial value problem (i.e., Banach-space-valued) linear initial value problem of the form:

$$\left.\begin{array}{l} \dot{u}(t) = Au(t), \text{ for } t \geq 0 \\ u(0) = u_0 \end{array}\right\} \qquad (1.20)$$

Where the independent variable t represents time, u(.) is a function with values in a Banach space X, $A:D(A)\subset X \longrightarrow X$ is a linear operator and $u_0 \in X$ the initial value.

We start by introducing the following terminology:

Definition 1.9.1 [Klaus,00]:

(i) The initial value problem given by (1.20):

Is called the abstract Cauchy problem denoted by (ACP) associated to (A,D(A)) and the initial value u_0,

(ii) A function $u: R^+ \longrightarrow X$ is called a (classical) solution of (ACP) on [0,a),where a is a fixed number if u is continuous on [0,a), continuously differentiable on (0,a), $u(t) \in D(A)$ for $0 < t < a$ and (1.20) is satisfied on [0,a).

Remark 1.9.2 [Klaus,00]:

If the operator A is the generator of a strongly continuous semigroup. It follows from proposition (1.5.3.ii) that the semigroup yields solutions of the associated abstract Cauchy problem given by (1.20).

Remark 1.9.3 [Paz,83]:

If A is the infinitesimal generator of a C_0 semigroup, which is not differentiable and does not satisfy (1.20), then in general, if $x \notin D(A)$, the initial value problem (ACP) does not have a solution. The function $t \longrightarrow T(t)x$ is then a "generalized solution" of the initial value problem (ACP) which call it a "mild solution".

Next, we consider the inhomogeneous initial value problem:

$$\left.\begin{aligned} &\dot{u}(t) = Au(t) + f(t), \text{ for } t \geq 0 \\ &u(0) = x \end{aligned}\right\} \qquad (1.21)$$

Where the independent variable t represents the time, u(.) is a function with values in Banach space X, $A : D(A) \subset X \longrightarrow X$, a linear operator and a function $f : R^{+} \longrightarrow X$ and $u_0 \in X$ the initial value.

Next, we introducing the necessary terminology:

Definition 1.9.4 [Paz,83]:

(i) The inhomogeneous initial values problem given by (1.21) is called inhomogeneous abstract Cauchy problem denoted by (ICAP).

(ii) A function $u: R^{+} \longrightarrow X$ is called a (classical) solution of (IACP) on [0,a),where a is a fixed number if u is continuous on [0,a), continuously differentiable on (0,a), $u(t) \in D(A)$ for $0< t <a$ and (1.21) is satisfied on [0,a).

Remark 1.9.5 [Paz,83]:

Let T(t) be the C_0 semigroup generated by A and let u be a solution of the (IACP). Then the X valued function g(s) = T(t –s)u(s) is differentiable for 0 < s < t and:

$$\frac{dg}{ds} = -AT(t-s)u(s) + T(t-s)u'(s)$$

$$= -AT(t-s)u(s) + T(t-s)Au(s) + T(t-s)f(s)$$

$$= T(t-s)f(s) \qquad (1.22)$$

If f ∈ L^1(0,a):X), then T(t – s)f(s) is integrable and integrating equation(1.22) from 0 to t yields, then:

$$u(t) = T(t)x + \int_{s=0}^{t} T(t-s)f(s)\,ds$$

We introduce the following definition of a mild solution.

Definition 1.9.6 [Paz,83]:

Let (A,D(A)) be the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X and take x∈ X and f∈ $L^1(R^+,X)$. Then the function u∈ C([0,a]:X) given by:

$$u(t)= T(t)x + \int_{s=0}^{t} T(t-s)f(s)\,ds, \quad 0\leq t\leq a,$$

is the mild solution of the (ICAP) on [0, a].

Remark 1.9.7 [Paz,83]:

It is not difficult to show that every classical solution of (IACP) is also a mild solution.

3.1 Introduction

In this chapter, by using the theory of semigroup and “Banach contraction principle”, the local existence, uniqueness of the mild solution to the semilinear initial value control problem has been developed in an arbitrary Banach space X, associated with the unbounded linear operator generating strongly continuous semigroup $\{S(t)\}_{t\geq 0}$.

3.2 Local Existence and Uniqueness of the Mild Solution to the Semilinear Initial Value Control Problem

Consider the following semilinear initial value control problem:

$$\left.\begin{aligned} &\frac{du}{dt} + Au(t) = f(t,u(t)) + \int_{s=0}^{t} h(t-s)\,g(s,u(s))\,ds + (Bw)(t),\ t > 0 \\ &u(0) = u_0 \end{aligned}\right\} \qquad (3.1)$$

The mild solution of equation (3.1) defines as follows:

Definition (3.2.1)

A continuous function u_w is said to be a mild solution to the semilinear initial value problem (3.1) given by:

$$u_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\Big[(Bw)(s) + f(s,u_w(s)) + \int_{s=0}^{t} h(s-\tau)\,g(s,u_w(\tau))\,d\tau\Big]ds \quad \forall\, w \in L^p([0, r):O) \qquad (3.2)$$

The local existence and uniqueness of a mild solution to the semilinear initial value control problem have been developed by assuming the following assumptions:

(a) $-A$ be the infinitesimal generator of bounded analytic semigroup $\{S(t)\}_{t\geq 0}$ and $0 \in \rho(-A)$.where the operator $-A$ define from $D(-A)\subset X$ into X, (X is a Banach space).

(b) Let U be an open subset of $[0, r)\times X_{\alpha}$, for $0 < r \leq \infty$.Where X_{α} is a Banach space being dense in X.

(c) For every $(t, x)\in U$, there exists a neighborhood $G \subset U$ of (t, x),the nonlinear maps f, $g:[0,r)\times X_{\alpha} \longrightarrow X$ satisfy the locally Lipchitize condition with respect to second argument,

$$\|f(t, u) - f(s, v)\|_{x} \leq L_0\|v - u\|_{\alpha}$$

$$\|g(t, u) - g(s, v)\|_{x} \leq L_1\|v - u\|_{\alpha}, \text{ for all } (t, u) \text{ and } (s, v) \in G.$$

(d) For $t''>0$, $\|f(t,v)\|_{x}\leq B_1$, $\|g(t,v)\|_{x}\leq B_2$, for $0\leq t\leq t''$ and for every $v\in X_{\alpha}$.

(e) For $t'''> 0$, $\|S(t) - I\|\,\|A^{\alpha}u_0\| \leq \delta'$, Where $\delta' < \delta$, $0 \leq t \leq t'''$.

(f) h is continuous function which at least $h\in L^1([0,r):R)$,Where R is the real number.

(g) $w(.)$ be the arbitrary control function is given in $L^p([0,r):O)$, a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X and $\|w(t)\|_{O}\leq k_1$, for $0 \leq t < r$.

(h) Let $t_1>0$ such that $t_1=\min\{t', t'', t''', r\}$,satisfy the condition

(h.i) $t_1 \leq \left\{[K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_{\alpha}^{-1}(1-\alpha)(\delta - \delta')\right\}^{\frac{1}{1-\alpha}}$

$$\Rightarrow \quad t_1^{1-\alpha} \leq [K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_{\alpha}^{-1}(1-\alpha)(\delta - \delta')$$

(i) there exist $C_2\geq 0$ and $0<\vartheta\leq 1$ such that:

$$|h(t) - h(s)| \leq C_2 |t - s|^{\vartheta}, \text{ for all } t, s \in [0, t_1].$$

(g) There exist $R_0 \geq 0$ and $0 < \xi \leq 1$ such that:

$$\|w(t) - w(s)\|_O \leq R_0 |t - s|^{\xi}, \text{ for all } t, s \in [0, t_1].$$

Remark (3.2.2):

The same problem (3.1) is being taken (in chapter2), but the conditions on the semigroup and the functions, operators are changed and hence, this problem shows it. The comparison between the conditions of the same problem of chapter 2 and chapter 3, present as follow:

Chapter two	Chapter three
1. A is the infinitesimal generator of C_0 compact semigroup $\{T(t)\}_{t\geq 0}$.	1. $-A$ is the infinitesimal generator of bounded analytic semigroup $\{S(t)\}_{t\geq 0}$.
2. The operator A is defining from $D(A)\subseteq X$ into X. Where X is a Banach Space.	2. The operator $-A$ is defining from $D(A)\subseteq X$ into X, $0\in \rho(-A)$. Where X is a Banach space.
3. U is an open subset of X.	3. U be an open subset of $[0, r)\times X_\alpha$, for $0 < r \leq \infty$. Where X_α is a Banach space being dense in X.
4.the nonlinear maps $f,g:[0,r)\times U \longrightarrow X$ Satisfy the locally Lipchitize condition $\|f(t, v_1) - f(t, v_2)\|_X \leq L_0\|v_1 - v_2\|$ $\|g(t, v_1) - g(t, v_2)\|_X \leq L_1\|v_1 - v_2\|$	4.The nonlinear maps $f,g:[0,r)\times X_\alpha \longrightarrow X$ satisfy the locally Lipchitize condition $\|f(t, u) - f(s, v)\|_x \leq L_0\|v - u\|_\alpha$ $\|g(t, u) - g(s, v)\|_x \leq L_1\|v - u\|_\alpha$
5. For $t''>0$, $\|T(t)u_0 - u_0\|_X \leq \rho'$, Where $\rho'<\rho$, $0 \leq t \leq t''$.	5. For $t''' > 0$, $\|S(t) - I\| \, \|A^\alpha u_0\| \leq \delta'$, where $\delta' < \delta$, $0 \leq t \leq t'''$, A^α is the fractional power Of A.

Theorem (3.2.3):

Assume that hypotheses (a)-(g) are hold, then for every $u_0 \in X_\alpha$, there exists a fixed number t_1, $0 < t_1 < r$, such that the initial value control problem has a unique local mild solution $u_w \in C((0,t_1]:X)$, for every control function $w(.) \in L^p([0, r):O)$.

Proof:

Without loss of generality, we may suppose $r<\infty$, because we are concerned here with the local existence only.

For a fixed point $(0,u_0)$ in the open subset U of $[0,r)\times X_\alpha$, we choose $\delta > 0$ such that the neighborhood G of the point $(0,u_0)$ define as follow:

$G = \{(t, x) \in U : 0 \le t \le t', \|x - u_0\|_\alpha \le \delta\} \subset U$ {since U is an open subset of $[0,r)\times X_\alpha$}.

$\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$, for $t > 0$ { by theorem (1.8.7.c }

Where C_α is a positive constant depending on α.

And assume $h_r = \int_0^r |h(s)|ds$

Set $Y = C([0,t_1]:X)$, then Y is a Banach space with the supremum norm:

$$\|y\|_Y = \sup_{0\le t\le t_1} \|y(t)\|_x$$

Let S_w be the nonempty subset of Y, define as follow:

$$S_w = \{u_w \in Y : u_w(0) = A^\alpha u_0, \|u_w(t) - A^\alpha u_0\|_x \le \delta, 0 \le t \le t_1\} \qquad (3.3)$$

To prove the closedness of S_w as a subset of Y. Let $u_w^n \in S_w$, such that

$u_w^n \xrightarrow{P.C.} u_w$ as $n \longrightarrow \infty$, we must prove that $u_w \in S_w$, where (P.C) stands for point wise convergence.

Since $u_w^n \in S_w \Rightarrow u_w^n \in Y, u_w^n(0) = A^{\alpha}u_0$ and $\| u_w^n(t) - A^{\alpha}u_0\|_x \le \delta, 0 \le t \le t_1$.

Since $u_w^n \xrightarrow{U.C.} u_w$ {see appendix D}, hence $u_w \in Y$. where (U.C) stands for the uniform convergence, and also

Since $u_w^n \xrightarrow{U.C.} u_w \Rightarrow \| u_w^n - u_w\|_Y \longrightarrow 0$, as $n \longrightarrow \infty$

Since $\| u_w^n - u_w\|_Y = \sup\limits_{0 \le t \le t_1} \| u_w^n(t) - u_w(t)\|_X \longrightarrow 0$, as $n \longrightarrow \infty$,

{By $\|y\|_Y = \sup\limits_{0 \le t \le t_1} \|y(t)\|_x$}.

Which implies that $\| u_w^n(t) - u_w(t)\|_x \longrightarrow 0$, as $n \longrightarrow \infty$, for every $0 \le t \le t_1$, i.e.,

$$\lim_{n \to \infty} u_w^n(t) = u_w(t),\ \forall\ 0 \le t \le t_1 \tag{3.6}$$

$\Rightarrow \lim\limits_{n \to \infty} u_w^n(0) = u_w(0)$ {by (3.6)}

$\Rightarrow \lim\limits_{n \to \infty} A^{\alpha}u_0 = u_w(0)$ {since $u_w^n \in S$}

$\Rightarrow A^{\alpha}u_0 = u_w(0)$

Notice that:

$$\|u_w(t) - A^{\alpha} u_0\|_x = \| \lim_{n \to \infty} u_w^n(t) - A^{\alpha} u_0\|_x \quad \text{\{by (3.6)\}}$$

$$= \| \lim_{n \to \infty} u_w^n(t) - \lim_{n \to \infty} A^{\alpha}u_0\|_x$$

$$= \| \lim_{n \to \infty} (u_w^n(t) - A^{\alpha} u_0)\|_x = \lim_{n \to \infty} \| u_w^n(t) - A^{\alpha} u_0\|_x$$

$\Rightarrow \|u_w(t) - A^{\alpha} u_0\|_x \leq \lim\limits_{n\to\infty} \delta$ {since $u_w^n \in S_w$}

$\Rightarrow \|u_w(t) - A^{\alpha} u_0\|_x \leq \delta$, for $0 \leq t \leq t_1$.

We have got S_w is closed subset of Y.

Now, define a map $F_w : S_w \longrightarrow Y$, given by:

$$(F_w u_w)(t) = S(t)A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\Big[f(s, A^{-\alpha}u_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}u_w(\tau))\,d\tau\Big]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds \quad (3.7)$$

To show that $F_w(S_w) \subseteq S_w$, let u_w be arbitrary element in S_w and let

$F_w u_w \in F_w(S_w)$, to prove $F_w u_w \in S_w$ for arbitrary element u_w in S_w.

From (3.3), notice that $F_w u_w \in Y$ {by the definition of the map F_w}

And $(F_w u_w)(0) = A^{\alpha}u_0$ {by (3.7)}, notice that:

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x = \Big\|S(t)A^{\alpha}u_0 - A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\Big[f(s, A^{-\alpha}u_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}u_w(\tau))\,d\tau\Big]ds\Big\|_x$$

$\Rightarrow$

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x = \Big\| S(t)A^{\alpha}u_0 - A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}u_w(s)) + \int_{s=0}^{t} h(s-\tau)\,g(\tau, A^{-\alpha}u_w(\tau))\,d\tau \right] ds + \int_{s=0}^{t} A^{\alpha}S(t-s)f(s,u_0)\,ds - \int_{s=0}^{t} A^{\alpha}S(t-s)f(s,u_0)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)\,g(\tau,u_0)\,d\tau \right) ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)\,g(\tau,u_0)\,d\tau \right) ds \Big\|_x$$

$\Rightarrow$

$$\|(F_w u_w)(t) - A^{\alpha}u_0\| = \Big\| S(t)A^{\alpha}u_0 - A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}u_w(s)) - f(s,u_0) \right] ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)\left[g(\tau, A^{-\alpha}u_w(\tau)) - g(\tau,u_0) \right] d\tau \right) ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(f(s,u_0) + \int_{\tau=0}^{s} h(s-\tau)\,g(\tau,u_0)\,d\tau \right) ds \Big\|_x$$

$\Rightarrow$

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \le \|S(t)A^{\alpha}u_0 - A^{\alpha}u_0\|_X + \int_{s=0}^{t} \| A^{\alpha}S(t-s)\| \, \|(Bw)(s)\| ds \ +$$

$$\int_{s=0}^{t} \| A^{\alpha}S(t-s)\| \, \| f(s, A^{-\alpha}u_w(s)) - f(s,u_0)\| ds \ +$$

$$\int_{s=0}^{t} \| A^{\alpha}S(t-s)\| \left(\int_{s=0}^{t} |h(s-\tau)| \, \| g(\tau, A^{-\alpha}u_w(\tau)) - g(\tau,u_0)\| d\tau \right) ds \ +$$

$$\int_{s=0}^{t} \| A^{\alpha}S(t-s)\| \left(\| f(s,u_0)\| + \int_{\tau=0}^{s} |h(s-\tau)| \, \| g(\tau,u_0)\| d\tau \right) ds$$

$\Rightarrow$

By the conditions c, d, e, we get:

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \le \delta' + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} K_0 K_1 \, ds \ +$$

$$\int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} L_0 \| A^{-\alpha}u_w(s) - u_0\|_{\alpha} \, ds \ +$$

$$\int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} h_{t_1} L_0 \| A^{-\alpha}u_w(s) - u_0\|_{\alpha} \, ds \ +$$

$$\int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} [B_1 + h_{t_1} B_2] \, ds$$

$\Rightarrow$

By using $\|x\|_{\alpha} = \|A^{\alpha}x\|_X$, we get:

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \le \delta' + C_{\alpha}K_0K_1(1-\alpha)^{-1} t_1^{1-\alpha} + \delta C_{\alpha}L_0(1-\alpha)^{-1} t_1^{1-\alpha} + \delta C_{\alpha} L_1 h_{t_1} (1-\alpha)^{-1} t_1^{1-\alpha} + (B_1 + h_{t_1} B_2)C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \le \delta' + [K_0K_1 + \delta L_0 + \delta L_1 h_{t_1} + (B_1 + h_{t_1} B_2)]C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \leq \delta' + [K_0K_1 + (\delta L_0 + B_1) + h_{t_1} (\delta L_0 + B_2)]C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha}$$

By using condition (h.i)

$\Rightarrow$

$\|(F_w u_w)(t) - A^{\alpha}u_0\|_x \leq \delta$, for $0 \leq t \leq t_1$.

So one can select $t_1>0$, such that:

$$t_1 = \min\{t', t'', t''', r, \left\{[K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_{\alpha}^{-1}(1-\alpha)\ (\delta - \delta')\right\}^{\frac{1}{1-\alpha}}\}$$

Thus, we have that $F_w : S_w \longrightarrow S_w$

Now, to show that F_w is a strict contraction on S_w, this will ensure the existence of a unique mild solution to the semilinear initial value control problem.

Let $\bar{\bar{u}}_w, \bar{u}_w \in S_w$, then:

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x = \Big\| S(t)A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}\bar{\bar{u}}_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}\bar{\bar{u}}_w(\tau))\,d\tau\right]ds - S(t)A^{\alpha}u_0 - \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}\bar{u}_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}\bar{u}_w(\tau))\,d\tau\right]ds \Big\|_x$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t)-(F_w\bar{u}_w)(t)\|_X =$$

$$\Big\| \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}\bar{\bar{u}}_w(s))+\int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}\bar{\bar{u}}_w(\tau))d\tau\right]ds\ -$$

$$\int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}\bar{u}_w(s))+\int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}\bar{u}_w(\tau))d\tau\right]ds\,\Big\|_X$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t)-(F_w\bar{u}_w)(t)\|_X \le$$

$$\int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\,\|f(s,A^{-\alpha}\bar{\bar{u}}_w(s))-f(s,A^{-\alpha}\bar{u}_w(s))\|_X ds\ +$$

$$\int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\left[\int_{s=0}^{t} |h(s-\tau)|\,\|g(\tau,A^{-\alpha}\bar{\bar{u}}_w(\tau))-g(\tau,A^{-\alpha}\bar{u}_w(\tau))\|d\tau\right]ds$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t)-(F_w\bar{u}_w)(t)\|_X \le \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}L_0\|A^{-\alpha}\bar{\bar{u}}_w(s)-A^{-\alpha}\bar{u}_w(s)\|ds\ +$$

$$\int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}h_{t_1}L_1\|A^{-\alpha}\bar{\bar{u}}_w(\tau)-A^{-\alpha}\bar{u}_w(\tau)\|ds$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t)-(F_w\bar{u}_w)(t)\|_X \le C_{\alpha}(1-\alpha)^{-1}L_0\|\bar{\bar{u}}_w(s)-\bar{u}_w(s)\|_X\ t_1^{1-\alpha}+C_{\alpha}(1-\alpha)^{-1}$$
$$h_{t_1}L_0\|\bar{\bar{u}}_w(\tau)-\bar{u}_w(\tau)\|_X\ t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w\bar{\ddot{u}})(t) - (F_w\bar{u}_w)(t)\|_x \leq C_\alpha(1-\alpha)^{-1}L_0 \sup_{0\leq t\leq t_1} \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \ t_1^{1-\alpha} + C_\alpha$$

$$(1-\alpha)^{-1}\ h_{t_1} L_1 \sup_{0\leq t\leq t_1} \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \ t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq C_\alpha(1-\alpha)^{-1}L_0 \|\bar{\bar{u}}_w - \bar{u}_w\|_Y \ t_1^{1-\alpha} +$$

$$C_\alpha(1-\alpha)^{-1}\ h_{t_1} L_1 \ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y \ t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq (L_0 + h_{t_1} L_1)C_\alpha(1-\alpha)^{-1}\ t_1^{1-\alpha} \ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq \frac{1}{\delta}\delta\,(L_0+L_1 h_{t_1})C_\alpha(1-\alpha)^{-1}\ t_1^{1-\alpha} \ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq \frac{1}{\delta}[\,\delta L_0+\delta\, h_{t_1} L_1]\ C_\alpha(1-\alpha)^{-1}\ t_1^{1-\alpha} \ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq \frac{1}{\delta}[\,\delta L_0+\delta\, h_{t_1} L_1+K_0K_1+B_1+h_{t_1} B_2]\ C_\alpha(1-\alpha)^{-1}$$

$$t_1^{1-\alpha} \ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \leq \frac{1}{\delta}[K_0K_1 + (\delta L_0 + B_1) + \ h_{t_1}(\delta L_1 + B_2)]\ C_\alpha$$

$$(1-\alpha)^{-1} \|\bar{\bar{u}}_w - \bar{u}_w\|_Y \ t_1^{1-\alpha}$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \le \frac{1}{\delta}[K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]\, C_\alpha(1 - \alpha)^{-1}[K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_\alpha^{-1}(1 - \alpha)(\delta - \delta')\ \|\bar{\bar{u}}_w - \bar{u}_w\|_Y \quad \{\text{by the condition (h.i)}\}$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \le \left(1 - \frac{\delta'}{\delta}\right) \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \le \left(1 - \frac{\delta'}{\delta}\right) \|\bar{\bar{u}}_w - \bar{u}_w\|_Y \tag{3.8}$$

Taking the supremum over [0, t_1] of both sides to (3.8), we get:

$$\sup_{0\le t\le t_1} \|(F_w\bar{\ddot{u}})(t) - (F_w\bar{u}_w)(t)\|_x \le \left(1 - \frac{\delta'}{\delta}\right) \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$$\|(F_w\bar{\bar{u}}_w)(t) - (F_w\bar{u}_w)(t)\|_x \le \left(1 - \frac{\delta'}{\delta}\right) \|\bar{\bar{u}}_w - \bar{u}_w\|_Y\ ,\ \{\text{by } \|y\|_Y = \sup_{0\le t\le t_1} \|y(t)\|_x\}$$

Thus, F_w is a strict contraction map {see appendix C for the definition} from S_w into S_w and therefore by the Banach contraction principle {see appendix C for the definition} there exist a unique fixed point u_w of F_w in S_w, i.e., there is a unique $u_w \in S_w$, such that $F_w u_w = u_w$.

The fixed point satisfies the integral equation:

$$u_w(t) = S(t)A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\Big[f(s, A^{-\alpha}u_w(s)) +$$

$$\int_{\tau=0}^{s} h(s-\tau)g(\tau, A^{-\alpha}u_w(\tau))d\tau\Big]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds,$$

for $0 \le t \le t_1$, $\forall\, w(.)\in L^p((0,t_1):O)$ (3.9)

For simplification, we set $\tilde{f}(t)=f(t,A^{-\alpha}u_w(t))$, $\tilde{g}(t)=g(t,A^{-\alpha}u_w(t))$. Then equation (3.9) can be rewritten as:

$$u_w(t) = S(t)A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[\tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds +$$

$$\int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds, \text{ for } 0 \le t \le t_1,\ \forall\, w(.)\in L^p((0,t_1):O) \quad (3.10)$$

To show that $\tilde{f}(t)$ is locally Hölder continuous {see appendix C for the definition} on $(0, t_1]$.

For this, we first show that $u_w(t)$ given by (3.10) is locally Hölder continuous on $(0, t_1]$.

Notice that, from the theorem (IV.7), it follows that for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have:

$$\|(S(h) - I)A^{\alpha}S(t-s)\| \le C_{\beta}\, h^{\beta}\, \|A^{\alpha+\beta}S(t-s)\| \le Ch^{\beta}(t-s)^{-(\alpha+\beta)}$$

Which is useful for proving $u_w(t)$ given by (3.10) is locally Hölder continuous on $(0, t_1]$.

Next, we have for $0 < t < t + h \le t_1$

$$\|u_w(t+h)-u_w(t)\|_x=\|S(t+h)A^{\alpha}u_0+\int_{s=0}^{t+h} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau\right]ds$$

$$+\int_{s=0}^{t+h} A^{\alpha}S(t+h-s)Bw(s)\,ds-S(t)A^{\alpha}u_0-$$

$$\int_{s=0}^{t+h} A^{\alpha}S(t-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau\right]ds-\int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)\,ds\,\|_x$$

$\Rightarrow$

$$\|u_w(t+h)-u_w(t)\|_x=\|S(t+h)A^{\alpha}u_0-S(t)A^{\alpha}u_0+$$

$$\int_{s=0}^{t} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau\right]ds+$$

$$\int_{s=t}^{t+h} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau\right]ds+$$

$$\int_{s=0}^{t} A^{\alpha}S(t+h-s)Bw(s)\,ds+\int_{s=t}^{t+h} A^{\alpha}S(t+h-s)Bw(s)\,ds-$$

$$\int_{s=0}^{t} A^{\alpha}S(t-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau\right]ds-\int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)\,ds\,\|_x$$

$\Rightarrow$

$$\|u_w(t+h)-u_w(t)\|_x\le\|(S(h)-I)S(t)A^{\alpha}u_0\|_x+$$

$$\int_{s=0}^{t}\|(S(h)-I)A^{\alpha}S(t-s)\left[\|\tilde{f}(s)\|_X+\int_{\tau=0}^{s}|h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\,d\tau\right]ds+$$

$$\int_{s=0}^{t}\|(S(h)-I)A^{\alpha}S(t-s)\|_X\,\|Bw(s)\|_X\,ds+$$

$$\int_{s=t}^{t+h} \| A^{\alpha}S(t+h-s)\|_X \left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\, d\tau \right] ds \; +$$

$$\int_{s=t}^{t+h} \| A^{\alpha}S(t+h-s)\|_X \, \| Bw(s)\|_X \, ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5. \qquad (3.11)$$

We estimate each of the terms of (3.11) separately.

$I_1 = \|S(h) - I)S(t)A^{\alpha}u_0\|_x \le C_{\beta}h^{\beta}\|A^{\alpha+\beta}S(t)\|_X \,\|u_0\| \le Ch^{\beta}\,\|u_0\|t^{-(\alpha+\beta)}$

$\Rightarrow$

$I_1 \le M_1h^{\beta}$, where M_1 depends on t for $0 \le t \le t_1$.

$$I_2 = \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\, d\tau \right] ds$$

$\Rightarrow$

$$I_2 \le \int_{s=0}^{t} (B_1 + h_{t_1}B_2)Ch^{\beta}(t-s)^{-(\alpha+\beta)}\,ds \le (B_1 + h_{t_1}B_2)Ch^{\beta}\int_{s=0}^{t} (t-s)^{-(\alpha+\beta)}\,ds$$

$\Rightarrow$

$$I_2 \le \frac{(B_1 + h_{t_1}B_2)Ch^{\beta}}{1-(\alpha+\beta)}\,t^{-(\alpha+\beta)+1} \le \frac{(B_1 + h_{t_1}B_2)Ch^{\beta}}{1-(\alpha+\beta)}\,t_1^{-(\alpha+\beta)+1}$$

$\Rightarrow$

$I_2 \le M_2\, h^{\beta}$, where $M_2 = \dfrac{(B_1 + h_{t_1}B_2)Ch^{\beta}t_1^{-(\alpha+\beta)+1}}{1-(\alpha+\beta)}$ is independent of t for $0 \le t \le t_1$.

$$I_3 = \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\|_X \,\|Bw(s)\|_X\, ds$$

$\Rightarrow$

$$I_3 \leq \int_{s=0}^{t} Ch^{\beta}(t-s)^{-(\alpha+\beta)} K_0 K_1 \, ds \leq Ch^{\beta} K_0 K_1 \int_{s=0}^{t} (t-s)^{-(\alpha+\beta)} \, ds$$

$\Rightarrow$

$$I_3 \leq \frac{Ch^{\beta} K_0 K_1}{1-(\alpha+\beta)} t^{1-(\alpha+\beta)} \leq \frac{Ch^{\beta} K_0 K_1}{1-(\alpha+\beta)} t_1^{1-(\alpha+\beta)}$$

$\Rightarrow$

$I_3 \leq M_3 h^{\beta}$, where $M_3 = \dfrac{Ch^{\beta} K_0 K_1 t_1^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}$ is independent of t for $0 \leq t \leq t_1$.

$$I_4 = \int_{s=t}^{t+h} \| A^{\alpha} S(t+h-s) \|_X \left[\| \tilde{f}(s) \|_X + \int_{\tau=0}^{s} | h(s-\tau) | \, \| \tilde{g}(\tau) \|_X \, d\tau \right] ds$$

$\Rightarrow$

$$I_4 \leq (B_1 + h_{t_1} B_2) C_{\alpha} \int_{s=t}^{t+h} (t+h-s)^{-\alpha} \, ds \leq \frac{(B_1 + h_{t_1} B_2) C_{\alpha}}{1-\alpha} h^{1-\alpha}$$

$\Rightarrow$

$I_4 \leq M_4 h^{1-\alpha}$, where $M_4 = \dfrac{(B_1 + h_{t_1} B_2) C_{\alpha}}{1-\alpha}$ is independent of t for $0 \leq t \leq t_1$.

$\Rightarrow$

$I_4 \leq M_4 h^{\beta}$

$$I_5 = \int_{s=t}^{t+h} \| A^{\alpha} S(t+h-s) \|_X \, \| Bw(s) \|_X \, ds$$

$\Rightarrow$

$$I_5 \le C_\alpha K_0 K_1 \int_{s=t}^{t+h} (t+h-s)^{-\alpha}\, ds \le \frac{C_\alpha K_0 K_1}{1-\alpha} h^{1-\alpha}$$

$\Rightarrow$

$I_5 \le M_5 h^{1-\alpha}$, where $M_5 = \dfrac{C_\alpha K_0 K_1}{1-\alpha}$ is independent of $t \in [0, t_1]$

$\Rightarrow I_5 \le M_5 h^\beta$

Combining (3.11) with these estimates, it follows that there is a constant C_1 such that:

$\|u_w(t+h) - u_w(t)\|_x \le C_1 h^\beta \le C_1 \left|h^\beta\right|$

Where $C_1 = M_1 + M_2 + M_3 + M_4 + M_5$.

$\Rightarrow$Let $k=t+h \Rightarrow h=k-t$

$\Rightarrow \|u_w(k)-u_w(t)\|_x \le C_1|k-t|^\beta$. For $0 < t < k \le t_1$.

And therefore u_w is locally Hölder continuous on $(0, t_1]$.

Now, to show that $\tilde{f}(t)$ is locally Hölder continuous on $(0, t_1]$, we have,

For $t > s$:

$\|\tilde{f}(t) - \tilde{f}(s)\|_X = \|f(t, A^{-\alpha}u_w(t)) - f(s, A^{-\alpha}u_w(s))\|_x$

$\Rightarrow$

$\|\tilde{f}(t) - \tilde{f}(s)\|_X \le L_0 [|t-s|^\theta + \| A^{-\alpha}u_w(t) - A^{-\alpha}u_w(s)\|_\alpha]$ for $0 < \theta \le 1$

{By using assumption (c)},

$\Rightarrow$

$\|\tilde{f}(t) - \tilde{f}(s)\|_X \le L_0 [|t-s|^\theta + \|u_w(t) - u_w(s)\|_x]$ {by $\|x\|_\alpha = \|A^\alpha x\|_x$}

$\Rightarrow$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \le L_0\,[|t-s|^{\theta} + C_1|t-s|^{\beta}]$

$\Rightarrow$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \le L_0\,[|t-s|^{\gamma} + C_1|t-s|^{\gamma}]$, where $\gamma = \min\{\theta,\beta\}$

$\Rightarrow$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \le L_0\,(1 + C_1)\,|t-s|^{\gamma}$

$\Rightarrow$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \le C_2|t-s|^{\gamma}$, where $C_2 = L_0\,(1 + C_1)$ is a positive constant.

Let $\tilde{h}(t) = \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)\,d\tau + Bw(t)$

To show that $\tilde{h}(t)$ is locally Hölder continuous on $(0, t_1]$.

For $t > s$, we have:

$$\|\tilde{h}(t)-\tilde{h}(s)\|_X = \left\| \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)\,d\tau - \tilde{f}(s) - \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau + Bw(t) - Bw(s) \right\|_X$$

$\Rightarrow$

$$\|\tilde{h}(t) - \tilde{h}(s)\|_X = \left\| \tilde{f}(t) - \tilde{f}(s) + \int_{\tau=0}^{s} h(t-\tau)\tilde{g}(\tau)\,d\tau + \int_{\tau=s}^{t} h(t-\tau)\tilde{g}(\tau)\,d\tau - \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau + Bw(t) - Bw(s) \right\|_X$$

$\Rightarrow$

$$\|\tilde{h}(t) - \tilde{h}(s)\|_X \leq \|\tilde{f}(t) - \tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(t-\tau) - h(s-\tau)| \, \|\tilde{g}(\tau)\|_X \, d\tau + \int_{\tau=s}^{t} |h(t-\tau)| \, \|\tilde{g}(\tau)\|_X \, d\tau + \|B(w(t) - w(s))\|_X$$

$\Rightarrow$

$$\|\tilde{h}(t) - \tilde{h}(s)\|_X \leq C_2|t-s|^{\gamma} + C_3 B_2 |t-s|^{\beta} t_1 + K_0 R_0 |t-s|^{\xi}$$

Consider the following initial value control problem:

$$\left.\begin{array}{l} \dfrac{dv(t)}{dt} + Av(t) = \tilde{h}(t),\ t > 0 \\ v(0) = u_0 \end{array}\right\} \qquad (3.12)$$

Where $\tilde{h}(t) = \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)\, d\tau + Bw(t)$.

Which implies that the initial value control problem (3.12) has a unique mild solution $v_w \in C((0, t_1]:X)$, {see appendix D for the state}, given by:

$$v_w(t) = S(t)u_0 + \int_{0}^{t} S(t-s)\tilde{h}(s)\, ds,$$

$\Rightarrow$

$$v_w(t) = S(t)u_0 + \int_{0}^{t} S(t-s)\left[\tilde{f}(s) + \int_{\tau=0}^{s} \tilde{h}(s-\tau)\tilde{g}(\tau)\, d\tau + Bw(s)\right] ds$$

$\Rightarrow$

$$v_w(t) = S(t)u_0 + \int_0^t S(t-s)\left[f(s, A^{-\alpha}u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau, A^{-\alpha}u_w(\tau))\,d\tau + Bw(s) \right] ds \tag{3.13}$$

For $0 < t \le t_1$, $\forall\, w \in L^p((0, t_1]:O)$.

Operating on both sides of equation (3.13) with A^α, we find:

$$A^\alpha v_w(t) = S(t)A^\alpha u_0 + \int_0^t S(t-s)A^\alpha \left[f(s, A^{-\alpha}u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau, A^{-\alpha}u_w(\tau))\,d\tau + Bw(s) \right] ds \tag{3.14}$$

But by (3.9), the right hand side of (3.14) equals $u_w(t)$ and therefore:

$A^\alpha v_w(t) = u_w(t)$

$\Rightarrow$

$v_w(t) = A^{-\alpha}u_w(t)$, for $0 < t \le t_1$, $\forall\, w(.) \in L^p((0, t_1]:O)$

$\Rightarrow$ $v_w(t) = S(t)u_0 + \int_0^t S(t-s)\left[f(s, v_w(s)) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)\,d\tau + Bw(s) \right] ds$ For $0<t\le t_1$, $\forall\, w(.)\in L^p((0, t_1]:O)$. So, we have a unique mild solution $v_w \in C((0, t_1]:X)$.

2.1 Introduction

In this chapter, by using the theory of semigroup and “schauder fixed point theorem”, the local existence, uniqueness and the exact controllable of the mild solution to the semilinear initial value control problem has been developed in an arbitrary Banach space X. Sufficient conditions for the global existence of the mild solution to the semilinear initial value control problem has also been developed.

The global existence of the mild solution to the semilinear initial value problem has been studied see [Paz, 83],

$$\left.\begin{aligned} &\frac{du}{dt} + Au(t) = f(t, u(t)) \\ &u(0) = u_0 \end{aligned}\right\} \qquad (2.1)$$

where A is the infinitesimal generator of a C_0 semigroup define from $D(A) \subset X$ into X and f is a nonlinear continuous map define from $[0,r) \times X$ into X.

Byszewski in 1991 [Byszewski, 91], has study the local existence and uniqueness of the mild solution to the semilinear initial value problem given by (2.1).

Definition(2.1.1) [Pazy,83]

A continuous function u is said to be a mild solution to the semilinear initial value problem (2.1) given by:

$$u(t) = T(t)\, u_0 + \int_{s=0}^{t} T(t-s) f(s, u(s)) ds \qquad (2.2)$$

Bahuguna.D in 1997 [Bahuguna, 97], has studied the local existence without uniqueness of the mild solution to the semilinear initial value problem:

$$\left.\begin{array}{l} \frac{du}{dt} + Au(t) = f(t,u(t)) + \int\limits_{s=0}^{t} h(t-s)g(s,u(s)\,ds\; t>0 \\ u(0) = u_0 \end{array}\right\} \qquad (2.3)$$

where A is the infinitesimal generator of a C_0 semigroup define from $D(A)\subset X$ into X , f and g are a nonlinear continuous maps define from $[0,r)\times X$ into X and h is the real valued continuous function define from [0,r) into R where R is the real number.

Pavel in 1999 [Pavel, 99] has studied the uniqueness of the mild solution to the semilinear initial value problem given by (2.3).

<u>Definition(2.1.2) [Bahuguna,97]:</u>

A continuous function u is said to be a mild solution to the semilinear initial value problem (2.3) given by:

$$u(t) = T(t)u_0 + \int\limits_{s=0}^{t} T(t-s)\left[f(s,u(s)) + \int\limits_{\tau=0}^{s} h(s-\tau)g(\tau,u(\tau))d\tau \right]ds \qquad (2.4)$$

Our work is concerned the semilinear initial value control problem:

$$\left.\begin{array}{l} \frac{du}{dt} + Au(t) = f(t,u(t)) + \int\limits_{s=0}^{t} h(t-s)g(s,u(s)\,ds + Bw(t), t>0 \\ u(0) = u_0 \end{array}\right\} \qquad (2.5)$$

where A is the infinitesimal generator of a C_0 semigroup define from $D(A)\subset X$ into X , f and g are a nonlinear continuous maps define from $[0,r)\times X$ into X, h is the real valued continuous function define from [0,r) into R where R is the real number and B is a bounded linear operator define from O into X. Where O is a Banach space and $w(.)$ be the arbitrary control function is given in $L^p([0, r):O)$, a Banach space of control functions with $\|w(t)\|_O \leq k_1$, for $0 \leq t < r$. The local existence, uniqueness, controllable and global existence of the mild solution to the semilinear initial value control problem has been developed.

The following subtitle (2.1.3) shows the scope of applicability of the present work. The present work guarantees the existence, uniqueness, controllability of some important class of control problems in infinite dimensional spaces and that is very important in real life applications in mathematics and one can see the following applicable examples taken from the literature. The reformulation of the follows examples to general problem in infinite dimensional spaces is out of the scope of this thesis, so we refer the reader to study this reformulation. {See [Klaus, 2000]}.

Practical scope of Problem(2.1.3)

We introduce some basic general concepts which are useful here.

Note: We set $\|.\| = |.|$

Definition(2.1.4)[Rektorys,72]:

The symbol D^α, $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_N)$ is an N-dimensional vector whose coordinates are non-negative integers is defined by the relation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}, \text{ where } |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N.$$

Remark (2.1.5 [Rektorys,72]:

Let $a_{\alpha\mu}(x)$ be a real or complex valued function defined on Ω and let

$\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_N)$, $\mu=(\mu_1, \mu_2,\ldots, \mu_N)$ be N-dimensional vectors with non-negative integer coordinates, $|\alpha|=\alpha_1 + \alpha_2 + \ldots + \alpha_N$, $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_N$.

The symbol:

$$A(x, D) = \sum_{|\alpha|,|\mu|\le K} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,\mu}(x)D^{\mu}) \qquad (2.6)$$

Define a partial differential operator of order 2K.

Here, the summation over $|\alpha|$, $|\mu| \le K$ means that addition is carried out over all N-dimensional vectors α, μ for which $\alpha_1 + \alpha_2 + \ldots + \alpha_N \le K$ and $\mu_1 + \mu_2 + \ldots + \mu_N \le K$.

We rewrite equation (2.6) as follows:

$$A(x, D) = \sum_{|\alpha|,|\mu|\le K} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}\left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x)\frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}\ldots\partial x_N^{\mu_N}}\right] \qquad (2.7)$$

Next, we give some examples:

Examples(2.1.6) [Rektorys,72] :

(i) Consider the operator given by (2.7), for k=1, i.e., an operator of the second order, the summation of (2.7) in detail, we obtain:

$$A(x, D) = \sum_{|\alpha|,|\mu|\le K} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}\left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x)\frac{\partial^{|\mu|}u}{\partial x_1^{\mu_1}\ldots\partial x_N^{\mu_N}}\right]$$

$$= \sum_{|\alpha|=1,|\mu|=1} (-1)^1 \frac{\partial}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x) \frac{\partial u}{\partial x_1^{\mu_1} \ldots \partial x_N^{\mu_N}} \right] +$$

$$\sum_{|\alpha|=1,|\mu|=0} (-1)^1 \frac{\partial}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x) \frac{\partial^0 u}{\partial x_1^{\mu_1} \ldots \partial x_N^{\mu_N}} \right] +$$

$$\sum_{|\alpha|=0,|\mu|=1} (-1)^0 \frac{\partial^0}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x) \frac{\partial u}{\partial x_1^{\mu_1} \ldots \partial x_N^{\mu_N}} \right] +$$

$$\sum_{|\alpha|=0,|\mu|=0} (-1)^0 \frac{\partial^0}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \left[a_{\alpha_1,\ldots,\alpha_N,\mu_1,\ldots,\mu_N}(x) \frac{\partial^0 u}{\partial x_1^{\mu_1} \ldots \partial x_N^{\mu_N}} \right] \quad (2.8)$$

If N = 2 (i.e., if we consider a plane problem), then the first sum includes generally four forms, namely for the pair of vectors, $\alpha=(\alpha_1,\alpha_2)$ and $\mu=(\mu_1,\mu_2)$:

From the relation $|\alpha| = 1 \Rightarrow \alpha_1 + \alpha_2 = 1$, for α_1,α_2 are nonnegative integers and

The relation $|\mu| = 1 \Rightarrow \mu_1 + \mu_2 = 1$, for μ_1, μ_2 are nonnegative integers. We have that:

$$\left.\begin{aligned} \alpha &= (1,0), \mu = (1,0) \\ \alpha &= (1,0), \mu = (0,1) \\ \alpha &= (0,1), \mu = (1,0) \\ \alpha &= (0,1), \mu = (0,1) \end{aligned}\right\} \quad (2.9)$$

The second sum includes two terms for the pairs of vectors, for $|\alpha| = 1, |\mu| = 0$ .We have that:

$$\left.\begin{aligned} \alpha &= (1,0), \mu = (0,0) \\ \alpha &= (0,1), \mu = (0,0) \end{aligned}\right\} \quad (2.10)$$

And the third sum also includes two terms for the pairs of vectors, for $|\alpha| = 0, |\mu| = 1$:

$$\left.\begin{array}{l}\alpha=(0,0),\mu=(1,0)\\ \alpha=(0,0),\mu=(0,1)\end{array}\right\} \qquad (2.11)$$

In the case of the last term in equation (2.8), there is only one possible pair of vectors, for $|\alpha|=0,|\mu|=0$:

$$\alpha=(0,\,0),\ \mu=(0,\,0) \qquad (2.12)$$

From equation (2.9), we get:

$$-\frac{\partial}{\partial x_1\partial x_2^0}\left[a_{1,0;1,0}(x)\frac{\partial u}{\partial x_1\partial x_2^0}\right]-\frac{\partial}{\partial x_1\partial x_2^0}\left[a_{1,0;0,1}(x)\frac{\partial u}{\partial x_1^0\partial x_2}\right]$$
$$-\frac{\partial}{\partial x_1^0\partial x_2}\left[a_{0,1;1,0}(x)\frac{\partial u}{\partial x_1\partial x_2^0}\right]-\frac{\partial}{\partial x_1^0\partial x_2^1}\left[a_{0,1;0,1}(x)\frac{\partial u}{\partial x_1^0\partial x_2}\right]$$

From equation (2.10), we get:

$$-\frac{\partial}{\partial x_1\partial x_2^0}\left[a_{1,0;0,0}(x)u\right]-\frac{\partial}{\partial x_1^0\partial x_2}\left[a_{0,1;0,0}(x)u\right]$$

From equation (2.11), we get:

$$a_{0,0;1,0}(x)\frac{\partial u}{\partial x_1\partial x_2^0}+a_{0,0;0,1}(x)\frac{\partial u}{\partial x_1^0\partial x_2}$$

From equation (2.12), we get:

$$a_{0,0;0,0}(x)u$$

Thus the differential operator (equation (2.6)) of the second order in two variables x_1, x_2 has in general nine terms. Some of these terms may naturally vanish (if the corresponding coefficient $a_{\alpha\mu}$ is zero).

(ii) Let $a_{\alpha,\mu}=1$, for $\alpha=(1,\,0)$, $\mu=(0,\,0)$, $\alpha=(0,\,1)$,$\mu=(0,\,1)$,

And $a_{\alpha,\mu}$=0, in the remaining seven cases. We get:

$$A(x,D)u = -\frac{\partial}{\partial x_1}\left(1\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(1\frac{\partial u}{\partial x_2}\right) = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = -\Delta u$$

Thus, in this case, the operator A is the familiar Laplace operator in two variables.

(iii) If k = 2 and N = 2, i.e., if we consider an operator of the fourth order in two variables and if we put:

$a_{\alpha\mu}$ =1 for $\alpha = (2, 0), \mu = (2, 0)$

$\alpha = (0, 2), \mu = (0,2)$,

$a_{\alpha\mu}$ =2 for $\alpha= (1,1), \mu= (1,1)$

And $a_{\alpha\mu} = 0$, in the other cases, then we obtain:

$$A(x,D)u = \frac{\partial^2}{\partial x_1^2}\left(1\frac{\partial^2 u}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(1\frac{\partial^2 u}{\partial x_2^2}\right) + \frac{\partial^2}{\partial x_1 \partial x_2}\left(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)$$

$\Rightarrow$

$$A(x,D)u = \frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} = \Delta^2 u .$$

Thus, in this case, the operator A is the biharmonic operator.

<u>Definition(2.1.7) [Rektorys,72]:</u>

The operator A given by (2.6) is said to be elliptic if for every real non-zero vector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$,

$$\sum_{|\alpha|,|\mu|=k} a_{\alpha\mu}(x)\xi_\alpha \xi_\mu \neq 0 \text{, where } \xi_\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\ldots\xi_N^{\alpha_N} \text{, } \xi_\mu = \xi_1^{\mu_1}\xi_2^{\mu_2}\ldots\xi_N^{\mu_N} .$$

Definition(2.1.8)[Rektorys,72]:

The operator A given by (2.6) is said to be strongly elliptic if there exist a number C > 0, such that:

$$\sum_{|\alpha|,|\mu|=k} a_{\alpha\mu}(x)\xi_\alpha\xi_\mu \geq C|\xi|^{2k},$$

Where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_N^2$.

Next we give some examples about the above definition.

Examples(2.1.9)[Rektorys,72:

(a) From example (2.1.6.ii), the Laplace operator $-\Delta$ is strongly elliptic, since:

K=1, N=2, $a_{\alpha,\mu}$=1, for $\alpha = (1, 0)$, $\mu = (0, 0)$, $\alpha = (0, 1)$,$\mu = (0, 1)$,

$|\alpha| = 1 \Rightarrow \alpha_1 + \alpha_2 = 1, |\mu| = 1 \Rightarrow \mu_1 + \mu_2 = 1.$

And $a_{\alpha,\mu}$=0, in the remaining seven cases. We get:

$$A(x,D)u = -\frac{\partial}{\partial x_1}\left(1\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(1\frac{\partial u}{\partial x_2}\right) = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = -\Delta u,$$

Then:

$$\sum_{|\alpha|,|\mu|=1} a_{\alpha\mu}\xi_\alpha\xi_\mu = 1.\ \xi_1^1\xi_2^0\xi_1^1\xi_2^0 + 1.\ \xi_1^0\xi_2^1\xi_1^0\xi_2^1$$

$$= \xi_1\xi_1 + \xi_2\xi_2$$

$$= \xi_1^2 + \xi_2^2$$

$$= |\xi|^2$$

So that it is sufficient to put C = 1.

(b) If K = 1, N = 2, and putting $a_{1,0;1,0} = 1$, $a_{0,1;0,1} = 1$, $a_{1,0;0,1} = C$, $a_{0,1;1,0} = -C$, with C being an arbitrary constant and $a_{\alpha\mu}=0$ in the other cases, we get:

$$A(x,D)u = -\frac{\partial}{\partial x_1}\left(1\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(1\frac{\partial u}{\partial x_2}\right) - \frac{\partial u}{\partial x_1}\left(C\frac{\partial u}{\partial x_2}\right) - \frac{\partial u}{\partial x_2}\left(-C\frac{\partial u}{\partial x_1}\right)$$

$$= -\Delta u$$

Also:

$$\sum_{|\alpha|,|\mu|=1} a_{\alpha\mu}\xi_\alpha\xi_\mu = 1.\xi_1^1\xi_2^0\xi_1^1\xi_2^0 + 1.\xi_1^0\xi_2^1\xi_1^0\xi_2^1 + C.\xi_1^1\xi_2^0\xi_1^0\xi_2^1 - C.\xi_1^0\xi_2^1\xi_1^1\xi_2^0$$

$$= \xi_1^2 + \xi_2^2 = |\xi|^2.$$

(c) If K = 1 and N = 2, $a_{1,0;1,0} = -1$, $a_{0,1;0,1} = -1$, then:

$$A(x,D)u = (-1)\frac{\partial}{\partial x_1}\left((-1)\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left((-1)\frac{\partial u}{\partial x_2}\right)$$

$\Rightarrow$

$$A(x,D)u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \Delta u.$$

So that:

$$\sum_{|\alpha|,|\mu|=1} a_{\alpha\mu}\xi_\alpha\xi_\mu = -\xi_1^2 - \xi_2^2 = -|\xi|^2,$$

Which implies A is not a strongly elliptic operator.

For the general example, we introduce the following semilinear parabolic integrodifferential equation. Let $\Omega \subset R^n$ be a bounded domain with smooth boundary$\partial\Omega$.

$$\left.\begin{aligned}&\frac{\partial u(x,t)}{\partial t}+A(x,D)u(x,t)=f(x,t,u(x,t))+\int_{s=0}^{t}h(t-s)g(x,s,u(x,s))ds\\&t>0,x\in\Omega\end{aligned}\right\}\tag{2.13}$$

where

$$A(x, D) = \sum_{|\alpha|\le K} a_\alpha(x)D^\alpha \tag{2.14}$$

Is said to be partial differential operator, where $a_\alpha(x)$ is a real or complex valued function defined on Ω and D^α stands for any α-th order derivative, and assume that the partial differential operator given by (2.14) is strongly elliptic.

The parabolic integrodifferential equation (2.13) can be reformulated as the following abstract integrodifferential equation in $X= L^p(\Omega),1<p<\infty$.

{See [Bahuguna, 97]}.

$$\left.\begin{aligned}&\frac{du}{dt}+Au(t)=f(t,u(t))+\int_{s=0}^{t}h(t-s)g(s,u(s))ds,\ t>0\\&u(0)=u_0\end{aligned}\right\}\tag{2.15}$$

where the operator $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator C_0 compact semigroup and f, g : $[0, r)\times X \longrightarrow X$, are nonlinear continuous maps and h is continuous function which at least $h\in L^1([0,r):R)$,Where R is the real number. And in our system the following parabolic integrodifferential control equation define as follow:

$$\left.\begin{array}{l}\dfrac{\partial u(x,t)}{\partial t}+A(x,D)u(x,t)=f(x,t,u(x,t))+\displaystyle\int_{s=0}^{t}h(t-s)g(x,s,u(x,s))ds+Bw(t)\\ t>0,x\in\Omega\end{array}\right\} \tag{2.16}$$

where

$$A(x, D) = \sum_{|\alpha|\le K} a_{\alpha}(x)D^{\alpha} \tag{2.17}$$

Is said to be partial differential operator, where $a_{\alpha}(x)$ is a real or complex valued function defined on Ω and D^{α} stands for any α-th order derivative, and assume that this partial differential operator given by (2.17) is strongly elliptic.

The parabolic integrodifferential control equation given by (2.16) can be reformulated as the following abstract integrodifferential equation in $X = L^{p}(\Omega), 1<p<\infty$. {See [Bahuguna, 97]}.

$$\left.\begin{array}{l}\dfrac{du}{dt}+Au(t)=f(t,u(t))+\displaystyle\int_{s=0}^{t}h(t-s)g(s,u(s))ds+Bw(t),\ t>0\\ u(0)=u_0\end{array}\right\} \tag{2.18}$$

where the operator $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator C_0 compact semigroup and f, g : $[0, r)\times X \longrightarrow X$, are nonlinear continuous maps and h is continuous function which at least $h\in L^{1}([0,r):R)$,Where R is the real number. And $w(.)$ be the arbitrary control function is given in $L^{p}([0, r): O)$, a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X.

Next, the following useful examples have been discussed in details:

<u>Example(2.1.10.i) [Klaus, 00]</u>:

Consider the partial differential equation with delay:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial^2 x^2} - d(x)u(t,x) + b(x)u(t-r,x), t \geq 0, x \in [0,1],$$

$$\frac{\partial u(t,0)}{\partial x} = 0 = \frac{\partial u(t,1)}{\partial x}, t \geq 0,$$

$$u(s,x) = h(s,x),\ s \in [-r,0],\ x \in [0,1] \tag{2.19}$$

This equation can be interpreted as a model for the growth of a population in [0, 1].In fact, u(t,.) is the population density at time t, and the term d^2/dx^2 u(t,x) describes the internal migration. Moreover, the continuous functions $d,b:[0,1] \longrightarrow R^+$ represent space-dependent death and birth rates, respectively, and r is the delay due to pregnancy.

In order to reformulation (2.19) as an abstract delay differential equation of the following form:

$$\begin{aligned} &\dot{u}(t) = Bu(t) + \Phi u_t, t \geq 0, \\ &u_0 = h \in X, \end{aligned} \tag{2.20}$$

where u_t: $[-r, 0] \longrightarrow Y$ is define $u_t(s) = u(t+s)$.

We introduce the spaces Y:=C[0,1] and X:=C([–r,0],Y). Moreover, define the operators:

$$\Delta := \frac{d^2}{dx^2}, D(\Delta) := \{y \in C^2[0,1] : y'(0) = 0 = y'(1)\},$$
$$B := \Delta - M_d, D(B) := D(\Delta).$$
$$\Phi := M_b \delta_{-r} \in (X,Y),$$

where M_d and M_b are the multiplication operators induced by d and b, respectively.

Example(2.1.10.ii)[Klaus, 00]:

On the interval [0, 1], consider the second–order Cauchy problem:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = b\frac{\partial^3 u(t,x)}{\partial t \partial x^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x} \right) - c\frac{\partial^4 u(t,x)}{\partial x^4}, t \ge 0, x \in [0,1],$$

$$\begin{aligned} &u(t,x) = 0 = \frac{\partial^2 u(t,x)}{\partial x^2}, t \ge 0, x = 0,1, \\ &u(0,x) = u_0(x), \frac{\partial u(0,x)}{\partial t} = u_1(x), x \in [0,1], \end{aligned} \qquad (2.21)$$

For $a \in C^1[0,1], b \in \mathbb{C}$ with Re $b \geq 0$ and $c>0$.

In order, to reformulate (2.21) to as an abstract second–order Cauchy problem on X: $=L^2[0,1]$, of the following form:

$$\begin{aligned} &\ddot{u}(t) = B\dot{u}(t) + Au(t), t \ge 0, \\ &u(0) = x, \dot{u}(0) = y \end{aligned} \qquad (2.22)$$

To that purpose, we introduce the operators:

$$A := -c\Delta^2, D(A) := \left\{ f \in H_0^4[0,1] : f''(0) = 0 = f''(1) \right\},$$
$$B := b\Delta, D(B) := H_0^2[0,1],$$
$$C := \sqrt{c}\Delta, D(C) := H_0^2[0,1],$$
$$D := -D_m M_a D_0, D(D) := H_0^2[0,1],$$

Moreover D_m and D_0 denote the first derivative with maximal domain and the boundary conditions, respectively, $\Delta := D_m D_0$ is the Laplacian with boundary conditions, and M_a stands for the multiplication operator induced by the function a.

Example (2.1.10.iii)[Klaus, 00]:

(Heat Equation).consider a hot bar of length one that is insulated at its endpoints s = 0, 1. We assume that the bar can be heated around some point $s_0 \in (0,1)$ and that we can measure its average temperature around some other point $s_1 \in (0,1)$. Denoted by x(s, t) the temperature at position $s \in [0,1]$ and the time $t \geq 0$ and by $x_0(.)$ the initial temperature profile. This model can be described by the equations:

$$\begin{aligned}
&\frac{\partial x(s,t)}{\partial t} = \frac{\partial^2 x(s,t)}{\partial s^2} + b(s)u(t), t \geq 0, s \in [0,1], \\
&\frac{\partial x(0,t)}{\partial s} = 0 = \frac{\partial x(1,t)}{\partial s}, t \geq 0, \\
&x(s,0) = x_0(s), s \in [0,1], \\
&y(t) = \int_0^1 c(s)x(s,t)ds, t \geq 0,
\end{aligned} \tag{2.23}$$

Here b and c represent the functions around the point s_0 and the point s_1, respectively, i.e., we may take:

$$\begin{aligned}
&b(s) := \frac{1}{2\varepsilon_0} k_{[s_0 - \varepsilon_0, s_0 + \varepsilon_0]}(s), \\
&c(s) := \frac{1}{2\varepsilon_1} k_{[s_1 - \varepsilon_1, s_1 + \varepsilon_1]}(s)
\end{aligned}$$

For $\varepsilon_0, \varepsilon_1 > 0$, where k_J denotes the characteristic function of a subset $J \subset [0,1]$. In order to reformulate (2.23) to an abstract control problem of the following form:

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t), \\
&y(t) = Cx(t), t \geq 0, \\
&x(0) = x_0.
\end{aligned} \tag{2.24}$$

Where A generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on Banach space X, B is a bounded control operator from the control Banach space U to X, u: $R^+ \longrightarrow U$ is a locally integrable control function (also called the input), C is a bounded operator from X to the Banach space Y, the function y: $R^+ \to Y$ is the output of the system, and $x_0 \in X$ is its initial state.

We choose the space X: $= L^2[0,1]$, the control space U = □ ,and the space Y:=□ and define the operators:

$$A := \frac{d^2}{ds^2}, D(A) := \left\{ x \in H^2[0,1] : x'(0) = 0 = x'(1) \right\},$$
$$B \in (U, X), Bu := b(.)u,$$
$$C \in (X, Y), Cx := \int_0^1 c(x)x(s)ds$$

2.2 Local Existence and Uniqueness of the Mild Solution to the Semilinear Initial Value Control Problem

In this section, the local existence and uniqueness of the mild solution to (2.5) has been developed.

Define the mild solution of (2.5) for every given $w \in L^p([0, r):O)$.

Definition(2.2.1) :

A continuous function u_w will be called a mild solution of (2.5), given by:

$$u_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[Bw(s) + f(s, u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_w(\tau))d\tau \right] ds \quad (2.25)$$

For every given $w \in L^p([0, r):O)$.

The local existence and uniqueness of a mild solution of (2.5) have been developed, by assuming the following assumptions:

(a) A be the infinitesimal generator C_0 compact semigroup $\{T(t)\}_{t\geq 0}$, where A define from $D(A) \subset X$ into X. where X be a Banach space.

(b) Let $\rho > 0$ such that $\mathcal{B}_\rho(u_0) = \{x \in X \mid \|x - u_0\| \leq \rho\}$, (where $u_0 \in U$ {open subset of X}),

The nonlinear maps f, g define from $[0, r) \times U$ into X, satisfy the locally Lipschitze condition with respect to second argument, i.e.

$\|f(t, v_1) - f(t, v_2)\|_X \leq L_0\|v_1 - v_2\|$ and $\|g(t, v_1) - g(t, v_2)\|_X \leq L_1\|v_1 - v_2\|$

For $0 \leq t < r$ and $v_1, v_2 \in \mathcal{B}_\rho(u_0)$ and L_0, L_1 is Lipschitze constant.

(c) h is continuous function which at least $h \in L^1([0,r):R)$,Where R is the real number.

(d) Let $t' > 0$ such that $\|f(t, v)\|_X \leq N_1$, $\|g(t, v)\|_X \leq N_2$,for $0 \leq t \leq t'$ and $v \in \mathcal{B}_\rho(u_0)$.

Also let $t''>0$ such that $\|T(t)u_0 - u_0\|_X \leq \rho'$ for $0 \leq t \leq t''$ and $u_0 \in U$, where ρ' is a positive constant such that $\rho' < \rho$.

(e) $w(.)$ be the arbitrary control function is given in $L^p([0, r): O)$, a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X with $\|w(t)\|_O \leq k_1$, for $0 \leq t < r$.

(f) Let $t_1>0$ such that:

$t_1 = \min\{r, t', t''\}$ and satisfy the following conditions

(i) $$t_1 \leq \frac{\rho - \rho'}{(K_0K_1 + N_1 + h_{t_1}N_2)M}$$

(ii) $$t_1 < \frac{1}{M(L_0 + L_1h_{t_1})}$$

We introduce the following main theorem:

Theorem (2.2.2):

Assume the hypotheses (a)-(f) are hold. Then for every $u_0 \in U$, there exist a fixed number t_1 , $0 < t_1 < r$, such that the initial value control problem (2.5) has a unique local mild solution $u_w \in C([0, t_1]:X)$, for every control function $w(.) \in L^p([0, r): O)$.

Proof:

Without loss of generality, we may suppose $r<\infty$, because we are concerned here with the local existence only.

There exist $M \geq 0$ such that $\|T(t)\| \leq M$, $0 \leq t \leq r$. { since T(t) is a bounded linear operator on X}.

Let $\rho > 0$ be such that $\mathcal{B}_\rho(u_0) = \{v \in X \mid \|v - u_0\| \leq \rho\} \subset U$ { since U is an open subset of X}.

$$\text{Assume } h_r = \int_{s=0}^{r} |h(s)|ds \tag{2.26}$$

We set $Y= C([0,t_1]:X)$, where Y is a Banach space with the sup-norm defined as follows:

$$\|y\|_Y = \sup_{0 \leq t \leq t_1} \|y(t)\|_x$$

and we define

$$S_w = \{u_w \in Y \mid u_w(0) = u_0,\ u_w(t) \in \mathcal{B}_\rho(t_0), \text{ for a given } w(.) \in L^p([0, r): O),\ 0 \leq t \leq t_1\} \tag{2.27}$$

To prove S_w is bounded, convex and closed subset of Y.

First to prove the boundedness {See appendix C for the definition} of S_w as a subset of Y for a given $w(.) \in L^p([0, r):O)$, i.e. to prove there exist k >0, such that $\|u_w\|_Y \leq k$, for every $u_w \in S_w$

Let $u_w \in S_w \Rightarrow u_w \in Y$, $u_w(0) = u_0$ and $u_w(t) \in B(u_0)$, for $0 \leq t \leq t_1$

Since $\|u_w\|_Y = \sup\limits_{0 \leq t \leq t_1} \|u_w(t)\|_x$ (2.28)

$\|u_w(t)\|_X = \|u_w(t) - u_0 + u_0\| \leq \|u_w(t) - u_0\|_X + \|u_0\|_X \leq \rho + \|u_0\|_X$

Let $k = \rho + \|u_0\|_X > 0$, then:

$$\|u_w(t)\|_x \leq k, \text{ for any } u_w \in S_w \qquad (2.29)$$

Take the supremum over $[0, t_1]$ on both sides of equation (2.29), we get:

$$\sup_{0 \leq t \leq t_1} \|u_w(t)\|_x \leq k \Rightarrow \|u_w\|_Y \leq k, \text{ for any } u_w \in S_w \ \{\text{by (2.28)}\}.$$

Which implies that S_w is a bounded subset of Y.

Second, to prove the convexity {see appendix C for the definition} of S_w as a subset of Y. i.e. to prove

$\lambda \bar{\bar{u}}_w + (1-\lambda)\bar{u}_w \in S_w$, for every $\bar{\bar{u}}_w, \bar{u}_w \in S_w$, $\lambda \in [0, 1]$

Let $\bar{\bar{u}}_w, \bar{u}_w \in S_w$, then $\bar{\bar{u}}_w, \bar{u}_w \in Y$, $\bar{\bar{u}}_w(0) = \bar{u}_w(0) = u_0$ and $\bar{\bar{u}}_w(t)$, $\bar{u}_w(t) \in \mathcal{B}_\rho(u_0)$,

$0 \leq t \leq t_1$.

from (2.27), notice that $\lambda \bar{\bar{u}}_w + (1 - \lambda)\bar{u}_w \in Y$ { the properties of the Banach space Y} and it is clear that $\lambda \bar{\bar{u}}_w(0) + (1 - \lambda)\bar{u}_w(0) = u_0$.To prove $\lambda \bar{\bar{u}}_w(t) + (1 - \lambda)\bar{u}_w(t) \in \mathcal{B}_\rho(u_0)$, $0 \leq t \leq t_1$.

From the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $\lambda \bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) \in X$ {the properties of the Banach space X} and

$\|\lambda\bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) - u_0\|_X = \|\lambda\bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) - \lambda\bar{\bar{u}}_w(0) - (1 - \lambda)\bar{u}_w(0)\|_X$

$\Rightarrow$

$$\|\lambda\bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) - u_0\|_X = \|\lambda\bar{\bar{u}}_w(t) - \lambda\bar{\bar{u}}_w(0) + (1-\lambda)\bar{u}_w(t) - (1 - \lambda)\bar{u}_w(0)\|_X$$

$$\leq \|\lambda(\bar{\bar{u}}_w(t) - \bar{\bar{u}}_w(0))\|_X + \|(1-\lambda)(\bar{u}_w(t) - \bar{u}_w(0))\|_X$$

$$\leq \lambda\rho + (1-\lambda)\rho = \rho$$

$\Rightarrow$

$$\|\lambda\bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) - u_0\|_X \leq \rho, \text{ for } 0 \leq t \leq t_1$$

$\Rightarrow$

$$\lambda\bar{\bar{u}}_w(t) + (1-\lambda)\bar{u}_w(t) \in \mathcal{B}_\rho(u_0), \text{ for } 0 \leq t \leq t_1$$

Therefore $\lambda\bar{\bar{u}}_w + (1-\lambda)\bar{u}_w \in S_w$, for every $\bar{\bar{u}}_w, \bar{u}_w \in S_w$, $0 \leq \lambda \leq 1$

Hence S is convex subset of Y.

Third, to prove the closedness {see appendix C for the definition} of S_w as a subset of Y. Let $u_w^n \in S_w$, such that $u_w^n \xrightarrow{\text{P.C.}} u_w$ as $n \longrightarrow \infty$, to prove $u_w \in S_w$, where (P.C) stands for point wise convergence.

Since $u_w^n \in S_w \Rightarrow u_w^n \in Y$, $u_w^n(0) = u_0$ and $u_w^n(t) \in \mathcal{B}_\rho(u_0)$, for $0 \leq t \leq t_1$

Since $u_w^n \xrightarrow{\text{U.C.}} u_w$ {see appendix D}, hence $u_w \in Y$. where (U.C) stands for the uniform convergence, and also

Since $u_w^n \xrightarrow{\text{U.C.}} u_w \Rightarrow \|u_w^n - u_w\|_Y \xrightarrow{\text{U.C.}} 0$, as $n \longrightarrow \infty$

Since $\|u_w^n - u_w\| = \sup_{0\le t\le t_1} \|u_w^n(t) - u_w(t)\|_X \longrightarrow 0$, as $n \longrightarrow \infty$

which implies that $\|u_w^n(t) - u_w(t)\|_X \longrightarrow 0$, as $n \longrightarrow \infty$, for every $t_0 \le t \le t_1$, i.e.,

$$\lim_{n\to\infty} u_w^n(t) = u_w(t),\ \forall\ 0 \le t \le t_1 \qquad (2.30)$$

$$\Rightarrow \lim_{n\to\infty} u_w^n(0) = u_w(0) \Rightarrow \lim_{n\to\infty} u_0 = u_w(0) \Rightarrow u_0 = u_w(0).$$

to prove $u_w(t) \in \mathcal{B}_\rho(u_0)$, for $0 \le t \le t_1$, from the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $u_w(t) \in X$, for $0 \le t \le t_1$ {the definition of the Banach space Y} and

$$\begin{aligned}\|u_w(t) - u_0\|_X &= \|\lim_{n\to\infty} u_w^n(t) - u_0\|_X \qquad \text{by (2.30)}\\ &= \|\lim_{n\to\infty} u_w^n(t) - \lim_{n\to\infty} u_0\|_X\\ &= \|\lim_{n\to\infty} [u_w^n(t) - u_0]\|_X\\ &= \lim_{n\to\infty} \|u_w^n(t) - u_0\|_X \le \lim_{n\to\infty} \rho = \rho\end{aligned}$$

Hence $\|u_w(t) - u_0\|_X \le \rho$, for $0 \le t \le t_1$

We have got S_w is closed subset of Y.

Define a map $F_w : S_w \longrightarrow Y$, by:

$$(F_w u_w)(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_w(\tau))d\tau \right]ds +$$

$$\int_{s=0}^{t} T(t-s)Bw(s)ds, \text{ for arbitrary } w(.) \in L^p([t_0, r): O) \qquad (2.31)$$

To show that $F_w(S_w) \subseteq S_w$, let u_w be arbitrary element in S_w such that

$F_w u_w \in F_w(S_w)$, to prove $F_w u_w \in S_w$.

From (2.26), notice that $F_w u_w \in Y$ {by the definition of the map F_w}

and $(F_w u_w)(0) = u_0$ by (2.31), to prove $(F_w u_w)(t) \in \mathcal{B}_\rho(u_0)$, for any $u_w \in S_w$

From the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $(F_w u_w)(t) \in X$ { the definition of the Banach space Y} and

$$\|(F_w u_w)(t) - u_0\|_X = \|T(t)u_0 - u_0 + \int_{s=0}^{t} T(t-s)Bw(s)\, ds + \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau \right] ds\,\|_X$$

$\Rightarrow$

$$\|(F_w u_w)(t) - u_0\|_X \le \|T(t)u_0 - u_0\|_X + \int_{s=0}^{t} \|T(t-s)\|\,\|Bw(s)\|_X\, ds + \int_{s=0}^{t} \|T(t-s)\|\,\|f(s,u_w(s))\|_X\, ds + \int_{s=0}^{t} \|T(t-s)\|\left[\int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_w(\tau))\|_X\, d\tau \right] ds$$

$\Rightarrow$

Let $J = \int_{\tau=0}^{s} |h(s-\tau)|\, d\tau$,

Assume $k = s-\tau \Rightarrow dk = -d\tau$

$\Rightarrow$ for $\tau = 0 \Rightarrow k = s$ and for $\tau = s \Rightarrow k = 0$

$\Rightarrow J = -\int_{k=s}^{0} |h(k)|\,dk \Rightarrow J = \int_{k=0}^{s} |h(k)|\,dk \le \int_{k=0}^{t_1} |h(k)|\,dk = h_{t_1}$ {by (2.26)},

$$\Rightarrow J = \int_{\tau=0}^{s} |h(s-\tau)|\, d\tau \le h_{t_1}$$

Notice that:

$$\int_{s=0}^{t} \|T(t-s)\| \left[\int_{\tau=0}^{s} |h(s-\tau)| \|g(\tau, u_w(\tau))\|_X \, d\tau \right] ds \le$$

$$\int_{s=0}^{t} \|T(t-s)\| \left[h_{t_1} \|g(\tau, u_w(\tau)\|_X \right] ds$$

$\Rightarrow$

$$\|(F_w u_w)(t) - u_0\|_X \le \|T(t)u_0 - u_0\|_X + \int_{s=0}^{t} \|T(t-s)\| \|Bw(s)\|_X \, ds + \int_{s=0}^{t} \|T(t-s)\| \|f(s, u_w(s))\|_X \, ds + \int_{s=0}^{t} \|T(t-s)\| \left[h_{t_1} \|g(\tau, u_w(\tau)\|_X \right] ds$$

$\Rightarrow$

$$\|(F_w u_w)(t) - u_0\|_X \le \rho' + MK_0K_1t_1 + MN_1t_1 + M h_{t_1} N_2 t_1$$

$$\le \rho' + (K_0K_1 + N_1 + h_{t_1} N_2)\, M\, t_1$$

By using the assumption (f.i), we get:

$\|(F_w u_w)(t) - u_0\|_X \le \rho$, for $0 \le t \le t_1$, i.e., $(F_w u_w)(t) \in \mathcal{B}_\rho(u_0)$, for $0 \le t \le t_1$

Hence $F_w u_w \in S_w$, for arbitrary $u_w \in S_w$, which implies that $F_w : S_w \longrightarrow S_w$

So one can select the time t_1 such that:

$$t_1 = \min \left\{ t', t'', r, \frac{\rho - \rho'}{(K_0K_1 + N_1 + h_{t_1} N_2)M} \right\}$$

To complete the prove, we have to show that $F_w : S_w \longrightarrow S_w$ is a continuous map:

Given $\| u_w^n - u_w \|_Y \longrightarrow 0$, as $n \longrightarrow \infty$

To prove $\| F_w u_w^n - F_w u_w \|_Y \longrightarrow 0$, as $n \longrightarrow \infty$

Notice that:

$$\| F_w u_w^n - F_w u_w \|_Y = \sup_{0 \le t \le t_1} \| (F_w u_w^n)(t) - (F_w u_w)(t) \|_X$$

$\Rightarrow$

$$\| F_w u_w^n - F_w u_w \|_Y = \sup_{0 \le t \le t_1} \| T(t)u_0 + \int_{s=0}^{t} T(t-s)(Bw)(s)\, ds + \int_{s=0}^{t} T(t-s)\left[f(s, u_w^n(s)) + \int_{\tau=t_0}^{s} h(s-\tau) g(\tau, u_w^n(\tau))\, d\tau \right] ds - T(t)u_0 - \int_{s=0}^{t} T(t-s)(Bw)(s)\, ds - \int_{s=0}^{t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{s} h(s-\tau) g(\tau, u_w(\tau))\, d\tau \right] ds \|_X$$

$\Rightarrow$

$$\| F_w u_w^n - F_w u_w \|_Y = \sup_{0 \le t \le t_1} \| \int_{s=0}^{t} T(t-s)\left[f(s, u_w^n(s)) - f(s, u_w(s)) \right] + \int_{s=0}^{t} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)\left[g(\tau, u_w^n(\tau)) - g(\tau, u_w(\tau)) \right] d\tau \right] ds \|_X$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le \sup_{0\le t\le t_1}$$

$$\int_{s=0}^{t} \|T(t-s)\| \left[\|f(s,u_w^n(s)) - f(s,u_w(s))\| + \int_{\tau=0}^{s} |h(s-\tau)| \|g(\tau,u_w^n(\tau)) - g(\tau,u_w(\tau))\|_X \, d\tau \right] ds$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le \sup_{0\le t\le t_1} \int_{s=0}^{t} M\left[L_0 \|u_w^n(s) - u_w(s)\|_X + h_{t_1} L_1 \|u_w^n(\tau) - u_w(\tau)\|_X \right] ds$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le$$

$$\sup_{0\le t\le t_1} \int_{s=0}^{t} M\left[L_0 \sup_{0\le t\le t_1} \|u_w^n(t) - u_w(t)\|_X + h_{t_1} L_1 \sup_{0\le t\le t_1} \|u_w^n(t) - u_w(t)\|_X \right] ds$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le \sup_{0\le t\le t_1} \left\{ \int_{s=0}^{t} M\left[L_0 \|u_w^n - u_w\|_Y + h_{t_1} L_1 \|u_w^n - u_w\|_Y \right] ds \right\}$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le \left\{ \int_{s=0}^{t_1} M\left[L_0 \|u_w^n - u_w\|_Y + h_{t_1} L_1 \|u_w^n - u_w\|_Y \right] ds \right\}$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le M\left[L_0 \|u_w^n - u_w\|_Y + h_{t_1} L_1 \|u_w^n - u_w\|_Y \right] t_1$$

$\Rightarrow$

$$\|F_w u_w^n - F_w u_w\|_Y \le M\left[L_0 + h_{t_1} L_1 \right] \|u_w^n - u_w\|_Y \; t_1$$

Since $\|u_w^n - u_w\|_Y \to 0$, as $n \to \infty$

$\Rightarrow$

$\lim\limits_{n\to\infty} \|F_w u_w^n - F_w u_w\|_Y = 0$, i.e. $\|F_w u_w^n - F_w u_w\|_Y \to 0$, as $n \to \infty$

Assume that $\tilde{S} = F_w(S)$, and for fixed $t \in [0, t_1]$, let

$\tilde{S}(t) = \{F_w u_w)(t) : u_w \in S_w\}$.

To show that $\tilde{S}(t)$ is a precompact set for every fixed $t \in [0, t_1]$,

for $t = 0 \Rightarrow \tilde{S}(0) = \{(F_w u_w)(0) : u_w \in S_w\} = \{u_0\}$ which is a precompact set in X { see appendix B}

Now for $t > 0$, $0 < \varepsilon < t$, define:

$$(F_w^{\varepsilon} u_w)(t) = T(t)u_0 + \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau \right] ds$$

$$+ \int_{s=0}^{t-\varepsilon} T(t-s)(Bw)(s)\,ds \text{, for arbitrary } u_w \in S_w$$

$\Rightarrow$

$(F_w^{\varepsilon} u_w)(t) =$

$$T(t)u_0 + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s) \right] ds,$$

$$\text{for arbitrary } u_w \in S_w \qquad (2.32)$$

To prove that for every ε, $0 < \varepsilon < t$,

The set $\tilde{S}_{\varepsilon}(t) = \left\{ \left(F_w^{\varepsilon} u_w \right)(t) \right\}_{u_w \in S_w} = \{(F_w^{\varepsilon} u_w)(t) : u_w \in S_w\}$ is precompact set in X.

Let:

$$J = \left\{ \int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s) \right] ds : u_w \in S_w \right\}$$

To prove the set J is bounded in X, i.e., there exists L>0 such that $\|j\|_X \le L, \forall j \in J$. $\forall\ u_w \in S_w$,

Notice that, for arbitrary $u_w \in S_w$:

$$\left\| \int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s) \right] ds \right\|_X$$

$\Rightarrow$

$$\le \int_0^{t-\varepsilon} \left\| T(t-s-\varepsilon)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s) \right] \right\|_X ds$$

$\Rightarrow$

$$\le \int_0^{t-\varepsilon} \|T(t-s-\varepsilon)\| \left[\|f(s,u_w(s))\|_X + h_{t_1}\|g(\tau,u_w(\tau))\|_X + K_0\|w(s)\|_O \right] ds$$

$\Rightarrow$

$$\le M\,(N_1 + h_{t_1} N_2 + K_0K_1)\,(t-\varepsilon) \le M\,(N_1 + h_{t_1} N_2 + K_0K_1)\,t_1$$

Let $C = M\,(N_1 + h_{t_1} N_2 + K_0K_1)\,t_1 > 0$

$\Rightarrow$

$$\left\| \int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s) \right] ds \right\|_X \le C,$$

for $\forall\ u_w \in S_w$.

$\Rightarrow$ J is a bounded set in X.

Let $A = \{T(\varepsilon)J : u_w \in S_w\}$ which Is precompact set, $\forall\ u_w \in S_w$ {the compactness of the semigroup}, also let B $= T(t)u_0$ is precompact set

$\forall\, u_w \in S_w$, which implies that A+B is precompact set, for $\forall\, u_w \in S_w$.[see appendix B].

$\Rightarrow$

The set $\tilde{S}_\varepsilon(t) = \left\{\left(F_w^\varepsilon u_w\right)(t)\right\}_{u_w \in S_w} = \{(F_w^\varepsilon u_w)(t) : u_w \in S_w\}$ is precompact set in X. Moreover for any $u_w \in S_w$, we have:

$$\|(F_w u_w)(t) - (F_w^\varepsilon u_w)(t)\|_X = \|T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds + \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds - T(t)u_0 - \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds - \int_{s=0}^{t-\varepsilon} T(t-s)(Bw)(s)\,ds\|_X$$

$\Rightarrow$

$$\|(F_w u_w)(t) - (F_w^\varepsilon u_w)(t)\|_X = \left\| \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds + \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds + \int_{s=t-\varepsilon}^{0} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds + \int_{s=t-\varepsilon}^{0} T(t-s)(Bw)(s)\,ds \right\|_X$$

$\Rightarrow \|(F_w u_w)(t) - (F_w^{\varepsilon} u_w)(t)\|_X =$

$$\left\| \int_{s=t-\varepsilon}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau \right] ds + \int_{s=t-\varepsilon}^{t} T(t-s)(Bw)(s)\,ds \right\|_X$$

$\Rightarrow$

$$\|(F_w u_w)(t) - (F_w^{\varepsilon} u_w)(t)\|_X \le MN_1(t-t+\varepsilon) + M h_{t_1} N_2(t-t+\varepsilon) + MK_0K_1\,(t-t+\varepsilon)$$
$$\le (N_1 + h_{t_1} N_2 + K_0K_1)M\varepsilon$$

$\Rightarrow$

$$\|(F_w u_w)(t) - (F_w^{\varepsilon} u_w)(t)\|_X \le (N_1 + h_{t_1} N_2 + K_0K_1)M\varepsilon$$

$\Rightarrow$

$$\|(F_w u_w)(t) - (F_w^{\varepsilon} u_w)(t)\|_X \longrightarrow 0 \quad, \text{as} \quad \varepsilon \longrightarrow 0$$

i.e., $\lim_{\varepsilon\to\infty} \left(F_w^{\varepsilon} u_w\right)(t) = \left(F_w u_w\right)(t)$

Which imply that $\tilde{S}(t)$ is precompact set in X, for every fixed t > 0 {see [Bahuguna, 97], [Balachandran, 02]}.

To prove that $\tilde{S} = F_w(S_w)$ is an equicontinuous family of functions {see appendix C for the definition}. We have:

$$\|(F_w u_w)(r_1) - (F_w u_w)(r_2)\|_X = \left\| T(r_1)u_0 + \int_{s=0}^{r_1} T(r_1-s)(Bw)(s)\,ds + \int_{s=0}^{r_1} T(r_1-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau \right] ds - T(r_2)u_0 - \int_{s=0}^{r_2} T(r_2-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau \right] ds - \int_{s=0}^{r_2} T(r_2-s)(Bw)(s)\,ds \right\|_X$$

$\Rightarrow$

$$\|(F_w u_w)(r_1) - (F_w u_w)(r_2)\|_X = \|(T(r_1) - T(r_2))u_0 + \int_{s=0}^{r_1} T(r_1 - s)(Bw)(s)\,ds + \int_{s=r_2}^{0} T(r_2 - s)(Bw)(s)\,ds + \int_{s=0}^{r_1} T(r_1 - s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds + \int_{s=r_2}^{0} T(r_2 - s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right] ds\Big\|_X$$

$\Rightarrow$

$$\|(F_w u_w)(r_1) - (F_w u_w)(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + \int_{s=0}^{r_1} \|T(r_1 - s)\|\,\|Bw(s)\|_X\,ds + \int_{s=r_2}^{0} \|T(r_2 - s)\|\,\|Bw(s)\|_X\,ds + \int_{s=0}^{r_1} \|T(r_1 - s)\|\left[\|f(s,u_w(s))\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_w(\tau))\|_X\,d\tau\right] ds + \int_{s=r_2}^{0} \|T(r_2 - s)\|\left[\|f(s,u_w(s))\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_w(\tau))\|_X\,d\tau\right] ds$$

$\Rightarrow$

$$\|(F_w u_w)(r_1) - (F_w u_w)(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M K_0 K_1 r_1 + M K_0 K_1(-r_2) + M(N_1 + h_{r_1} N_2)\, r_1 + M(N_1 + h_{r_1} N_2)\,(-r_2)$$

$\Rightarrow$

$$\|(F_w u_w)(r_1) - (F_w u_w)(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M K_0 K_1 (r_1 - r_2) + M(N_1 + h_{r_1} N_2)\,(r_1 - r_2)$$

Since $\{T(t)\}_{t\geq 0}$ is a compact semigroup which implies T(t) is continuous in the uniform operator topology for t > 0 {see (1.7.1.4) Theorem}, therefore the right hand side of the above inequality tends to zero as $r_1 - r_2$ tends to zero. Thus $\tilde{S}$ is equicontinuous family of functions.

It follows from the theorem "Arzela-Ascoli's theorem" {see appendix C for the state} that is $\tilde{S}=F_w(S)$ be relatively compact in Y and by Applying "Schauder fixed point theorem" {see appendix C for the state}, which implies $F_w : S_w \longrightarrow S_w$ has a fixed point, i.e., $F_w u_w = u_w$, Hence the initial value control problem given by equation(2.5) has a local mild solution $u_w \in C([0,t_1]:X)$.

To show the uniqueness,

Let $\bar{\bar{u}}_w(t)$, $\bar{u}_w(t)$ be two local mild solutions of the initial value control problem given by equation(2.5) on the interval $[0, t_1]$. We must prove

$\| \bar{\bar{u}}_w(t) - \bar{u}_w(t) \|_X = 0$, Assume $\| \bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \neq 0$, notice that:

$$\| \bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)(Bw)(s)\, ds +$$

$$\int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_w(\tau))\,d\tau \right] ds - T(t)u_0 -$$

$$\int_{s=0}^{t} T(t-s)\left[f(s,\bar{u}_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\bar{u}_w(t))\,d\tau \right] ds -$$

$$\int_{s=0}^{t} T(t-s)(Bw)(s)\, ds \Bigg\|_X$$

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X = \left\| \int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_w(s)) - f(s,\bar{u}_w(s))\right] ds + \int_{s=0}^{t} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)\big(g(\tau,\bar{\bar{u}}_w(\tau)) - g(\tau,\bar{u}_w(\tau))\big)d\tau\right] ds \right\|_X$$

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \leq M\|f(s, \bar{\bar{u}}_w(s)) - f(s, \bar{u}_w(s))\|_X\, t_1 + M h_{t_1} \|g(\tau, \bar{\bar{u}}_w(\tau)) - g(\tau,\bar{u}_w(\tau)\|_X\, t_1$$

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \leq ML_0\|\bar{\bar{u}}_w(s) - \bar{u}_w(s)\|_X\, t_1 + M L_1 h_{t_1} \|\bar{\bar{u}}_w(\tau) - \bar{u}_w(\tau)\|_X\, t_1$$

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \leq ML_0 \sup_{0\leq t\leq t_1} \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X\, t_1 + M h_{t_1} L_1 \sup_{0\leq t\leq t_1} \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X\, t_1$$

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \leq ML_0\|\bar{\bar{u}}_w - \bar{u}_w\|_Y\, t_1 + M h_{t_1} L_1\|\bar{\bar{u}}_w - \bar{u}_w\|_Y\, t_1$$

{By using $\|u_w\|_Y = \sup_{0\leq t\leq t_1} \|u_w(t)\|_X$ }

$\Rightarrow$

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \leq M(L_0 + h_{t_1} L_1)\|\bar{\bar{u}}_w - \bar{u}_w\|_Y\, t_1$$

By using assumption (g.ii) which implies:

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X < M(L_0 + h_{t_1} L_1) * \frac{1}{M(L_0 + h_{t_1} L_1)}\|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow$

$\| \bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X < \| \bar{\bar{u}}_w - \bar{u}_w \|_Y$

By taking the suprumun over $[0, t_1]$ of the both sides of the above inequality, we get:

$\| \bar{\bar{u}}_w - \bar{u}_w \|_Y < \| \bar{\bar{u}}_w - \bar{u}_w \|_Y$, which implies to a contradiction

$\Rightarrow \| \bar{\bar{u}}_w(t) - \bar{u}_w(t) \|_X = 0 \Rightarrow \bar{\bar{u}}_w(t) = \bar{u}_w(t)$ for $0 \le t \le t_1$.

Hence we have a unique local mild solution $u_w \in C([0, t_1]:X)$, for arbitrary $w(.) \in L^p([0, t_1]: O)$.

So one can select $t_1>0$ such that:

$$t_1 = \min \left\{ t', t'', r, \frac{\rho - \rho'}{(K_0 K_1 + N_1 + h_{t_1} N_2)M}, \frac{1}{M(L_0 + L_1 h_{t_1})} \right\}$$

<u>*2.3 Exact Controllable of the Mild Solution to the Semilinear Initial Value Control Problem*</u>

In section, the controllable of the mild solution given by (2.25) to the semilinear initial value control problem given by (2.5) has been established.

We introduce the following general concept of the "exact controllability"

<u>***Definition(2.3.2) [Balakrishnan,78],[Balachandran,02]:***</u>

Given any two points u_0 , $u_\gamma \in X$ (X is a Banach space) , we say that the mild solution given by (2.25) to the semilinear initial value control problem given by (2.5) is exactly controllable on $J_0 = [0,\gamma]$, if there exist a control $\underline{w} \in L^p(J_0:O)$ such that the mild solution $u_w(.)$ of equation (2.25) satisfy the following conditions $u_{\underline{w}}(0)=u_0$ and $u_{\underline{w}}(\gamma)=u_\gamma$.

Remark(2.3.3)

Our aim is, the mild solution $u_{\underline{w}}(.)$ to the semilinear initial value control problem can be steered to a subspace of X denoted by V and it can may be transfer from $u_{\underline{w}}(0) = u_0$ to $u_{\underline{w}}(\gamma) = u_\gamma = v_0 \in V$.

The following remark which is useful for finding the control $\underline{w} \in L^p(J_0:O)$

Remark(2.3.4)

Let O be a reflexive Banach space define as follow:

$O = \{w(t) : w \in L^p(J_0:O)\}$, and X be a Banach space.

Define a linear operator W: O $\to$ X as follow:

$$Ww(t) = \int_0^t T(t-s)Bw(s)ds, \text{ for } 0 \le t \le \gamma, \forall\ w(.) \in L^p([0, r):O).$$

Or equivalently as, define $W = L(t)B = \int_0^t T(t-s)B.ds$, for $0 \le t \le \gamma$

Where B: $O \to X$ and L(t) :$X \to X$, where $L(t) = \int_0^t T(t-s)..ds$, for $0 \le t \le \gamma$

For the special case when $t=\gamma$, define a linear operator $G=L(\gamma)B$,

G: $O \to X$, as follows:

$$Gw(\gamma) = \int_0^\gamma T(\gamma - s)Bw(s)ds\ ,\ \forall\, w(.) \in L^p([0,\gamma):O).$$

Actually, we can assume, without losing generality that RangW=V and we cam construct an invertible operator $\tilde{W}$ define on O/kerW.

We are trying to simplify and discuss the construction of an invertible operator $\tilde{W}$ define on O/kerW. [See appendix A].

The controllable of the local mild solution to the semilinear initial value control problem will be developed by using the following assumptions:

(a) A be the infinitesimal generator C_0 compact semigroup $\{T(t)\}_{t\geq 0}$, where A defined from $D(A) \subset X$ into X. where X be a Banach space.

(b) For $\rho > 0$, we define $\mathcal{B}_\rho(u_0) = \{x \in X \mid \|x - u_0\|_X \leq \rho\}$, where $u_0 \in U$ (open subset of X) , The nonlinear maps f, g define from $[0,r)\times U$ into X, satisfy the local Lipschitz condition with respect to the second arguments , i.e.

$\|f(t, v_1) - f(t, v_2)\|_X \leq L_0\|v_1 - v_2\|$ and $\|g(t, v_1) - g(t, v_2)\|_X \leq L_1\|v_1 - v_2\|$

For $0 \leq t \leq t_1$ and $v_1, v_2 \in \mathcal{B}_\rho(u_0)$ and L_0, L_1 is a Lipschitz constants.

(c) h is continuous function which at least $h \in L^1([0,t_1):R)$,Where R is the real number.

(d) Let $t' > 0$ such that $\|f(t,a)\|_X \leq N_1$, $\|g(t,a)\|_X \leq N_2$, for $0 \leq t \leq t'$ and $a \in \mathcal{B}_\rho(u_0)$.

Also let $t'' > 0$ such that $\|T(t)u_0 - u_0\|_X \leq \rho'$ for $0 \leq t \leq t''$ and $u_0 \in U$, where ρ' is a positive constant such that $\rho' < \rho$.

(e) $w(.)$ be the arbitrary control function is given in $L^p([0,t_1]:O)$, a Banach space of control function with O as a reflexive Banach space and here B is a bounded linear operator from O into X.

(f) The linear operator G from O into X defined by:

$$Gw(\gamma) = \int_{s=0}^{\gamma} T(\gamma - s)Bw(s)\,ds \ , \ \forall \ w(.) \in L^p([0,\gamma): O).$$

Induces an invertible operator $\tilde{G}$ defined on O/kerG.

(g) There exist a positive constant I_1, such that $\|\tilde{G}^{-1}\| \leq I_1$.

(h) Let γ=min $\{t', t'', t_1\}$ and satisfy the following conditions

(h.i) $\gamma \le \dfrac{\rho - \rho' - I_0 I_1(\|v\| + M\|u_0\|)}{(1 + I_0 I_1)M(N_1 + h_\gamma N_2)}$

(h.ii) $\gamma < \dfrac{1}{(L_0 + h_\gamma L_1)(1 + I_0 I_1)M}$

Remark(2.3.5):

The condition (f) in our assumption can be satisfied {see appendix A}.

Theorem (2.3.6):

Assume that the hypotheses (a)-(h) are hold. Then for every u_0, $v_0 \in V \subseteq U$, there exists a fixed number,γ ,$0<\gamma<t_1$, such that (2.25) is exactly controllable on $J_0 = [0,\gamma]$.

Proof:

Using the condition (f), define the control:

$$\underline{w}(t) = \tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau) g(\tau, u(\tau)) d\tau \Big] ds \right] \tag{2.33}$$

Define the following map, given by:

$$(\phi_w u_w)(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\Big[(Bw)(s) + f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau\Big]ds \quad , \forall\ w(.) \in L^p([0, t_1): O) \quad (2.34)$$

By using (2.33) and (2.34), we have to show that (2.34) has a fixed point.

We can rewrite equation (2.34) as follows:

$$(\phi_w u_w)(t)=T(t)u_0+ \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau \right]ds + L(t)Bw(t) \quad , \forall\, w(.) \in L^p([0, t_1): O) \qquad (2.35)$$

By using (2.33) and (2.35), we obtain:

$$(\phi_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \right]ds$$

$$+ L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big]ds \right] \text{ For } \underline{w} \in L^p([0, t_1):O)$$

There exist $M \geq 0$, such that $\|T(t)\| \leq M$, for $0 \leq t \leq t_1$ {since T(t) is a bounded linear operator on X}.

Let $\rho > 0$ be such that $\mathcal{B}_\rho(u_0) = \{x \in X : \|x - u_0\|_x \leq \rho\} \subset U$ { since U is an open subset of X}.

To guarantee the fixed point property, we have done as follow:

Assume $h_r = \int_{s=0}^{r} |h(s)|ds$

We set Z=C(J_0 :X), where Z is a Banach space with the supremum defined as follows:

$$\|z\|_Z = \sup_{0\le t\le\gamma} \|z(t)\|_X$$

And define $Z_0 = \{ u_{\underline{w}} \in Z : u_{\underline{w}}(0) = u_0,\ u_{\underline{w}}(t) \in \mathcal{B}_\rho(u_0), \text{ for } 0\le t \le \gamma\}$

We note that Z_0 is bounded, Closed and convex subset of Z, (see theorem (2.2.2)).

Define a nonlinear map $\phi_{\underline{w}} : Z_0 \longrightarrow Z$, by:

$$(\phi_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \right]ds$$

$$+ L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big]ds \right], \text{ for } \underline{w} \in L^p([0,t_1):O) \quad (2.36)$$

To prove that $\phi_{\underline{w}}(Z_0) \subseteq Z_0$

Let $u_{\underline{w}}$ be arbitrary element in Z_0 such that $\phi_{\underline{w}} u_{\underline{w}} \in \phi_{\underline{w}}(Z_0)$, to prove $\phi_{\underline{w}} u_{\underline{w}} \in Z_0$ for arbitrary element $u_{\underline{w}} \in Z_0$.

from the definition of Z_0, notice that $\phi_{\underline{w}} u_{\underline{w}} \in Z$ {the definition of $\phi_{\underline{w}}$}and $(\phi_{\underline{w}} u_{\underline{w}})(0) = u_0${by equation(2.36)}, to prove $(\phi_{\underline{w}} u_{\underline{w}})(t) \in \mathcal{B}_\rho(u_0)$, for $0\le t \le \gamma$. From the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $(\phi_{\underline{w}} u_{\underline{w}})(t) \in X$ {the definition of the Banach space Z} and we have :

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X = \Bigg\| T(t)u_0 - u_0 + \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds + L(t)B\tilde{G}^{-1}\Big[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds\Big]\Bigg\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X \le \|T(t)u_0 - u_0)\|_X + \int_{s=0}^{t} \|T(t-s)\|_X \|f(s,u_{\underline{w}}(s))\|_X\, ds + \int_{s=0}^{t} \|T(t-s)\|_X \left(\int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_{\underline{w}}(\tau))\|_X\, d\tau\right)ds + \Bigg\| L(t)B\tilde{G}^{-1}\Big[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds\Big]\Bigg\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X \le \|T(t)u_0 - u_0)\|_X + \int_{s=0}^{t} \|T(t-s)\|_X \|f(s,u_{\underline{w}}(s))\|_X\, ds + \int_{s=0}^{t} \|T(t-s)\|\left(\int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_{\underline{w}}(\tau))\|_X\, d\tau\right)ds + I_0\Bigg\| \tilde{G}^{-1}\Big[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds\Big]\Bigg\|$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X \le \|T(t)u_0-u_0)\|_X + \int_{s=0}^{t}\|T(t-s)\|_X\|f(s,u_{\underline{w}}(s))\|_X\,ds + \int_{s=0}^{t}\|T(t-s)\|\left(\int_{\tau=0}^{s}|h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|_X\,d\tau\right)ds + I_0\|\tilde{G}^{-1}\|\left\|\left[v_0-T(t)u_0-\int_{s=0}^{t}T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds\right]\right\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X \le \|T(t)u_0-u_0)\|_X + \int_{s=0}^{t}\|T(t-s)\|_X\|f(s,u_{\underline{w}}(s))\|_X\,ds + \int_{s=0}^{t}\|T(t-s)\|\left(\int_{\tau=0}^{s}|h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|_X\,d\tau\right)ds + I_0I_1\left[\|v_0\|+\|T(t)u_0\|+\int_{s=0}^{t}\|T(t-s)\|\left[\|f(s,u_{\underline{w}}(s))\|+\int_{\tau=0}^{s}|h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|d\tau\right]ds\right.$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t)-u_0\|_X \le \rho' + MN_1\gamma + M\,h_\gamma N_2\,\gamma + I_0I_1\|v_0\| + I_0I_1M\|u_0\| + I_0I_1M(N_1 + h_\gamma N_2)\,\gamma$$

$\Rightarrow$

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho' + (N_1 + h_\gamma N_2) M\gamma + I_0 I_1 [M\|u_0\| + \|v_0\|] + (N_1 + h_\gamma N_2) M I_0 I_1 \gamma$$

$\Rightarrow$

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho' + [1 + I_0 I_1](N_1 + h_\gamma N_2) M\gamma + I_0 I_1 [M\|u_0\| + \|v_0\|]$$

By using the condition (h.i), we get:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho$$

Therefore $\phi_{\underline{w}} u_{\underline{w}} \in Z_0$, for any $u_{\underline{w}} \in Z_0$

$\Rightarrow \phi_{\underline{w}} : Z_0 \longrightarrow Z_0$

So, one can select $\gamma > 0$, such that:

$$\gamma = \text{Min}\left\{ t', t'', t_1, \frac{\rho - \rho' - I_0 I_1(\|v_0\| + M\|u_0\|)}{(1 + I_0 I_1) M (N_1 + h_\gamma N_2)} \right\}$$

To complete the prove, to show that $\phi_{\underline{w}} : Z_0 \longrightarrow Z_0$ is a continuous map

Given $\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z \longrightarrow 0$, as $n \longrightarrow \infty$

To prove $\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z \longrightarrow 0$, as $n \longrightarrow \infty$

Notice that:

$$\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z = \sup_{0 \le t \le \gamma} \|(\phi_{\underline{w}} u^n_{\underline{w}})(t) - (\phi_{\underline{w}} u_{\underline{w}})(t)\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_Z = \sup_{0\le t\le\gamma} \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau \Big] ds + L(t)B\tilde{G}^{-1} \Big[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau \Big] ds \Big] - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big] ds - L(t)B\tilde{G}^{-1} \Big[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big] ds \Big] \Bigg\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\| = \sup_{0\le t\le\gamma} \Bigg\| \int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau \Big] ds + L(t)B\tilde{G}^{-1} \Big[-\int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau \Big] ds \Big] - \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big] ds - L(t)B\tilde{G}^{-1} \Big[-\int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau \Big] ds \Big] \Bigg\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z = \sup_{0\le t\le\gamma} \Bigg\| \int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) - f(s,u_{\underline{w}}(s))\Big]ds \; +$$

$$\int_{s=0}^{t} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)\Big(g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau))\Big)d\tau\right]ds \; +$$

$$L(t)B\tilde{G}^{-1}\left[-\int_{s=0}^{t} T(t-s)\Big[f(s,u^n_{\underline{w}}(s)) + f(s,u_{\underline{w}}(s))\,ds\Big] \; +\right.$$

$$L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\left(\int_{\tau=0}^{s} h(s-\tau)(g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau)))d\tau\right)ds\right]\Bigg\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z =$$

$$\sup_{0\le t\le\gamma}\Bigg\|\int_{s=0}^{t} T(t-s)\left[\Big[f(s,u^n_{\underline{w}}(s)) - f(s,u_{\underline{w}}(s))\Big] + \int_{\tau=0}^{s} h(s-\tau)\Big(g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau))\Big)d\tau\right]ds$$

$$+ \, L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\left[\Big[f(s,u^n_{\underline{w}}(s)) + f(s,u_{\underline{w}}(s))\Big] + \int_{\tau=0}^{s} h(s-\tau)(g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau)))d\tau\right]ds\right]\Bigg\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \le$$

$$\sup_{0\le t\le\gamma}\Bigg\|\int_{s=0}^{t} \|T(t-s)\|\left[\Big\|f(s,u^n_{\underline{w}}(s)) - f(s,u_{\underline{w}}(s))\Big\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\Big\|g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau))\Big\|d\tau\right]ds \; +$$

$$\|L(t)B\|\;\|\tilde{G}^{-1}\|$$

$$\left[\int_{s=0}^{t} \|T(t-s)\|\left[\Big\|f(s,u^n_{\underline{w}}(s)) + f(s,u_{\underline{w}}(s))\Big\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\Big\|g(\tau,u^n_{\underline{w}}(\tau)) - g(\tau,u_{\underline{w}}(\tau))\Big\|_X\, d\tau\right]ds\right]\Bigg\|_X$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq \sup_{0\leq t\leq\gamma} \left\{ \int_{s=0}^{t} M\left[L_0 \left\| u^n_{\underline{w}}(s) - u_{\underline{w}}(s) \right\|_X + h_\gamma L_1 \left\| u^n_{\underline{w}}(\tau) - u_{\underline{w}}(\tau) \right\|_X \right] ds \right.$$

$$\left. + I_0 I_1 \left[\int_{s=0}^{t} M\left[L_0 \left\| u^n_{\underline{w}}(s) - u_{\underline{w}}(s) \right\|_X + h_\gamma L_1 \left\| u^n_{\underline{w}}(\tau) - u_{\underline{w}}(\tau) \right\|_X \right] ds \right] \right\}$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq \sup_{0\leq t\leq\gamma} \left\{ (1 + I_0 I_1) M \int_{s=0}^{t} \left[L_0 \left\| u^n_{\underline{w}}(s) - u_{\underline{w}}(s) \right\|_X + h_\gamma L_1 \left\| u^n_{\underline{w}}(\tau) - u_{\underline{w}}(\tau) \right\|_X \right] ds \right\}$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq \sup_{0\leq t\leq\gamma} \left\{ (1 + I_0 I_1) M \int_{s=0}^{t} \left[L_0 \sup_{0\leq t\leq\gamma} \left\| u^n_{\underline{w}}(t) - u_{\underline{w}}(t) \right\|_X + h_\gamma L_1 \sup_{0\leq t\leq\gamma} \left\| u^n_{\underline{w}}(t) - u_{\underline{w}}(t) \right\|_X \right] ds \right\}$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq \sup_{0\leq t\leq\gamma} \left\{ (1 + I_0 I_1) M \int_{s=0}^{t} \left[L_0 \left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_X + h_\gamma L_1 \left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_X \right] ds \right\}$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq (1 + I_0 I_1) M \int_{s=0}^{\gamma} \left[L_0 \left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_Z + h_\gamma L_1 \left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_Z \right] ds$$

$\Rightarrow$

$$\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_z \leq (1+I_0 I_1)\,(L_0 + h_\gamma L_1) \left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_Z \gamma$$

Since $\left\| u^n_{\underline{w}} - u_{\underline{w}} \right\|_Z \to 0$, as $n \to \infty$

$\Rightarrow$

$\lim_{n\to\infty} \|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_Z = 0$, i.e. $\|\phi_{\underline{w}}\, u^n_{\underline{w}} - \phi_{\underline{w}}\, u_{\underline{w}}\|_Z \to 0$, as $n \to \infty$

Assume $\tilde{R} = \phi_{\underline{w}}(Z_0)$, let $\tilde{R}(t) = \{(\phi_{\underline{w}}\, u_{\underline{w}})(t) : u_{\underline{w}} \in Z_0\}$, to show that $\tilde{R}(t)$ is a precompact set in X, for every fixed $t \in J_0$, when $t = 0$

$\Rightarrow \tilde{R}(0) = \{(\phi_{\underline{w}}\, u_{\underline{w}})(0) : u_{\underline{w}} \in Z_0\} = \{u_0\}$ which is a precompact set in X {see appendix B}.

Now, for $t > 0$, $0 < \varepsilon < t$, define:

$$(\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + \int_{s=0}^{t-\varepsilon} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds$$

$\Rightarrow$

$$(\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t) = T(t)u_0 + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds \qquad (2.37)$$

To show that for any ε, 0<ε< t.

The set $\tilde{R}_{\varepsilon}(t) = \left\{(\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t)\right\}_{u_{\underline{w}} \in Z_0} = \{(\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})\,(t) : u_{\underline{w}} \in Z_0\}$ is precompact set in X.

Let Q =

$$\left\{\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau + B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.\right.\right.$$
$$\left.\left.\left.\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds : u_{\underline{w}} \in Z_0\right\}$$

To prove the set Q is bounded in X, i.e. there exists $L' > 0$ such that $\|q\|_X \le L', \forall q \in Q.\ \forall\ u_{\underline{w}} \in Z_0$, notice that:

$$\left\|\left\{\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau + B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left[f(\theta,u_w(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_w(\tau))d\tau\right]d\theta\right]\right]ds\right\|_X \le$$

$$\int_{s=0}^{t-\varepsilon}\left\|T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau + B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.\right.\right.$$
$$\left.\left.\left.\left[f(\theta,u_w(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_w(\tau))d\tau\right]d\theta\right]\ \right]\right\|_X ds$$

$\Rightarrow$

$$\le \int_{s=0}^{t-\varepsilon}\|T(t-s-\varepsilon)\|\left[\|f(s,u_{\underline{w}}(s))\|_X + \int_{\tau=0}^{s}|h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|d\tau + K_0\|\tilde{G}^{-1}\|\left[\|v_0\| + \right.\right.$$
$$\left.\left.M\|u_0\| + \int_{\theta=0}^{s}\|T(s-\theta)\|\|f(\theta,u_{\underline{w}}(\theta))\|_X + \int_{\tau=0}^{\theta}|h(\theta-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|d\tau\right]d\theta\right]\right]ds$$

$\Rightarrow$

$$\leq \int_{s=0}^{t-\varepsilon} M\left[N_1 + h_\gamma N_2 + K_0 I_1\left[\|v_0\| + M\|u_0\| + \int_{\theta=0}^{s} M\left(N_1 + h_\gamma N_2\right)d\theta\right]\right]ds$$

$\Rightarrow$

$$\leq \int_{s=0}^{t_1} M\left[N_1 + h_\gamma N_2 + K_0 I_1\left[\|v_0\| + M\|u_0\| + \int_{\theta=0}^{t_1} M\left(N_1 + h_\gamma N_2\right)d\theta\right]\right]ds$$

$\Rightarrow$

$$\leq \int_{s=0}^{t_1} M\left[N_1 + h_\gamma N_2 + K_0 I_1\left[\|v_0\| + M\|u_0\| + M\left(N_1 + h_\gamma N_2\right)t_1\right]\right]ds$$

$\Rightarrow$

$$\leq M\left[N_1 + h_\gamma N_2 + K_0 I_1\left[\|v_0\| + M\|u_0\|\right]\right]t_1 + M\left(N_1 + h_\gamma N_2\right)t_1^2$$

Let P= $M\left[N_1 + h_\gamma N_2 + K_0 I_1\left[\|v_0\| + M\|u_0\|\right]\right]t_1 + M\left(N_1 + h_\gamma N_2\right)t_1^2 > 0$

$\Rightarrow$

$$\left\|\left\{\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau + B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.\right.\right.\right.$$
$$\left.\left[f(\theta,u_w(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta]]ds\right\|_X \leq P,\ \forall\, u_{\underline{w}} \in Z_0$$

$\Rightarrow$ Q is a bounded set in X.

Let $A' = \{T(\varepsilon)Q : u_{\underline{w}} \in Z_0\}$ which Is precompact set $\forall\, u_{\underline{w}} \in Z_0$ {the compactness of the semigroup}, also let B $=T(t)u_0$ is precompact set $\forall\, u_{\underline{w}} \in Z_0$, which implies that A+B is precompact set, $\forall\, u_{\underline{w}} \in Z_0$.[see appendix B].

$\Rightarrow$

The set $\tilde{R}_{\varepsilon}(t) = \left\{\{(\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t)\right\}_{u_{\underline{w}} \in Z_0} = \{(\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})\,(t) : u_{\underline{w}} \in Z_0\}$ is precompact set in X.

Moreover for any $u_{\underline{w}} \in Z_0$, we have:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds +$$
$$\int_{s=0}^{t} T(t-s)B\tilde{G}^{-1}\Big[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\Big[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]d\theta\Big]ds$$
$$- T(t)u_0 - \int_{s=0}^{t-\varepsilon} T(t-s)\Big[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds$$
$$- \int_{s=0}^{t-\varepsilon} T(t-s)B\tilde{G}^{-1}\Big[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\Big[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]d\theta\Big]ds\Bigg\|_X$$

$\Rightarrow$

$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}u_{\underline{w}})(t)\|_X =$

$$\Bigg\| \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds \; + \int_{s=0}^{t} T(t-s)B\tilde{G}^{-1}\Bigg[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\Bigg]ds \; - \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds - \int_{s=0}^{t-\varepsilon} T(t-s)B\tilde{G}^{-1}\Bigg[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\Bigg]ds\Bigg\|_X$$

$\Rightarrow$

$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}u_{\underline{w}})(t)\|_X =$

$$\Bigg\| \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds \; + \int_{s=0}^{t} T(t-s)B\tilde{G}^{-1}\Bigg[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\Bigg]ds + \int_{s=t-\varepsilon}^{0} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds +$$

$$\int_{s=t-\varepsilon}^{0} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.$$
$$\left.\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds\Bigg\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X =$$
$$\Bigg\|\int_{s=t-\varepsilon}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds \;+$$
$$\int_{s=t-\varepsilon}^{t} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s-t_0)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds\Bigg\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le$$
$$\int_{s=t-\varepsilon}^{t} \|T(t-s)\|\left[\|f(s,u_{\underline{w}}(s))\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,u_{\underline{w}}(\tau))\|d\tau\right]ds \;+$$
$$\int_{s=t-\varepsilon}^{t} \|T(t-s)\|\,\|B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]\|\,ds$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le \int_{s=t-\varepsilon}^{t} M(N_1 + h_\gamma N_2)ds \;+$$
$$\int_{s=t-\varepsilon}^{t} MK_0I_1\left[\|v_0\| + M\|u_0\| + \int_{\theta=0}^{s} M(N_1 + h_\gamma N_2)d\theta\right]ds$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le \int_{s=t-\varepsilon}^{t} M(N_1 + h_\gamma N_2)ds \;+$$
$$\int_{s=t-\varepsilon}^{t} MK_0I_1\big[\|v_0\| + M\|u_0\|\big]ds + M^2K_0I_1(N_1 + h_\gamma N_2)\left(\int_{s=t-\varepsilon}^{t}\int_{\theta=0}^{s} d\theta\right)ds$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le \int_{s=t-\varepsilon}^{t} M(N_1 + h_\gamma N_2)\,ds \; +$$

$$\int_{s=t-\varepsilon}^{t} MK_0I_1\left[\|v_0\| + M\|u_0\|\right]ds + M^2K_0I_1(N_1 + h_\gamma N_2)\left(\int_{s=t-\varepsilon}^{t} s\,ds\right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le \int_{s=t-\varepsilon}^{t} M(N_1 + h_\gamma N_2)\,ds \; +$$

$$\int_{s=t-\varepsilon}^{t} MK_0I_1\left[\|v_0\| + M\|u_0\|\right]ds + M^2K_0I_1(N_1 + h_\gamma N_2)\left(\left.\frac{s^2}{2}\right|_{t-\varepsilon}^{t}\right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le M(N_1 + h_\gamma N_2)(t - t + \varepsilon) +$$

$$MK_0I_1\left[\|v_0\| + M\|u_0\|\right](t - t + \varepsilon) + M^2K_0I_1(N_1 + h_\gamma N_2)\left(t\varepsilon - \frac{\varepsilon^2}{2}\right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \le M(N_1 + h_\gamma N_2)\,\varepsilon + MK_0I_1\left[\|v_0\| + M\|u_0\|\right]\varepsilon +$$

$$M^2K_0I_1(N_1 + h_\gamma N_2)\left(t\varepsilon - \frac{\varepsilon^2}{2}\right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}}\, u_{\underline{w}})(t)\|_X \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0.$$

Which implies that $\widetilde{R}(t)$ is precompact set in X for every fixed $t > 0$ {see [Bahuguna, 97], [Balachandran, 02]}.

To prove that $\tilde{R} = \phi_{\underline{w}}(Z_0)$ is an equicontinuous family of functions.

Notice that:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X = \Bigg\| T(r_1)u_0 + \int_{s=0}^{r_1} T(r_1 - s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau\Big]ds + \int_{s=0}^{r_1} T(r_1 - s)B\tilde{G}^{-1}\Big[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\Big[f(\theta, u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau\Big]d\theta\Big]ds$$

$$- T(r_2)u_0 - \int_{s=0}^{r_2} T(r_2 - s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau\Big]ds - \int_{s=0}^{r_2} T(r_2 - s)B\tilde{G}^{-1}\Big[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\Big[f(\theta, u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau\Big]d\theta\Big]ds\Bigg\|_X$$

$\Rightarrow$

$\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X =$

$$\Bigg\| T(r_1)u_0 + \int_{s=0}^{r_1} T(r_1 - s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_{\underline{w}}(\tau))d\tau\Big]ds$$

$$+\int_{s=0}^{r_1} T(r_1-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.$$

$$\left.\left[f(\theta,u\_(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds$$

$$-T(r_2)u_0 + \int_{s=r_2}^{0} T(r_2-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds +$$

$$\int_{s=r_2}^{0} T(r_2-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.$$

$$\left.\left.\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds\right\|_X$$

$\Rightarrow$

$$\|(\phi_{\underline{w}}u_{\underline{w}})(r_1) - (\phi_{\underline{w}}u_{\underline{w}})(r_2)\|_X \le \|(T(r_1)-T(r_2))u_0\|_X +$$

$$\int_{s=0}^{r_1} \|T(r_1-s)\|\left[\|f(s,u_{\underline{w}}(s))\| + \int_{\tau=0}^{s} |h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|d\tau\right]ds$$

$$+\int_{s=r_2}^{0} \|T(r_2-s)\|\left[\|f(s,u_{\underline{w}}(s))\| + \int_{\tau=0}^{s} |h(s-\tau)|\|g(\tau,u_{\underline{w}}(\tau))\|d\tau\right]ds$$

$$+\int_{s=0}^{r_1} \|T(r_1-s)\|\left\|B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\right.\right.$$

$$\left.\left.\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]\right\|ds + \int_{s=r_2}^{0} \|T(r_2-s)\|\left\|B\tilde{G}^{-1}\right.$$

$$\left.\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]\right\|ds$$

$\Rightarrow$

$$\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M(N_1 + h_{r_1} N_2)\, r_1$$

$$+ M\,(N_1 + h_{r_1} N_2)\,(-r_2) + MK_0 I_1 \left[\|v_0\| + M\|u_0\|\right] r_1$$

$$+ MK_0 I_1 \left[\|v_0\| + M\|u_0\|\right] (-r_2)$$

$$+ M^2 K_0 I_1 (N_1 + h_{r_1} N_2) \left(\int_{s=0}^{r_1} \left(\int_{\theta=0}^{s} d\theta \right) ds \right)$$

$$+ M^2 K_0 I_1 (N_1 + h_{r_1} N_2) \left(\int_{s=r_2}^{0} \left(\int_{\theta=0}^{s} d\theta \right) ds \right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M(N_1 + h_{r_1} N_2)\,(r_1 - r_2)$$

$$+ MK_0 I_1 \left[\|v_0\| + M\|u_0\|\right] (r_1 - r_2)$$

$$+ M^2 K_0 I_1 (N_1 + h_{r_1} N_2)\, \frac{r_1^2}{2} + M^2 K_0 I_1 (N_1 + h_{r_1} N_2) \left(\frac{-r_2^2}{2} \right)$$

$\Rightarrow$

$$\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M(N_1 + h_{r_1} N_2)\,(r_1 - r_2)$$

$$+ MK_0 I_1 \left[\|v_0\| + M\|u_0\|\right] (r_1 - r_2)$$

$$+ \frac{M^2 K_0 I_1 (N_1 + h_{t_1} N_2)}{2} \left(r_1^2 - r_2^2 \right)$$

Since $\{T(t)\}_{t \ge 0}$ is a compact semigroup, which implies T(t) is continuous in the uniform operator topology for t > 0, therefore the right hand side tends to zero as $r_1 - r_2$ tends to zero.

Thus $\tilde{R}$ is equicontinuous family of functions

It follows from the theorem {Arzela-Ascoli's theorem} that is $\tilde{R} = \phi_{\underline{w}}(Z_0)$ be relatively compact in Z.

By applying "schauder fixed point theorem", which implies $\phi_{\underline{w}}$ has a fixed point, i.e. $\phi_{\underline{w}} u_{\underline{w}} = u_{\underline{w}}$

Now, to show that the uniqueness:

Let $\bar{\bar{u}}_{\underline{w}}(t)$ and $\bar{u}_{\underline{w}}(t)$ be two mild solution of equation (2.5) on the interval J_0, we must prove that $\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_x = 0$. Assume that $\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X \neq 0$. Notice that:

$$\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_x = \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_{\underline{w}}(s)(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)(\tau))d\tau \right] ds$$

$$+ L(t)B\tilde{G}^{-1}\Bigg[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Bigg[f(s,\bar{\bar{u}}_{\underline{w}}(s)) +$$

$$\int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau))d\tau \Bigg] ds \Bigg] - T(t)u_0 -$$

$$\int_{s=0}^{t} T(t-s)\left[f(s,\bar{u}_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{u}_{\underline{w}}(\tau))d\tau \right] ds - L(t)B\tilde{G}^{-1}$$

$$\Bigg[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\Big[f(s,\bar{u}_{\underline{w}}(s)) +$$

$$\int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{u}_{\underline{w}}(\tau))d\tau \Bigg] ds \Bigg] \Bigg\|_X$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_X = \Big\| \int_{s=0}^{t} T(t-s)\big[f(s,\bar{\bar{u}}_{\underline{w}}(s)) - f(s,\bar{u}_{\underline{w}}(s))\big]ds \; +$$

$$\int_{s=0}^{t} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)\big(g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) - g(\tau,\bar{u}_{\underline{w}}(\tau))\big)d\tau\right]ds \; + L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\big[f(s,\bar{\bar{u}}_{\underline{w}}(s)) - f(s,\bar{u}_{\underline{w}}(s))ds\big]\right] + L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\left(\int_{\tau=0}^{s} h(s-\tau)(g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) - g(\tau,\bar{u}_{\underline{w}}(\tau)))d\tau\right)ds\right]\Big\|_X$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_X \le \int_{s=0}^{t} \|T(t-s)\|\|f(s,\bar{\bar{u}}_{\underline{w}}(s)) - f(s,\bar{u}_{\underline{w}}(s))\|ds \; + \int_{s=0}^{t} \|T(t-s)\|$$

$$\left[\int_{\tau=0}^{s} |h(s-\tau)|\|g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) - g(\tau,\bar{u}_{\underline{w}}(\tau))\|d\tau\right]ds \; + \|L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\big[f(s,\bar{\bar{u}}_{\underline{w}}(s)) + f(s,\bar{u}_{\underline{w}}(s))ds\big]\right]\|_X + \|L(t)B\tilde{G}^{-1}$$

$$\left[-\int_{s=0}^{t} T(t-s)\left(\int_{\tau=0}^{s} h(s-\tau)(g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) - g(\tau,\bar{u}_{\underline{w}}(\tau)))d\tau\right)ds\right]\|_X$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_X \le M\,\|f(s,\bar{\bar{u}}_{\underline{w}}(s)) - f(s,\bar{u}_{\underline{w}}(s))\|_X\,\gamma + M\,h_\gamma\|g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) -$$

$$g(\tau,\bar{u}_{\underline{w}}(\tau))\|_X\,\gamma + I_0I_1\left[\int_{s=0}^{t} \|T(t-s)\|\|f(s,\bar{\bar{u}}_{\underline{w}}(s)) - f(s,\bar{u}_{\underline{w}}(s))\|_X ds\right]$$

$$+I_0I_1\left[\int_{s=0}^{t} \|T(t-s)\|\left(\int_{\tau=0}^{s} |h(s-\tau)|\|g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)) - g(\tau,\bar{u}_{\underline{w}}(\tau))\|_X\,d\tau\right)ds\right]$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le ML_0\,\|\bar{\bar{u}}_{\underline{w}}(s)-\bar{u}_{\underline{w}}(s)\|_x\,\gamma + M\,h_\gamma\,L_1\,\|\bar{\bar{u}}_{\underline{w}}(\tau)-\bar{u}_{\underline{w}}(\tau)\|_x\,\gamma$$
$$+ I_0I_1ML_0\,\|\bar{\bar{u}}_{\underline{w}}(s)-\bar{u}_{\underline{w}}(s)\|_x\,\gamma + I_0I_1Mh_\gamma L_1\|\bar{\bar{u}}_{\underline{w}}(\tau)-\bar{u}_{\underline{w}}(\tau)\|_x\,\gamma$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le ML_0 \sup_{0\le t\le\gamma} \|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x\,\gamma + M\,h_\gamma L_1 \sup_{0\le t\le\gamma} \|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x\,\gamma$$
$$+ I_0I_1ML_0 \sup_{0\le t\le\gamma} \|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x\,\gamma + I_0I_1Mh_\gamma L_1 \sup_{0\le t\le\gamma} \|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x\,\gamma$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le ML_0\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma + M\,h_\gamma L_1\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma$$
$$+ I_0I_1ML_0\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma + I_0I_1M\,h_\gamma L_1\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le (1+I_0I_1)\,ML_0\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma + (1+I_0I_1)\,M\,h_\gamma L_1\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma$$

$\Rightarrow$

$$\|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le (L_0+h_\gamma L_1)\,(1+I_0I_1)\,M\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma$$

Take the spremum over $0\le t\le\gamma$ of the above inequality, we obtain:

$$\sup_{0\le t\le\gamma} \|\bar{\bar{u}}_{\underline{w}}(t)-\bar{u}_{\underline{w}}(t)\|_x \le (L_0+h_\gamma L_1)\,(1+I_0I_1)\,M\,\|\bar{\bar{u}}_{\underline{w}}-\bar{u}_{\underline{w}}\|_Z\,\gamma$$

$\Rightarrow$

$\| \bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}} \|_Z \leq (L_0 + h_\gamma L_1)(1 + I_0 I_1)\, M \, \| \bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}} \|_Z \;\gamma$

By using the condition (h.ii), we get:

$\| \bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}} \|_Z < \| \bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}} \|_Z$, which get a contradiction

$\Rightarrow \| \bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t) \|_x = 0$

$\Rightarrow \bar{\bar{u}}_{\underline{w}}(t) = \bar{u}_{\underline{w}}(t)$, for $0 \leq t \leq \gamma$

Therefore, we have a unique local mild solution $u_{\underline{w}} \in C(J_0 : X)$

So one can select the time γ Such that:

$$\gamma = \text{Min}\left\{ t', t'', t_1, \frac{\rho - \rho' - I_0 I_1 (\| v_0 \| + M \| u_0 \|)}{(1 + I_0 I_1) M (N_1 + h_\gamma N_2)}, \frac{1}{(L_0 + h_\gamma L_1)(1 + I_0 I_1) M} \right\}$$

Notice that $(\phi_{\underline{w}}\, u_{\underline{w}})(0) = u_0$ and

$$(\phi_{\underline{w}}\, u_{\underline{w}})(\gamma) = T(\gamma)u_0 + \int_{s=0}^{\gamma} T(\gamma - s)\left[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \right] ds \; +$$

$$L(\gamma) B \tilde{G}^{-1}\left[v_0 - T(\gamma)u_0 - \int_{s=0}^{\gamma} T(\gamma - s)\left[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \right] ds \right]$$

$\Rightarrow (\phi_{\underline{w}}\, u_{\underline{w}})(\gamma) = v_0$

Thus equation (2.25) is exactly controllable on J_0.

2.4 Global Existence of the Mild Solution to the Semilinear Initial Value Control Problem

In this section the global existence of the mild solution to the semilinear initial value control problem has been developed.

The following remark which is useful here:

Remark (2.4.1)

(1) As a result of the theorem (2.2.2), we get a first unique local mild solution u_w define on $[0,t_1]$ for every $w \in L^p([0, t_1]:O)$, given by:

$$u_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[Bw(s) + f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau \right]ds,$$

with t_1 is defining as follow:

$$t_1 = \min\left\{ t', t'', r, \frac{\rho-\rho'}{(K_0K_1+N_1+h_{t_1}N_2)M}, \frac{1}{M(L_0+L_1h_{t_1})} \right\}$$

(2) To extend the local mild solution u_w give by (2.25) to a maximal interval $[0, t_{max}]$, we define the 2^{nd} problem as follow:

$$\left.\begin{array}{l} \dfrac{dv}{dt} + Av(t) = f(t,v(t)) + \displaystyle\int_{s=t_1}^{t} h(t-s)g(s,v(s))\,ds + (Bw)(t),\ t > t_1 \\ v(t_1) = u(t_1) \end{array}\right\} (2.38)$$

We have to show that the existence of the unique local mild solution

$v_w \in C([t_1, t_2]:X)$ for some t_2, $t_1 < t_2 < \infty$.

The local existence and uniqueness of the mild solution v_w given by (2.38) have been developed, by assuming the following assumptions:

(a) A be the infinitesimal generator of C_0 compact semigroup $\{T(t)\}_{t\geq 0}$, where A is defined from $D(A) \subset X$ into X. where X be a Banach space.

(b) Let $\xi > 0$, such that $\mathcal{B}_\xi(u_1) = \{x \in X : \|x - u_1\|_x \leq \xi\}$.

The nonlinear maps f, g define from $[t_1,\infty)\times X$ into X, satisfy the Lipschitz condition with respect to the second argument, i.e.

$\|f(t, v_1) - f(t, v_2)\|_X \leq L'_1\|v_1 - v_2\|_X$ and $\|g(t, v_1) - g(t, v_2)\|_X \leq L'_2\|v_1 - v_2\|_X$

For $t_1 \leq t < \infty$ and $v_1, v_2 \in \mathcal{B}_\xi(u_1)$, and L'_1, L'_2 are Lipschitz constants.

(c) h is continuous function which at least $h \in L^1([t_1,\infty):R)$,Where R is the real number.

(d) Let $\tau' > t_1$, $\|f(t, v)\|_x \leq N'_1$, $\|g(t, v)\|_x \leq N'_2$, for $t_1 \leq t \leq \tau'$, where N'_1 and N'_2 are positive constants. also let $\tau'' > t_1$, $\|T(t - t_1)u_1 - u_1\|_X \leq \xi'$, for $t_1 \leq t \leq \tau''$, where ξ' is a positive constant, such that $\xi' < \xi$.

(e) $w(.)$ be the control function is given in $L^p([t_1,\infty]:O)$, a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X with $\|w(t)\|_O \leq K'_1$, for $t_1 \leq t < \infty$.

(f) Let $t_2 > t_1$ such that $t_2 = \min\{\tau',\tau''\}$ satisfy the following conditions:

(f.i) $t_2 \leq t_1 + \dfrac{\xi - \xi'}{(K'_0K'_1 + N'_1 + h_{t_2}N'_2)M}$

(f.ii) $t_2 < t_1 + \dfrac{1}{M(L'_1 + h_{t_2}L'_2)}$

We introduce the following lemma:

Lemma (2.4.2)

Assume that hypotheses (a)-(f) are hold, then for every $u_1 \in X$, there exists a fixed number t_2, $t_1<t_2<\infty$, such that the semilinear initial value control problem given by (2.37) has a unique local mild solution $v_w \in C([t_1, t_2]:X)$, for every control function $w \in L^p([t_1, \infty]:O)$.

Proof:

Without loss of generality, we may suppose $r<\infty$, because we are concerned here with the local existence only.

There exist $M \geq 0$, such that $\|T(t)\| \leq M'$, for all $0 \leq t \leq t_2$ {since T(t) is a bounded linear operator on X}.

Assume $h_r = \int_{s=0}^{r} |h(s)| ds$

Set $H = C([t_1,t_2]:X)$, where H is a Banach space with the sup-norm defined as follows:

$$\|\Upsilon\|_H = \sup_{t_1 \leq t \leq t_2} \|\Upsilon(t)\|_X$$

and define $H_0 = \{v_w \in H \mid v_w(t_1) = u_1,\ v_w(t) \in B_\xi(u_1)$, for $w \in L^p([t_1, \infty]: O)$,

$t_1 \leq t \leq t_2\}$

It is clear that H_0 is bounded, convex and closed subset of H {see theorem (2.2.2)}.

Define a map $A_w : H_0 \longrightarrow H$, by the following:

$$(A_w v_w)(t) = T(t-t_1)u_1 + \int_{s=t_1}^{t} T(t-s)\Big[(Bw)(s) + f(s, v_w(s)) + \int_{\tau=t_1}^{s} |h(s-\tau)| \, \|g(\tau, v_w(\tau))\| d\tau\Big] ds$$

To show that $A_w(H_0) \subseteq H_0$, let v_w be an arbitrary element in H_0 such that $A_w v_w \in A_w(H_0)$, to prove $A_w v_w \in H_0$.

From the definition of H_0, notice that $A_w v_w \in H$ and $(A_w v_w)(t_1) = u_1$ {see theorem (2.2.2.)}, to prove $(A_w v_w)(t) \in \mathcal{B}_\xi(u_1)$, for $t_1 \le t \le t_2$

From the definition of the closed ball $\mathcal{B}_\xi(u_1)$, notice that:

$(A_w v_w)(t) \in X$ and $\|(A_w v_w)(t) - u_1\|_x \le \xi$, for $t_1 \le t \le t_2$ { see theorem (2.2.2.)}.

Therefore $A_w(H_0) \subseteq H_0$, for an arbitrary element v_w in H_0,

$\Rightarrow A_w: H_0 \to H_0$

So one can select $t_2 > t_1$ such that $t_2 = \min\left\{\tau', \tau'', t_1 + \dfrac{\xi - \xi'}{(K_0' K_1' + N_1' + h_{t_2} N_2')M}\right\}$

To complete the prove, the map $A_w: H_0 \to H_0$ is continuous {see theorem (2.2.2)}

Suppose that $\tilde{P} = A_w(H_0)$, and for fixed $t \in [t_1, t_2]$,

Let $\tilde{P}(t) = \{(A_w v_w)(t) : v_w \in H_0\}$

To show that $\tilde{P}(t)$ is a precompact set for every fixed $t \in [t_1, t_2]$,

When $t = t_1$, then $\tilde{P}(t_1) = \{(A_w v_w)(t_1) : v_w \in H_0\} = \{u_1\}$ which is precompact set in X {see theorem (2.2.2)}.

Now, for $t > t_1$, $0 < \varepsilon < t - t_1$, define:

$$(A_w^{\varepsilon} v_w)(t) = T(t-t_1)u_1 + \int_{s=t_1}^{t-\varepsilon} T(t-s)\Big[f(s,v_w(s)) + \int_{\tau=t_1}^{s} |h(s-\tau)|\|g(\tau,v_w(\tau))\|d\tau + (Bw)(s)\Big]ds \qquad (2.39)$$

To show that for every ε, $0 < \varepsilon < t - t_1$, the set $\tilde{P}_{\varepsilon}(t) = \{(A_w^{\varepsilon} v_w)(t) : v_w \in H_0\}$ is precompact set in X. {see theorem (2.2.2)}.

Moreover for any $v_w \in H_0$, we have:

$$\|(A_w v_w)(t) - (A_w^{\varepsilon} v_w)(t)\|_X = \Big\| T(t-t_1)u_1 + \int_{s=t_1}^{t} T(t-s)\Big[f(s,v_w(s)) + \int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))d\tau + (Bw)(s)\Big]ds - T(t-t_1)u_1 - \int_{s=t_1}^{t-\varepsilon} T(t-s)\Big[f(s,v_w(s)) + \int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))d\tau + (Bw)(s)\Big]ds\Big\|_X$$

$$\Rightarrow \|(A_w v_w)(t) - (A_w^{\varepsilon} v_w)(t)\|_X = \Big\| \int_{s=t-\varepsilon}^{t} T(t-s)\Big[f(s,v_w(s)) + \int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))d\tau + (Bw)(s)\Big]ds\Big\|_X$$

$\Rightarrow$

$$\|(A_w v_w)(t) - (A_w^{\varepsilon} v_w)(t)\|_X \le (N'_1 + h_{t_1} N'_2 + K'_0 K'_1)M\varepsilon$$

$\Rightarrow$

$\|(A_w v_w)(t)-(A_w^{\varepsilon} v_w)(t)\|_X \to 0$, as $\varepsilon \to 0$.

Which imply that $\tilde{P}(t)$ is precompact in X for every fixed $t > t_1$. {See [Bahuguna, 97], [Balachandran, 02]}.

To prove $\tilde{P} = A_w(H_0)$ is an equicontinuous family of functions {see theorem (2.2.2)}.

It follows from "Arzela-Ascoli's theorem" that is $\tilde{P} = A_w(H_0)$ be relatively compact in H_w.

Applying "Schauder fixed point theorem" which implies

$A_w : H_0 \longrightarrow H_0$ has a fixed point, i.e., $A_w v_w = v_w$, $v_w \in H_0$

Hence the initial value control problem given by (2.37) has a local mild solution $v_w \in C([t_1, t_2]:X)$, for every $w \in L^p([t_1, \infty]: O)$.

To show that the uniqueness:

Let $\bar{v}_w(t)$ and $\bar{\bar{v}}_w(t)$ be two local mild solutions of the initial value control problem given by (2.37) on the interval $[t_1, t_2]$.

We must prove $\|\bar{v}_w(t) - \bar{\bar{v}}_w(t)\|_x = 0$ {see theorem (2.2.2)}.

Hence we have a unique local mild solution $v_w \in C([t_1, t_2]:X)$

For t_2, $t_1 < t_2 < \infty$.

So one can select $t_2 > t_1$ such that

$$t_2 = \min \left\{\tau', \tau'', t_1 + \frac{\xi - \xi'}{(K_0' K_1' + N_1' + h_{t_t} N_2')M}, t_1 + \frac{1}{M(L_1' + h_{t_1} L_2')}\right\}$$

So, we conclude that there exist two unique local mild solutions u_w , v_w to the semilinear initial value control problem (2.5) and (2.38) respectively. Given as follow:

$$u_w(t)=T(t)u_0+\int_{s=0}^{t}T(t-s)\left[Bw(s)+f(s,u_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds$$

For $0\leq t\leq t_1$ and every given $w(.)\in L^p([0,t_1]:O)$. And

$$v_w(t)=T(t-t_1)u_1+\int_{s=t_1}^{t}T(t-s)\big[(Bw)(s)+f(s,v_w(s))+$$

$$\left.\int_{\tau=t_1}^{s}h(s-\tau)g(\tau,v_w(\tau))d\tau\right]ds$$, For $t_1\leq t\leq t_2$ and every given $w\in L^p([t_1,t_2]:O)$.

Now, we define a function $\tilde{u}_w:[0,t_2]\longrightarrow X$, by:

$$\tilde{u}_w(t)=\begin{cases}u_w(t), & 0\leq t\leq t_1\\ v_w(t), & t_1\leq t\leq t_2\end{cases}$$

Then $\tilde{u}_w\in C([0,t_2]:X)$ and $\tilde{u}_w$ satisfies the integral equation:

$$\tilde{u}_w(t)=T(t)u_0+\int_{s=0}^{t}T(t-s)\left[f(s,\tilde{u}_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,\tilde{u}_w(\tau))d\tau+Bw(s)\right]ds$$

For $0\leq t\leq t_2$, $\forall\ w(.)\in L^p([0,t_2]:O)$.

To see this, for $0\leq t\leq t_1$, we have:

$\tilde{u}_w(t)=u_w(t)=$

$$T(t)u_0+\int_{s=0}^{t}T(t-s)\left[Bw(s)+f(s,\tilde{u}_w(s))+\int_{\tau=0}^{s}h(s-\tau)g(\tau,\tilde{u}_w(\tau))d\tau\right]ds$$

Also, for $t_1\leq t\leq t_2$, we have:

$$v_w(t)=T(t-t_1)u_1+\int_{s=t_1}^{t} T(t-s)\Big[(Bw)(s)+f(s,v_w(s))+\int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))\,d\tau\Big]ds$$

$\Rightarrow$

$$v_w(t)=T(t-t_1)\Big[T(t_1)u_0+\int_{s=0}^{t_1} T(t_1-s)\Big[f(s,u_w(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau+Bw(s)\Big]ds+\int_{s=t_1}^{t} T(t-s)\Big[(Bw)(s)+f(s,v_w(s))+\int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))\,d\tau\Big]ds$$

$\Rightarrow$

$$v_w(t)=T(t)u_0+\int_{s=0}^{t_1} T(t-s)\Big[f(s,u_w(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau+Bw(s)\Big]ds+\int_{s=t_1}^{t} T(t-s)\Big[f(s,v_w(s))+\int_{\tau=t_1}^{s} h(s-\tau)g(\tau,v_w(\tau))\,d\tau+(Bw)(s)\Big]ds$$

For $t_1 \le t \le t_2$, notice that : $\tilde{u}_w(t) = v_w(t) = T(t)u_0 + \int_{s=0}^{t_1} T(t-s)\Big[f(s,\tilde{u}_w(s)) +$

$$\int_{\tau=0}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau))\,d\tau + Bw(s)\Big]ds + \int_{s=t_1}^{t} T(t-s)\Big[f(s,\tilde{u}_w(s)) +$$

$$\int_{\tau=t_1}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau))\,d\tau + (Bw)(s)\Big]ds$$

$\Rightarrow$

$\tilde{u}_w(t) = T(t)u_0 +$

$$\int_{s=0}^{t} T(t-s)\big[f(s,\tilde{u}_w(s)) + Bw(s)\big]ds +$$

$$\int_{s=0}^{t_1} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau))d\tau\right]ds +$$

$$\int_{s=t_1}^{t} T(t-s)\left[\int_{\tau=t_1}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau))d\tau\right]ds$$

$\Rightarrow$

$\tilde{u}_w(t) = T(t)u_0 +$

$$\int_{s=0}^{t} T(t-s)\big[f(s,\tilde{u}_w(s)) + Bw(s)\big]ds +$$

$$\int_{s=0}^{t_1}\int_{\tau=0}^{s} T(t-s)h(s-\tau)g(\tau,\tilde{u}_w(\tau)d\tau ds +$$

$$\int_{s=t_1}^{t}\int_{\tau=t_1}^{s} T(t-s)h(s-\tau)g(\tau,\tilde{u}_w(\tau)d\tau ds$$

$$\text{Let } J = \int_{s=0}^{t_1}\int_{\tau=0}^{s} T(t-s)h(s-\tau)g(\tau,\tilde{u}_w(\tau)d\tau ds + \int_{s=t_1}^{t}\int_{\tau=t_1}^{s} T(t-s)h(s-\tau)g(\tau,\tilde{u}_w(\tau)d\tau ds$$

$\Rightarrow$

$$J = \int_{s=0}^{t} T(t-s)\left[\int_{\tau=0}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau)d\tau\right]ds \quad , \{\text{see [Bahuguna, 97]}\}$$

$\Rightarrow$

$$\tilde{u}_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,\tilde{u}_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\tilde{u}_w(\tau))d\tau + Bw(s)\right]ds$$

for $t_1 \le t \le t_2$.

so we have a unique local mild solution $\tilde{u}_w$ to the semilinear initial value control problem given by (2.5) on the interval $[0, t_2]$.

(3) From the steps {(1) and (2)}, we can extend the local mild solution $\tilde{u}_w$ to a maximal interval $[0, t_{max}]$, with $0 < t_{max} < \infty$.

(4) We impose some assumptions to obtain a global existence of the mild solution $\tilde{u}_w \in C([0,\infty]:X)$.

The following theorem which is useful for obtaining the global existence of the mild solution.

Theorem (2.4.3):

Let A is the infinitesimal generator of a C_0 compact semigroup T(t), t > 0 on X. If f, g: $[t_0,\infty)\times X \longrightarrow X$ are continuous and map bounded sets in

$[0,\infty)\times X$ into bounded sets in X, h is a real–valued continuous function which at least $h\in L^1([0,\infty):R)$ and $w(.)$ be an arbitrary control function is given in $L^P([0,\infty):O)$, a Banach space of control functions with O as a Banach space and here B is a bonded linear operator from O into X with $\|w(t)\|_O \le I_1$ for $0\le t<\infty$,Then

For $t_{max}<\infty$, $\lim_{t\uparrow t_{max}} \|u_w(t)\|_X$. For arbitrary control function $w\in L^P([0,\infty):O)$.

Proof:

To show that $\lim_{t\uparrow t_{max}} \| u_w(t) \|_X = \infty$, if this is false, then there is a sequence $t_n\uparrow t_{max}$ and a constant K, such that $\|u_w(t_n)\|_x \le K$, for all n

$\exists$ M>0 such that $\|T(t)\| \le M$, for $0\le t\le t_{max}$ {T(t) is a bounded linear operator} and let:

$$N_1 = \sup_{0\le t\le t_{max}} \{\|f(t,v)\|_x : 0 \le t \le t_{max}, \|v\| \le M(K + 1)\}$$

$$N_2 = \sup_{0\le t\le t_{max}} \{\|g(t,v)\|_x : 0 \le t \le t_{max}, \|v\| \le M(K + 1)\}$$

We can find a sequence $\{h_n\}$ with following properties:

$h_n \longrightarrow 0$ as $n \longrightarrow \infty$, $\|u_w(t)\|_x\le M(K + 1)$, for $t_n \le t \le t_n + h_n$ and $\|u_w(t_n + h_n)\|_X = M(K + 1)$ {see [Paz,83]}.But then we have:

$$M (K + 1) = \|u_w(t_n + h_n)\|_x = \Bigg\| T(t_n + h_n)u_0 + \int_{s=0}^{t_n+h_n} T(t_n+h_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau + (Bw)(s) \right]ds \Bigg\|_X$$

$\Rightarrow$

$$M\,(K+1) = \|u_w(t_n+h_n)\|_X = \Bigg\| T(h_n)T(t_n)u_0 + \int_{s=0}^{t_n} T(t_n+h_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds + \int_{s=t_n}^{t_n+h_n} T(t_n+h_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds\Bigg\|_X$$

$\Rightarrow$

$$M\,(K+1) = \|u_w(t_n+h_n)\|_X = \Bigg\| T(h_n)[T(t_n)u_0 + \int_{s=0}^{t_n} T(t_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds + \int_{s=t_n}^{t_n+h_n} T(t_n+h_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds\Bigg\|_X$$

$\Rightarrow$

$$M\,(K+1) = \|u_w(t_n+h_n)\|_X \le \Bigg\| T(h_n)[T(t_n)u_0 + \int_{s=0}^{t_n} T(t_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds\Bigg\|_X + \left\|\int_{s=t_n}^{t_n+h_n} T(t_n+h_n-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + (Bw)(s)\right]ds\right\|_X$$

$\Rightarrow$

$$M\,(K+1) \le \|T(h_n)u_w(t_n)\|_X + \left\| \int_{s=t_n}^{t_n+h_n} T(t_n+h_n-s) \left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau) g(\tau,u_w(\tau))\,d\tau + (Bw)(s) \right] ds \right\|_X$$

$\Rightarrow$

$$M\,(K+1) \le \|T(h_n)u_w(t_n)\|_X + \int_{s=t_n}^{t_n+h_n} \|T(t_n+h_n-s)\| \left[\|f(s,u_w(s))\| + \int_{\tau=0}^{s} |h(s-\tau)|\, \|g(\tau,u_w(\tau))\|_X\,d\tau + \|(Bw)(s)\| \right] ds$$

$\Rightarrow$

$$M\,(K+1) = \|u_w(t_n+h_n)\|_x \le MK + M\,[N_1 + h_{max} N_2 + I_0 I_1](t_n+h_n) - t_n]$$

$$\le MK + M\,(N_1 + h_{max} N_2 + I_0 I_1)\, h_n$$

$$\le M\,[K + (N_1 + h_{max} N_2 + I_0 I_1)\, h_n]$$

Which gives a contradiction as $h_n \longrightarrow 0$

Hence $\lim\limits_{t \uparrow t_{max}} \|u(t)\|_X = \infty$.

This completes the proof.

Corollary(2.4.4):

Let A is the infinitesimal generator of a C_0 compact semigroup T(t), t > 0 on X. If f, g: $[t_0,\infty)\times X \longrightarrow X$ are continuous and map bounded sets in $[0,\infty)\times X$ into bounded sets in X, h is a real–valued continuous function which at least $h\in L^1([0,\infty):R)$ and $w(.)$ be an arbitrary control function is given in $L^P([0,\infty):O)$, a Banach space of control functions with O as a Banach space and here B is a bonded linear operator from O into X with $\|w(t)\|_O \le K_1$ for $0\le t<\infty$, Then for every $u_0\in X$, the initial value control problem given by (2.5), has a global solution $u_w\in C([0,\infty):X)$ if either one of the following conditions are satisfied:

(i) There exists a continuous function $\tau : [0,\infty) \longrightarrow [0,\infty)$, such that:

$\|u_w(t)\|_x \le \tau(t)$, for every t in the interval of existence of u_w.

(ii) There exist functions $K_i : [0,\infty) \longrightarrow (0,\infty)$, i = 1, 2, 3, such that K_1, K_2, K_3 are continuous functions on $[0,\infty)$, and for $0 \le t < \infty$, $v \in X$:

$\|f(t, v)\|_x \le K_1(t)\, \|v\| + K_2(t)$

$\|g(t, v)\|_x \le K_3(t)$

Proof:

Part (i), To prove that the solution u_w has a global solution on X, i.e., we must prove that the solution u_w exist on the interval $[0,\infty)$

Assume that the solution u_w does not has a global solution on X,

Since the solution u_w exist on the interval $[0, t_{max}]$, for $t_{max} < \infty$

By assumption (i), we get $\|u_w(t)\|_x \le \tau(t)$, for every $t \in [0,t_{max}]$

$\Rightarrow$

$$\lim_{t\uparrow t_{max}} \| u_w (t) \|_X \le \lim_{t\uparrow t_{max}} \tau(t)$$

$\Rightarrow$

$$\lim_{t\uparrow t_{max}} \| u_w(t) \|_X < \infty \quad C!$$

Therefore, u_w has a global solution on X.

Part (ii) There exist functions $K_i : [0, \infty) \longrightarrow [0,\infty)$, i = 1, 2, 3, such that K_1, K_2, K_3 are continuous functions on $[0,\infty)$, and for $0 \le t < \infty$, $v \in X$:

$$\|f(t, v)\|_x \le K_1(t)\ \|v\| + K_2(t) \tag{2.39}$$

$$\|g(t, v)\|_x \le K_3(t) \tag{2.40}$$

The Part (ii) can be reducing it to (i) as follow:

Assume that the solution u_w exist on the $[0, t)$ for $0 \le t < t_{max}$, given by:

$$u_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\Big[(Bw)(s) + f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\Big]ds \tag{2.41}$$

There exist $w \in R$ and $M \ge 1$, such that $\|T(t)\| \le Me^{tw}$, $\forall\ t \ge 0$ {by the proposition (1.4.4)}.

Taking the norm of equation (2.41) and then multiplying by e^{-wt}, we get:

$$\|u_w(t)\|_x \le \|T(t)\|\ \|u_0\| + \int_{s=0}^{t} \|T(t-s)\|\Big[\|(Bw)(s)\| + \|f(s,u_w(s))\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\ \|g(\tau,u_w(\tau))\|_X\, d\tau\Big]ds$$

$\Rightarrow$

$$\|u_w(t)\|_x \le M\|u_0\|e^{wt} + M\int_{s=0}^{t} e^{w(t-s)}\Big[K_0K_1 + K_1(s)\|u_w(s)\| + K_2(s) + \int_{\tau=0}^{s} |h(s-\tau)|\ K_3(\tau)\,d\tau\Big]ds \quad \text{\{by (2.39) and (2.40)\}}$$

And multiplying by e^{-wt}, we get:

$$e^{-wt}\ \|u_w(t)\| \le M\|u_0\| + M\int_{s=0}^{t} e^{-ws}\Big[K_0K_1 + K_1(s)\,\|u_w(s)\| + K_2(s)\ +$$

$$\int_{\tau=0}^{s} |h(s-\tau)|\ K_3(\tau)\,d\tau\Big]ds$$

For $0 \le t < \infty$, set:

$$\psi(t) = M\|u_0\| + M\int_{s=0}^{t} e^{-ws}\Big[K_0K_1 + K_2(s) + \int_{\tau=0}^{s} |h(s-\tau)|\ K_3(\tau)\,d\tau\Big]ds$$

The function $\psi:[0,\infty) \longrightarrow (0,\infty)$, thus defined is obviously continuous on $[0,\infty)$.

$$\Rightarrow\ e^{-wt}\ \|u_w(t)\|_x \le \psi(t) + M\int_{s=0}^{t} e^{-ws}K_1(s)\|u_w(s)\|_X\,ds$$

By Gronwall's inequality {see appendix C for the state} implies that:

$$e^{-wt}\|u_w(t)\|_x \le \psi(t)\,e^{M\int_{s=0}^{t} K_1(s)\,ds}$$

$\Rightarrow$

$$\|u_w(t)\|_x \le e^{tw}\,\psi(t)\ e^{M\int_{s=0}^{t} K_1(s)\,ds}$$

From the last inequality implies that there exists a continuous function

$Ч(t) = e^{tw}\,\psi(t)\ e^{M\int_{s=0}^{t} K_1(s)\,ds}$ such that $\|u_w(t)\|_x \le e^{tw}\,\psi(t)\ e^{M\int_{s=0}^{t} K_1(s)\,ds}$

For every t in the interval of existence of u_w.

CONCLUSIONS AND FUTURE WORK

Conclusions:

1. The basic preliminaries of understanding this subject are infinite dimensional spaces and theory of semi group and some non-linear functional analysis. This subject is very important in applications in control theory area.
2. The existence and uniqueness as well as controllability problems have represented the main objects of real life dynamical control system, so studying these titles providing as a good work studying of this subject.
3. The subject of the problem is based mainly on the semi group theory and the provided as a good approach or approach or tools to solve the problem not only to ensure the existence and uniqueness.
4. This present approach is difficult, but its scope is wider than others and covering a large class of non-linear integral, integro, distributed, differential system with the system with initial or even boundary conditions.

Future Work:

1. Developing the present approach to some optimum control in infinite dimensional spaces.
2. Developing numerical procedures to find the solution numerically or even exact in some infinite dimensional spaces.
3. Developing numerical procedures to find the solution numerically or even exact in approximate finite dimensional spaces.

Contents

Dedication

To the candles which burn themselves to illuminate my way

My Parents

To my Sister.

Manaf

EXAMINING COMMITTEE CERTIFICATION

We certify that we have read this thesis entitled **"Solvability and Controllability of Semilinear Initial Value Control Problem Via Semigroup Approch"** and as examining committee examined the student **(Manaf Adnan Salah)** in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Introduction

The theory of one parameter semigroups of linear operators on Banach spaces started earlier, acquired its core in 1948 with the Hille-Yosida generation theorem, and attained its first apex with the 1957 edition of “semigroups and functional analysis” by E. Hille and R. S. Phillips. In the 1970's and 80's, the theory reached a certain state of perfection, which is well represented in the monographs by [Dav, 80], [Gol, 85], [Paz, 83] and others. Today, the situation is characterized by manifold application of this theory not only to the traditional areas such as partial differential equations or stochastic processes. Semigroup has become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. The theory of control is one of the major areas of application of mathematics today. From its early inception to meet the demands of automatic control system design in engineering, it has grown steadily in scope and now has spread too many other far removed areas such as economics. The theory of semigroups of linear operators lends a convenient setting and offers many advantages. Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after a well-developed semigroup theory was at hand. A control system uses many ideas from the standard control methodology except in control it is often said that a formal mathematical model is assumed unavailable so that mathematical analysis is imposible.We emphasize, however, that mathematical analysis cannot alone provide the definitive answers about the control system since such analysis proves properties about the model of the process, not the actual physical process. It is important to note that the advantages of control often become most apparent for very complex problems where we have an intuitive idea about how to achieve high performance control. The work of this thesis is divided into three chapters; the first chapter entitled "Some basic concepts of semigroup theory" which is recalls some Definitions, basic concepts, propositions, theorems and some properties of the semigroup theory which are important for the discussion of our later results. In chapter two entitled "Existence, uniqueness and controllability of mild solution to the semilinear initial value control problem via "Schauder fixed point theorem" , by using the theory of semigroup and “schauder fixed point theorem”, the local existence, uniqueness and the exact controllable of the mild solution to the semilinear initial value control problem has been developed in an arbitrary Banach space X. Sufficient conditions for the global existence of the mild solution to the semilinear initial value control problem has also been developed. In chapter three entitled Existence and uniqueness of mild solution to the semilinear initial value control problem via "Banach contraction principle", by using the theory of semigroup and “Banach contraction principle”, the local existence and uniqueness of the mild solution to the semilinear initial value control problem has been developed in an arbitrary Banach space X, associated with the unbounded linear operator generating strongly continuous semigroup $\{T(t)\}_{t\geq 0}$.

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Supervisors Certification

We certify that this thesis was prepared under our supervision at the department of mathematics and computer applications in the College of Science, Al-Nahrian University as a partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics.

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In view of the available recommendations; I forward this thesis for debate by the examining committee.

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Table of Notations

$\{T(t)\}_{t\geq 0}$	Family of bounded linear operators
U	Open set
O, X, Y	Banach space
B	Bounded linear operator
w	Control function
u_w	Continuous function depend on w
$L^p([0, r):O)$	Banach space of p-integrable functions with its domain [0,r) into O such that $\|f\|_p = \left(\int_{t=0}^{r} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty$
u_w^n	Sequence of Continuous function depend on w
$\Im_F(E)$	The set of all continuous function define from E into F
$\mathcal{B}_\rho(u_0)$	Closed ball with center u_0 and radius ρ
$A^{-\alpha}$	bounded linear operator
X_α	Banach space depend on α
A^α	The inverse operator of $A^{-\alpha}$
$\{S(t)\}_{t\geq 0}$	Family of bounded linear operators
$H_0^k[0,1]$	Classical Sobolev space of order (k, 2)
$C^1[0,1]$	The space of 1-times continuously differentiable functions

$C([0,1]:X)$	The space of continuous functions define from [0, 1] into X
$\mathbb{C}$	The set of complex numbers
R^{+}	The set of positive real numbers
$\mathbb{C}^{n}$	Unitary space
$\ell(X)$	Banach algebra (the set of all bounded linear operators define from X into X)
$M_n(\mathbb{C})$	The space of all $n \times n$ matrices with components of complex number
$\rho(A)$	Resolvent set of the operator A
$\sigma(A)$	Spectrum set of the operator A
$R(\lambda; A)$	Resolvent operator
$C_0(\Omega)$	The space of continuous functions vanishing at infinity
Ω	Compact set
M_q	The multiplication operator
$D(M_q)$	The domain of the operator M_q
C_0	Strongly continuous semigroup
A_λ	Yosida approximation" of A
C_α	Positive constant depend on α
ACP	Abstract Cauchy problem
IACP	Inhomogeneous abstract Cauchy problem
S_w	The set depend on control function *w*
$\bar{\bar{u}}_w, \bar{u}_w$	Continuous function depend on *w*

F_w	Mapping depend on *w*
$\tilde{S}$	The set of continuous function
W	Linear operator
O/kerW	Quotient space
[*w* (t)]	An equivalent classes of *w*(t)
$\underline{w}$	Control function belong to [*w* (t)]
$\phi_{\underline{w}}$	Mapping depend on the control function $\underline{w}$
$u_{\underline{w}}^{n}$	Sequence of continuous function depend on the control function $\underline{w}$
ψ, Ч	Continuous map
V	Subspace of X
$\tilde{W}^{-1}$	The inverse linear operator of $\tilde{W}$
$+\partial U$	Smooth positively oriented boundary

$$\|\cdot\|_Y = \sup_{0 \le t \le t_1} \|.(t)\|_X$$

$$\|\cdot\|_\alpha = \left\|A^\alpha .\right\|_X$$

$$\|[\cdot]\|_{O/KerW} = \inf_{. \in [.]} \|\cdot\|_O$$

وزارة التعليم العالي والبحث العلمي
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كلية العلوم

أمكانية وجود حل و قابلية السيطرة لمسألة سيطرة شبه خطية ذات قيمة أبتدأية بواسطة أسلوب شبه الزمرة

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Solvability and Controllability of Semilinear Initial Value Control Problem Via Semigroup Approach

A Thesis

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