# *APPENDIX (A)*

## ***A.1 THE JORDAN FORM [32]***

We saw that the representation of the transition matrix can be facilitated by the diagonalizing the matrix A. This diagonalization is not possible if the $n \times n$ matrix A does not have $n$ linearly independent characteristic vectors. In this case, however it is possible to bring A into the so-called *Jordan normal form* which is almost diagonal and from which the transition matrix can easily be obtained.

We first recall a few facts from linear algebra. If M is a matrix, the *null spase* of M is defined as

$$\mathrm{N}(\mathrm{M}) = \{x : x \in \ell^n, \mathrm{M}x = 0\} \quad \text{(A.1)}$$

where $\ell^n$ is the $n$-dimensional complex vector space . Furthermore, if $\mathrm{M}_1$ and $\mathrm{M}_2$ are two linear subspaces of an $n$-dimensional space, a linear subspace $\mathrm{M}_3$ is said to be the *direct sum* of $\mathrm{M}_1$ and $\mathrm{M}_2$, written as

$$\mathrm{M}_3 = \mathrm{M}_1 \oplus \mathrm{M}_2 \quad \text{(A.2)}$$

if any vector $x_3 \in \mathrm{M}_3$ can be written in one and only one way as $x_3 = x_1 + x_2$ where $x_1 \in \mathrm{M}_1$ and $x_2 \in \mathrm{M}_2$.

***Theorem (A.1) :[32]***

Suppose that the $n \times n$ matrix A has $k$ distinct characteristic values $\lambda_i$, $i = 1,2,...,k$. Let the multiplicity of each characteristic value $\lambda_i$ in the characteristic polynomial of A be given by $m_i$.

Define

$$M_i = (\mathrm{A} - \lambda_i \mathrm{I})^{m_i} \quad \text{(A.3)}$$

and let

$$N_i = N(M_i) \quad \text{(A.4)}$$

Then

**(a)** The dimension of the linear subspace $N_i$ is $m_i$, $i = 1,2,...,k$;

**(b)** The whole *n*-dimensional complex space $\ell^n$ is the direct sum of the null spaces $N_i$, $i = 1,2,...,k$, that is

$$\ell^n = N_1 \oplus N_2 \oplus ... \oplus N_k \tag{A.5}$$

When the matrix A has *n* distinct characteristic values, the null spaces $N_i$ reduced to one-dimensional subspaces each of which is spanned by a characteristic vector of A.

***<u>Theorem (A.2) :[32]</u>***

Consider the matrix A with the same notation as in the theorem (A.1). Then it is always possible to find a nonsingular transformation matrix $T$ which can be partitioned as

$$T = (T_1, T_2, \ldots, T_k) \tag{A.6}$$

such that

$$A = TJT^{-1} \tag{A.7}$$

where

$$J = diag\,(J_1, J_2, \ldots, J_k) \tag{A.8}$$

The block $J_i$ has dimensions $m_i \times m_i$, $i = 1,2,...,k$ and the partitioning of $T$ matches that $J$. The columns of $T_i$ form a specially chosen basis for the null space $N_i$, $i = 1,2,...,k$. The blocks $J_i$ can be subpartitioned as

$$J_i = diag\,(J_{i1}, J_{i2}, \ldots, J_{il_i}) \tag{A.9}$$

where each subblock $J_{ij}$ is of the form

$$J_{ij} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & \cdots \\ 0 & \lambda_1 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix} \qquad \text{(A.10)}$$

$J$ is called the ***Jordan nomal form*** of A.

***<u>Theorem (A.3) :[32]</u>***

Consioder the matrix A with the same notation as in theorems (A.1) and (A.2). Then

**(a)** $e^{At} = Te^{Jt}T^{-1}$ (A.11)

**(b)** $e^{Jt} = diag(e^{J_1 t}, e^{J_2 t}, ..., e^{J_k t})$ (A.12)

**(c)** $e^{J_{i1} t} = diag(e^{J_{i1} t}, e^{J_{i2} t}, ..., e^{J_{il_i} t})$ (A.13)

**(d)** $$e^{J_{ij} t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_{ij}-1}}{(n_{ij}-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_{ij}-2}}{(n_{ij}-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix} \qquad \text{(A.14)}$$

where $n_{ij}$ is the dimension of $J_{ij}$.

It is seen from this theorem that the response of the system

$$\dot{x}(t) = Ax(t) \qquad \text{(A.15)}$$

may contain besides purely exponential terms of the form $\exp(\lambda_i t)$ also terms of the form $t \exp(\lambda_i t)$, $t^2 \exp(\lambda_i t)$, and so on.

***Theorem (A.4) :[32]***

Consider the time-invariant linear system

$$\dot{x}(t) = \mathrm{A}x(t) \qquad \text{(A.16)}$$

Express the initial state $x(0)$ as

$$x(0) = \sum_{i=1}^{k} v_i \quad \text{with} \quad v_i \in N_i,\ \ i = 1,2,...,k \qquad \text{(A.17)}$$

Write

$$T^{-1} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_k \end{pmatrix}, \qquad \text{(A.18)}$$

where the partitioning corresponds to that of $T$ in theorem (A.2).

Then the response of the system can be expressed as

$$x(t) = \sum_{i=1}^{k} T_i \exp(J_i) U_i v_i \qquad \text{(A.19)}$$

From this theorem we see that if the initial state is within one of the null spaces $N_i$, the nature of the response of the system to this initial state is completely determined by the corresponding characteristic value . We call the response of the system to any initial state within one of the null spaces a *mode* of the system.

# *APPENDIX (B)*

***Program ( 1 )***

***Computational matrices of observer when E is invertible matrix***

***[Illustrations (2.1)]***

```
N = [-5   0   0;0   -4     2;0    2   -10]
M = eig(N)     {Eigenvalues of N}
A1= [0   1;0   0;-0.6   -1.5]
A2 = [0;1;-0.9]
E1=[4  2;0  -2;0  0]
E2=[0;5;3]
J1=[1  0;0   1;0   0]
J2=[0;0;1]
B = [0;0;1]
F = E2'        {The Transpose of  matrix  E2}
U = F*E2
V =inv(U)
R=J2*V*F  {The computational matrix R}
D = R*E1
K = J1-D
W = R*A1
P = N*R*E1
L = W-P   {The computational matrix L}
G = R*B   {The computational matrix G}
```

## *Program ( 2 )*

## *Ricatti Algabraic equations when E is invertible matrix*

## *[Illustrations (2.1)]*

```
N = [-5  0  0;0  -4  2;0  2  -10]
V = eig(N)   {Eigenvalues of N}
J = N'           {The Transpose of the matrix N}
I = [1  0  0 ;0  1  0;0  0  1 ]
S = N+I
F = eig(S)   {Eigenvalues of S}
D = S'
T = eig(D)
B = [0;0;0]
C = [1  0  0 ;0  1  0]
H = C'{The Transpose of the matrix C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
          {The commands of algebraic Ricatti equation}

U = eig(P)
W = inv(P){Inverse of matrix P}
MM = det(P) {Determinant of matrix P}
```

## *Program(3)*

### *Compute the pseudo-inverse of the matrix E of illustrations (2.2)*

E=[1 2 3;-1 0 7;1 2 3]

J=det(E) **{ Determinant of the matrix E }**

[U,S,V]=svd(E) **{ The commands of singular value decomposition }**

F=E' **{ The transpose of the matrix E }**

T=E*F **{ Compute the matrix** $EE^T$ **}**

D=det(T) **{ Determinant of the matrix** $EE^T$ **}**

R=eig(T) **{ The eigenvalues of the matrix** $EE^T$ **}**

TT=F*E **{ Compute the matrix** $E^TE$ **}**

DD=det(TT) **{ Determinant of the matrix** $E^TE$ **}**

RR=eig(TT) **{ The eigenvalues of the matrix** $E^TE$ **}**

Q=U' **{ The transpose of the matrix U }**

QQ=V' **{ The transpose of the matrix V }**

W=[8.3275 0;0 2.9414] **{ Diagonal matrix of** $D_r$ **}**

HH=inv(W) **{ Inverse of diagonal matrix** $D_r$ **}**

HHH=[0.1201 0 0;0 0.34 0;0 0 0];

EE=QQ*HHH*Q **{ Compute the pseudo-inverse of the matrix E }**

## *Program ( 4 )*

## *Computational matrices of observer when E is singular value decomposition [Illustrations (2.2)]*

N = [-3 0 0;0 -2 1;0 1 -5]

M = eig(N) **{Eigenvalues of N}**

$A_1$ = [0 1;0 0;-0.3 -0.6]

$A_2$ = [0;1;0.4]

$E_1$ = [1 2;-1 0;1 2]

$E_2$ = [3;7;3]

$J_1$ = [1 0;0 1;0 0]

$J_2$ = [0;0;1]

B = [0;0;1]

F=$E_2$' **{TheTranspose of matrix $E_2$}**

U=F*$E_2$

V=inv(U)

R=$J_2$*V*F **{The computational matrix R}**

D=R*$E_1$

K=$J_1$-D

W=R*$A_1$

P=N*R*$E_1$

L=W-P **{The computational matrix L}**

G = R*B **{The computational matrix G}**

## *Program ( 5 )*

## *Ricatti Algabraic equations when E is singular value decomposition*

## *[Illustrations (2.2)]*

```
N = [-3  0  0;0  -2  1;0  1  -5]
V = eig(N) {Eigenvalues of N}
J = N' {The Transpose of the matrix N}
I = [1  0  0;0  1  0;0  0  1]
S = N+I
F = eig(S) {Eigenvalues of S}
D = S'
T = eig(D)
B = [0;0;0]
C = [1  0  0 ;0  1  0]
H = C'   {The Transpose of the matrix C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
          {The commands of algebraic Ricatti equation}
U = eig(P)
W = inv(P)   {Inverse of matrix P}
MM = det(P)   {Determinant of matrix P}
```

***Program(6)***

***Compute the pseudo-inverse of the matrix E of illustrations (2.3)***

E=[1 3 -4;2 -1 2;-9 15 0]

J=det(E) **{ Determinant of the matrix E }**

[U,S,V]=svd(E) **{ The commands of singular value decomposition }**

F=E' **{ The transpose of the matrix E }**

T=E*F **{ Compute the matrix $EE^T$ }**

D=det(T) **{ Determinant of the matrix $EE^T$ }**

R=eig(T) **{ The eigenvalues of the matrix $EE^T$ }**

TT=F*E **{ Compute the matrix $E^TE$ }**

DD=det(TT) **{ Determinant of the matrix $E^TE$ }**

RR=eig(TT) **{ The eigenvalues of the matrix $E^TE$ }**

Q=U' **{ The transpose of the matrix U }**

QQ=V' **{ The transpose of the matrix V }**

W=[17.7169 0;0 5.2069] **{ Diagonal matrix of $D_r$ }**

HH=inv(W) **{ Inverse of diagonal matrix $D_r$ }**

HHH=[0.0564 0 0;0 0.1921 0 ;0 0 0];

EE=QQ*HHH*Q **{ Compute the pseudo-inverse of the matrix E }**

## *Program ( 7 )*

## *Computational matrices of observer when E is singular value decomposition [Illustrations (2.3)]*

N = [-3 0 0;0 -6 3;0 3 -3]

M = eig(N) **{Eigenvalues of N}**

$A_1$ = [0 1;0 0;-0.3 -1.2]

$A_2$ = [0;1;-0.9]

$E_1$ = [1 3;-2 1;-9 15]

$E_2$ = [-4;2;0]

$J_1$ = [1 0;0 1;0 0]

$J_2$ = [0;0;1]

B = [0;0;1]

F=$E_2$' **{TheTranspose of matrix $E_2$}**

U=F*$E_2$

V=inv(U)

R=$J_2$*V*F **{The computational matrix R}**

D=R*$E_1$

K=$J_1$-D

W=R*$A_1$

P=N*R*$E_1$

L=W-P **{The computational matrix L}**

G = R*B **{The computational matrix G}**

## *Program ( 8 )*

## *Ricatti Algabraic equations when E is singular value decomposition*

## *[Illustrations (2.3)]*

```
N = [-3   0   0;0   -6   3;0   3   -3]
V = eig(N)      {Eigenvalues of N}
J = N'       {The Transpose of the matrix N}
I = [1  0  0;0  1  0;0  0  1]
S = N+I
F = eig(S)       {Eigenvalues of S}
D = S'
T = eig(D)
B = [0;0;0]
C = [1   0   0 ;0   1   0]
H = C'       {The Transpose of the matrix C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
            {The commands of algebraic Ricatti equation}
U = eig(P)
W = inv(P)       {Inverse of matrix P}
MM = det(P)    {Determinant of matrix P}
```

## *Program(9)*

### *Compute the pseudo-inverse of the matrix E of illustrations (2.4)*

E=[5 10 17;10 20 34;17 34 61]

J=det(E) **{ Determinant of the matrix E }**

[U,S,V]=svd(E) **{ The commands of singular value decomposition }**

F=E' **{ The transpose of the matrix E }**

T=E*F **{ Compute the matrix $EE^T$ }**

D=det(T) **{ Determinant of the matrix $EE^T$ }**

R=eig(T) **{ The eigenvalues of the matrix $EE^T$ }**

TT=F*E **{ Compute the matrix $E^TE$ }**

DD=det(TT) **{ Determinant of the matrix $E^TE$ }**

RR=eig(TT) **{ The eigenvalues of the matrix $E^TE$ }**

Q=U' **{ The transpose of the matrix U }**

QQ=V' **{ The transpose of the matrix V }**

W=[85.0595 0;0 0.9405] **{ Diagonal matrix of $D_r$ }**

HH=inv(W) **{ Inverse of diagonal matrix $D_r$ }**

HHH=[0.0118 0 0;0 1.0633 0;0 0 0];

EE=QQ*HHH*Q **{ Compute the pseudo-inverse of the matrix E }**

## *Program ( 10 )*

### *Computational matrices of observer when E is singular value decomposition [Illustrations (2.4)]*

N = [-5 0 0;0 -3 2;0 2 -7]

M = eig(N) **{Eigenvalues of N}**

$A_1$ = [0 1;0 0;-0.4 -1.6]

$A_2$ = [0;1;-0.2]

$E_1$ = [5 10;10 20;17 34]

$E_2$ = [17;34;61]

$J_1$ = [1 0;0 1;0 0]

$J_2$ = [0;0;1]

B = [0;0;1]

F=$E_2$' **{TheTranspose of matrix $E_2$}**

U=F*$E_2$

V=inv(U)

R=$J_2$*V*F **{The computational matrix R}**

D=R*$E_1$

K=$J_1$-D

W=R*$A_1$

P=N*R*$E_1$

L=W-P **{The computational matrix L}**

G = R*B **{The computational matrix G}**

## *Program ( 11)*

## *Ricatti Algabraic equations when E is singular value decomposition*

## *[Illustrations (2.4)]*

```
N = [-5   0   0;0   -3   2;0   2   -7]
V = eig(N)      {Eigenvalues of N}
J = N'      {The Transpose of the matrix N}
I = [1  0  0;0  1  0;0  0  1]
S = N+I
F = eig(S)      {Eigenvalues of S}
D = S'
T = eig(D)
B = [0;0;0]
C = [1   0   0 ;0   1   0]
H = C'      {The Transpose of the matrix C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
          {The commands of algebraic Ricatti equation}
U = eig(P)
W = inv(P)      {Inverse of matrix P}
MM = det(P)    {Determinant of matrix P}
```

***Program(12)***

***Compute the pseudo-inverse of the matrix E [ illustrations (2.5)]***

E=[1 -1 2 3 -1 1;4 5 0.1 0.2 0.25 0.3;5 4 0.2 0.1 0.3 0.25;0.9 0.3 0.4 0.125 0.75 0.15;1 -1 2 3 -1 1;0.3 0.9 0.125 0.65 0.85 0.75]

J=det(E) **{ Determinant of the matrix E }**

[U,S,V]=svd(E) **{ The commands of singular value decomposition }**

F=E' **{ The transpose of the matrix E }**

T=E*F **{ Compute the matrix** $EE^T$ **}**

D=det(T) **{ Determinant of the matrix** $EE^T$ **}**

R=eig(T) **{ The eigenvalues of the matrix** $EE^T$ **}**

TT=F*E **{ Compute the matrix** $E^TE$ **}**

DD=det(TT) **{ Determinant of the matrix** $E^TE$ **}**

RR=eig(TT) **{ The eigenvalues of the matrix** $E^TE$ **}**

Q=U' **{ The transpose of the matrix U }**

QQ=V' **{ The transpose of the matrix V }**

W=[9.1204 0 0 0 0;0 5.849 0 0 0;0 0 1.4065 0 0;0 0 0 1.1031 0;0 0 0 0 0.3189] **{ Diagonal matrix of** $D_r$ **}**

HH=inv(W) **{ Inverse of diagonal matrix** $D_r$ **}**

HHH=[0.1096 0 0 0 0 0;0 0.171 0 0 0 0;0 0 0.711 0 0 0;0 0 0 0.9065 0 0;0 0 0 0 3.1358 0;0 0 0 0 0 0];

EE=QQ*HHH*Q **{ Compute the pseudo-inverse of the matrix E }**

## *Program( 13 )*

## *Computational matrices of observer when E is singular value decomposition [Illustrations (2.5)]*

```
N=[-5 0 0 0 0 0;0 -6 1 2 3 4;0 1 -4 1 4 3;0  0 2 -3 2 5;0 0 0 0 -2 4;0 0 0 0 0 -9]
M = eig(N)          {Eigenvalues of N}
A1= [0 1 0 0;0 0 1 0;0 0 0 1;0 0 0 0;0  0 0 0;-0.25 -0.15 -0.3 -0.4]
A2 = [0 0;0 0;0 0;1 0;0 1;-0.175 -0.1285]
E1=[1 -1 2 3;4 5 0.1 0.2;5 4 0.2 0.1;0.9 0.3 0.4 0.125;1 -1 2 3;0.3 0.9 0.125
0.65]
E2=[-1 1;0.25 0.3;0.3 0.25;0.75 0.15;-1 1;0.85 0.75]
J1=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1;0 0 0 0;0 0 0 0]
J2=[0 0;0 0;0 0;0 0;1 0;0 1]
B = [0;0;0;0;0;1]
F=E2'          {TheTranspose of matrix E2}
U=F*E2
V=inv(U)
R=J2*V*F    {The computational matrix R}
D=R*E1
K=J1-D
W=R*A1
P=N*R*E1
L=W-P     {The computational matrix L}
G = R*B    {The computational matrix G}
```

## *Program ( 14 )*

## *Ricatti Algabraic equations when E is singular value decomposition*

## *[Illustrations (2.5)]*

```
N=[-5 0 0 0 0 0;0 -6 1 2 3 4;0 1 -4 1 4 3;0  0 2 -3 2 5;0 0 0 0 -2 4;0 0 0 0 0 -9]
SS=eig(N)        {Eigenvalues of N}
J = N'           {TheTranspose of matrix N}
I = [1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0;0 0 0 1 0 0;0 0 0 0 1 0;0 0 0 0 0 1]
S = N+I
F = eig(S)   {Eigenvalues  of  S}
D = S'
T = eig(D)
B = [0;0;0;0;0;0]
C = [1 0 0 0 0 0 ;0 1 0 0 0 0;0 0 1 0 0 0;0 0 0 1 0 0]
H = C'    {The  transpose  of  the  matrix  C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
            {The  commands  of algebraic  Ricatti equation}
U = eig(P)
W = inv(P)     {Inverse of  matrix P}
MM = det(P)   {Determinant  of  matrix P}
```

## *Program(15)*

### *Compute the pseudo-inverse of the matrix E of illustrations (2.6)*

E=[-1 -2 -3 -1 1;0.1 0.4 0.25 0.1 0.9;0.4 0.1 0.35 0.4 0.75;0.5 0.3 0.11 0.5 0.03;0.3 0.5 0.13 0.3 0.125]

J=det(E) **{ Determinant of the matrix E }**

[U,S,V]=svd(E) **{ The commands of singular value decomposition }**

F=E' **{ The transpose of the matrix E }**

T=E*F **{ Compute the matrix** $EE^T$ **}**

D=det(T) **{ Determinant of the matrix** $EE^T$ **}**

R=eig(T) **{ The eigenvalues of the matrix** $EE^T$ **}**

TT=F*E **{ Compute the matrix** $E^TE$ **}**

DD=det(TT) **{ Determinant of the matrix** $E^TE$ **}**

RR=eig(TT) **{ The eigenvalues of the matrix** $E^TE$ **}**

Q=U' **{ The transpose of the matrix U }**

QQ=V' **{ The transpose of the matrix V }**

W=[4.08 0 0 0;0 1.391 0 0;0 0 0.6341 0;0 0 0 0.3868]

**{ Diagonal matrix of** $D_r$ **}**

HH=inv(W) **{ Inverse of diagonal matrix** $D_r$ **}**

HHH=[0.2451 0 0 0 0;0 0.7189 0 0 0;0 0 1.577 0 0;0 0 0 2.5833 0;0 0 0 0 0 ];

EE=QQ*HHH*Q **{ Compute the pseudo-inverse of the matrix E }**

***Program( 16 )***

***Computation matrices of observer when E is singular value decomposition [Illustrations (2.6)]***

N=[-6 0 0 0 0;0 -7 2 3 4;0 2 -5 2 5;0 0 -3 -4 3;0 0 0 0 -9]

M = eig(N) **{Eigenvalues of N}**

$A_1$= [0 1 0 0;0.1 0.2 0.3 0.4;0.2 0.1 0.4 0.3;0.11 0.15 0.17 0.19;0.15 0.11 0.19 0.17]

$A_2$ = [0;0.5;0.6;0.1;0.12]

$E_1$=[-1 -2 -3 -1;0.1 0.4 0.25 0.1;0.4 0.1 0.35 0.4;0.5 0.3 0.11 0.5;0.3 0.5 0.13 0.3]

$E_2$=[1;0.9;0.75;0.03;0.125]

$J_1$=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1;0 0 0 0;]

$J_2$=[0;0;0;0;1]

B = [0;1;0.2;0.45;0.1]

F=$E_2$' **{TheTranspose of matrix $E_2$}**

U=F*$E_2$

V=inv(U)

R=$J_2$*V*F **{The computational matrix R}**

D=R*$E_1$

K=$J_1$-D

W=R*$A_1$

P=N*R*$E_1$

L=W-P **{The computational matrix L}**

G = R*B **{The computational matrix G}**

## *Program ( 17 )*

## *Ricatti Algabraic equations when E is singular value decomposition*

## *[Illustrations (2.6)]*

```
N=[-6 0 0 0 0;0  -7  2  3  4;0  2  -5  2  5;0  0  -3  -4  3;0  0 0  0 -9]
SS=eig(N)        {Eigenvalues of N}
J = N'            {TheTranspose of matrix N}
I = [1 0 0 0 0;0 1 0 0 0;0 0 1 0 0;0 0 0 1 0;0 0 0 0 1]
S = N+I
F = eig(S)    {Eigenvalues  of  S}
D = S'
T = eig(D)
B = [0;0;0;0;0]
C = [1 0 0 0 0;0 1 0 0 0;0 0 1 0 0;0 0 0 1 0]
H = C'    {The  transpose  of  the  matrix  C}
Q = 2*H*C
[P,L,G,rr] = care(S,B,Q)
                {The  commands  of algebraic  Ricatti equation}
U = eig(P)
W = inv(P)     {Inverse of  matrix P}
MM = det(P)   {Determinant  of  matrix P}
```

# APPENDIX ( C )

***Problem (1) of illustrations (2.1) case(1) to find the solutions of $x_1$, $x_2$ and $x_3$***

$$f = inline('[0.25*x(1)+0.875*x(2)+0.625*x(3)-0.4167+ 0.00025*\cos(x(1)+x(2))+0.0005*\sin(x(2))-0.0004167*\sin(x(3))+ 0.00075*\sin(x(1))*\cos(x(2))-0.0004167*\cos(x(3)); -0.5*x(1)-1.25*x(2)-1.25*x(3)+0.8333-0.001*\sin(x(2))+ 0.0008333*\sin(x(3))-0.0015*\sin(x(1))*\cos(x(2))+ 0.0008333*\cos(x(3));-0.2*x(1)-0.5*x(2)-0.3*x(3)+0.3333+ 0.0003333*\sin(x(3))+0.0003333*\cos(x(3))]','t','x');$$

$$[t,xa] = ode45(f,[0:0.01:30],[0.5 \quad -0.5 \quad 1.5])$$

***Problem (2) of illustrations (2.1) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$***

$$f1 = inline('[-9*e(1);-16*e(2)+2*e(3);56*e(2)-10*e(3)+ 0.0002942*\sin(-103.7635)-0.0002942*\sin(-103.7635-e(2))+ 0.0000882*\sin(161.4271)-0.0000882*\sin(161.4271-e(3))+ 0.0004413*\sin(-109.4921)*\cos(-103.7635)- 0.0004431*\sin(-109.4921-e(1))*\cos(-103.7635-e(2))+ 0.0000882*\cos(161.4271)-0.0000882*\cos(161.4271-e(3))]','t','e');$$

$$[t,ea] = ode45(f1,[0:0.01:30],[0.00001 \quad 0.00002 \quad 0.000001])$$

***Problem (3) of illustrations (2.1) case(2) to find the solutions of $x_1$, $x_2$ and $x_3$***

```
f = inline('[0.25*x(1)+0.875*x(2)+0.625*x(3)-0.4167*sin(t)+
    0.00025*sin(t)*cos(x(1)+x(2))+0.0005*sin(t)*sin(x(2))-
    0.0004167*sin(t)*sin(x(3))+0.00075*sin(x(1))*cos(x(2))-
    0.0004167*cos(x(3));-0.5*x(1)-1.25*x(2)-1.25*x(3)+
    0.8333*sin(t)-0.001*sin(t)*sin(x(2))+
    0.0008333*sin(t)*sin(x(3))-0.0015*sin(x(1))*cos(x(2))+
    0.0008333*cos(x(3));-0.2*x(1)-0.5*x(2)-0.3*x(3)+
    0.3333*sin(t)+0.0003333*sin(t)*sin(x(3))+
    0.0003333*cos(x(3))],'t','x');

[t,xa] = ode45(f,[0:0.01:4],[0.5 -0.5 1.5])
```

***Problem (4) 0f illustrations (2.1) case(2) to find the error solutions of $e_1$, $e_2$ and $e_3$***

```
f1 = inline('[-9*e(1);-16*e(2)+2*e(3);56*e(2)-10*e(3)+
    0.0002942*sin(t)*sin(-2.2586)-
    0.0002942*sin(t)*sin(-2.2586-e(2))+
    0.0000882*sin(t)*sin(2.2609)-
    0.0000882*sin(t)*sin(2.2609-e(3))+
    0.0004413*sin(0.0321)*cos(-2.2586)-
    0.0004431*sin(0.0321-e(1))*cos(-2.2586-e(2))+
    0.0000882*cos(2.2609)-0.0000882*cos(2.2609-e(3))]','t','e');

[t,ea] = ode45(f1,[0:0.01:4],[0.00001 0.00002 0.000001])
```

***Problem (5) 0f illustrations (2.2) case(1) to find the solutions***

***$x_1$, $x_2$ and $x_3$***

$$f = inline('[0.0115*x(1)-0.0153*x(2)+0.0212*x(3)-0.0382-0.00191*\sin(x(1))-0.0001146*\cos(x(3))-0.0001528*\cos(x(1))+0.0001825*\sin(x(1))*\cos(x(2));-0.0388*x(1)+0.0517*x(2)-0.159*x(3)+0.1293+0.006465*\sin(x(1))+0.0003879*\cos(x(3))+0.0005172*\cos(x(1))+0.0010535*\sin(x(1))*\cos(x(2));-0.046*x(1)+0.0613*x(2)+0.0306*x(3)+0.01533+0.007665*\sin(x(1))+0.0004599*\cos(x(3))+0.0006132*\cos(x(1))-0.0001535*\sin(x(1))*\cos(x(2))]','t','x');$$

$$[t,xa] = ode45(f,[0:0.01:6],[0.5 \quad -0.5 \quad 1.5])$$

***Problem (6) 0f illustrations (2.2) case(1) to find the error solutions***

***of $e_1$, $e_2$ and $e_3$***

$$f1 = inline('[-5*e(1);-7*e(2)+e(3);21*e(2)-5*e(3)+0.00224*\sin(0.6494)-0.00224*\sin(0.6494-e(1))+0.0001344*\cos(2.2189)-0.0003144*\cos(2.2189-e(3))+0.0001792*\cos(0.6494)-0.0001792*\cos(0.6494-e(1))+0.0005225*\sin(0.6494)*\cos(-1.9961)-0.0005225*\sin(0.6494-e(1))*\cos(-1.9961-e(2))]','t','e');$$

$$[t,ea] = ode45(f1,[0:0.01:6],[0.00001 \quad 0.00002 \quad 0.000001])$$

***Problem (7) 0f illustrations (2.2) case(2) to find the solutions of $x_1$, $x_2$ and $x_3$***

$$f = inline('[0.0115*x(1)-0.0153*x(2)+0.0212*x(3)-0.0382*\sin(t)- 0.00191*\sin(t)*\sin(x(1))-0.0001146*\sin(t)*\cos(x(3))- 0.0001528*\cos(x(1))+0.0001825*\sin(x(1))*\cos(x(2));-0.0388*x(1)+ 0.0517*x(2)-0.159*x(3)+0.1293*\sin(t)+ 0.006465*\sin(t)*\sin(x(1))+0.0003879*\sin(t)*\cos(x(3))+ 0.0005172*\cos(x(1))+0.0010535*\sin(x(1))*\cos(x(2));-0.046*x(1)+ 0.0613*x(2)+0.0306*x(3)+0.01533*\sin(t)+ 0.007665*\sin(t)*\sin(x(1))+0.0004599*\sin(t)*\cos(x(3))+ 0.0006132*\cos(x(1))-0.0001535*\sin(x(1))*\cos(x(2))]','t','x');$$

$$[t,xa] = ode45(f,[0:0.01:10],[0.5 \quad -0.5 \quad 1.5])$$

***Problem (8) 0f illustrations (2.2) case(2) to find the error solutions of $e_1$, $e_2$ and $e_3$***

$$f1 = inline('[-5*e(1);-7*e(2)+e(3);21*e(2)-5*e(3)+ 0.00224*\sin(t)*\sin(1.0996)- 0.00224*\sin(t)*\sin(1.0996-e(1))+ 0.0001344*\sin(t)*\cos(0.5897)- 0.0003144*\sin(t)*\cos(0.5897-e(3))+ 0.0001792*\cos(1.0996)-0.0001792*\cos(1.0996-e(1))+ 0.0005225*\sin(1.0996)*\cos(-3.6582)- 0.0005225*\sin(1.0996-e(1))*\cos(-3.652-e(2))]','t','e');$$

$$[t,ea] = ode45(f1,[0:0.01:10],[0.00001 \quad 0.00002 \quad 0.000001])$$

***Problem (9) 0f illustrations (2.3) case(1) to find the solutions of $x_1$, $x_2$ and $x_3$***

```
f =inline('[0.0115*x(1)+0.1897*x(2)−0.0438*x(3)−0.0383+
    0.0001437*sin(x(1))*cos(x(2))−0.000782*cos(x(2))*sin(x(3))−
    0.000766*sin(x(2))*cos(x(1))+0.0004311*cos(x(3))−
    0.000383*cos(x(1)+x(2));−0.0067*x(1)+0.019*x(2)−
    0.03914*x(3)+0.0222+0.0000456*sin(x(1))*cos(x(2))−
    0.000191*cos(x(2))*sin(x(3))+0.000444*sin(x(2))*cos(x(1))+
    0.0001368*cos(x(3))+0.000222cos(x(1)+x(2));−0.0107*x(1)+
    0.0374*x(2)−0.0664*x(3)+0.0358+
    0.0000803*sin(x(1))*cos(x(2))−0.000342*cos(x(2))*sin(x(3))+
    0.000716*sin(x(2))*cos(x(1))+0.0002409*cos(x(3))+
    0.000358*cos(x(1)+x(2))]','t','x');

[t,xa]=ode45(f,[0:0.01:100],[0.5 −1 −1.5])
```

***Problem (10) 0f illustrations (2.3) case(1) to find the error solutions of $e_1$ , $e_2$ and $e_3$***

```
f1=inline('[−5*e(1);−13*e(2)+3*e(3);7.6667*e(2)−3*e(3)−
    0.0002*sin(−2.1217)*cos(0.7487)+
    0.0002*sin(−2.1217−e(1))*cos(0.7487−e(2))+
    0.001*cos(0.7487)*sin(1.3182)−
    0.001*cos(0.7487−e(2))*sin(1.3182−e(3))−
    0.0006*cos(1.3182)+0.0006*cos(1.3182−e(3))]','t','e');

[t,ea]=ode45(f1=,[0:0.01:100],[0.00001 0.00002 0.000001])
```

***Problem (11) 0f illustrations (2.3) case(2) to find the solutions of $x_1$, $x_2$ and $x_3$***

```
f = inline('[0.0115*x(1)+0.1897*x(2)-0.0438*x(3)-
    0.0383*cos(t)+0.0001437*cos(t)*sin(x(1))*cos(x(2))-
    0.000782*cos(t)*cos(x(2))*sin(x(3))-
    0.000766*cos(t)*sin(x(2))*cos(x(1))+0.0004311*cos(x(3))-
    0.000383*cos(x(1)+x(2));-0.0067*x(1)+0.019*x(2)-
    0.03914*x(3)+0.0222*cos(t)+
    0.0000456*cos(t)*sin(x(1))*cos(x(2))-
    0.000191*cos(t)*cos(x(2))*sin(x(3))+
    0.000444*cos(t)*sin(x(2))*cos(x(1))+
    0.0001368*cos(x(3))+0.000222cos(x(1)+x(2));-0.0107*x(1)+
    0.0374*x(2)-0.0664*x(3)+0.0358*cos(t)+
    0.0000803*cos(t)*sin(x(1))*cos(x(2))-
    0.000342*cos(t)*cos(x(2))*sin(x(3))+
    0.000716*cos(t)*sin(x(2))*cos(x(1))+0.0002409*cos(x(3))+
    0.000358*cos(x(1)+x(2))]','t','x');
[t,xa]=ode45(f,[0:0.01:50],[0.5 -1 -1.5])
```

***Problem (12) 0f illustrations (2.3) case(2) to find the error solutions of $e_1$, $e_2$ and $e_3$***

```
f1=inline('[-5*e(1);-13*e(2)+3*e(3);7.6667*e(2)-3*e(3)-
    0.0002*cos(t)*sin(-2.5527)*cos(0.0827)+
    0.0002*cos(t)*sin(-2.5527-e(1))*cos(0.0827-e(2))+
    0.001*cos(t)*cos(0.0.0827)*sin(0.2)-
    0.001*cos(t)*cos(0.0827-e(2))*sin(0.2-e(3))-
    0.0006*cos(0.2)+0.0006*cos(0.2-e(3))]','t','e');
[t,ea]=ode45(f1=,[0:0.01:100],[0.00001  0.00002  0.000001])
```

***Problem (13) 0f illustrations (2.4) case(1) to find the solutions of $x_1$, $x_2$ and $x_3$***

```
f = inline('[0.1097*x(1)+0.6243*x(2)+0.4397*x(3)-0.2696-
    0.0003858*cos(x(3))*sin(x(2))+0.0011574*sin(x(1)+x(2))-
    0.00002696*cos(x(2))*sin(x(3))+0.001929*cos(x(1))+
    0.0007716*sin(x(1))*sin(x(3))-0.008088*cos(x(2))*cos(x(3));
    0.1704*x(1)+0.9863*x(2)+0.6947*x(3)-0.426+
    0.0006094*cos(x(3))*sin(x(2))+0.0018285*sin(x(1)+x(2))-
    0.000426*cos(x(2))*sin(x(3))+0.003047*cos(x(1))+
    0.001219*sin(x(1))*sin(x(3))-0.01278*cos(x(2))*cos(x(3));
    0.1053*x(1)+0.55983*x(2)+0.407*x(3)-0.2632+
    0.0003544*cos(x(3))*sin(x(2))+0.0010629*sin(x(1)+x(2))-
    0.0002632*cos(x(2))*sin(x(3))+0.001772*cos(x(1))+
    0.0007086*sin(x(1))*sin(x(3))-
    0.007896*cos(x(2))*cos(x(3))]','t','x')'

[t,xa] = ode45(f,[0:0.001:10],[0.5  -0.5  -1.5])
```

***Problem (14) 0f illustrations (2.4) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$***

```
f1=inline('[-9*e(1);-11*e(2)+2*e(3);26*e(2)-7*e(3)+
    0.0000066*cos(-2.7504*(1.0e+006))*sin(-4.5712*(1.0e+006))-
    0.0000066*cos(-2.7504*(1.0e+006)-e(3))*sin(-4.5712*
    (1.0e+006)-e(2))+0.0000198*sin(-2.8935*(1.0e+006)-4.5712*
    (1.0e+006))-0.0000198*sin(-2.8935*(1.0e+006)-e(1)+
    (-4.5712*(1.0e+006))-e(2))+0.0000118*cos(-4.5712*(1.0e+006))*
    sin(-2.7504*(1.0e+006))-0.0000118*cos(-4.5712*(1.0e+006)-
    e(2))*sin(-2.7504*(1.0e+006)-e(3))+0.000033*cos(-2.8935*
    (1.0e+006))-0.000033*cos(-2.8935*(1.0e+006)-e(1))+
    0.0000132*sin(-2.8935*(1.0e+006))*sin(-2.7504*(1.0e+006))-
    0.0000132*sin(-2.8935*(1.0e+006)-e(1))*sin(-2.7504*
    (1.0e+006)-e(3))+0.000354*cos(-4.5712*(1.0e+006))*
    cos(-2.7504*(1.0e+006))-0.000354*cos(-4.5712*(1.0e+006)-
    e(2))*cos(-2.7504*(1.0e+006)-e(3))]','t','e');

[t,ea]=ode45(f1,[0:0.001:10],[0.00001  0.00002   0.000001])
```

***Problem (15) 0f illustrations (2.4) case(2) to find the solutions of $x_1$, $x_2$ and $x_3$***

```
f = inline('[0.1097*x(1)+0.6243*x(2)+0.4397*x(3)-
    0.2696*cos(t)-0.0003858*cos(t)*cos(x(3))*sin(x(2))+
    0.0011574*cos(t)*sin(x(1)+x(2))-
    0.00002696*cos(t)*cos(x(2))*sin(x(3))+0.001929*cos(x(1))+
    0.0007716*sin(x(1))*sin(x(3))-0.008088*cos(x(2))*cos(x(3));
    0.1704*x(1)+0.9863*x(2)+0.6947*x(3)-0.426*cos(t)+
    0.0006094*cos(t)*cos(x(3))*sin(x(2))+
    0.0018285*cos(t)*sin(x(1)+x(2))-
    0.000426*cos(t)*cos(x(2))*sin(x(3))+
    0.003047*cos(x(1))+0.001219*sin(x(1))*sin(x(3))-
    0.01278*cos(x(2))*cos(x(3));0.1053*x(1)+0.55983*x(2)+
    0.407*x(3)-0.2632*cos(t)+
    0.0003544*cos(t)*cos(x(3))*sin(x(2))+
    0.0010629*cos(t)*sin(x(1)+x(2))-
    0.0002632*cos(t)*cos(x(2))*sin(x(3))+
    0.001772*cos(x(1))+0.0007086*sin(x(1))*sin(x(3))-
    0.007896*cos(x(2))*cos(x(3))]','t','x')

[t,xa] = ode45(f,[0:0.001:20],[0.5 -0.5 -1.5])
```

***Problem (16) 0f illustrations (2.4) case(2) to find the error solutions of $e_1$, $e_2$ and $e_3$***

$$f1 = inline('[-9*e(1);-11*e(2)+2*e(3);26*e(2)-7*e(3)+ 0.0000066*\cos(t)*\cos(-2.7504*(1.0e+006))* \sin(-4.5712*(1.0e+006))-0.0000066*\cos(t)* \cos(-2.7504*(1.0e+006)-e(3))*\sin(-4.5712* (1.0e+006)-e(2))+0.0000198*\cos(t)*\sin(-2.8935* (1.0e+006)-4.5712*(1.0e+006))- 0.0000198*\cos(t)*\sin(-2.8935*(1.0e+006)-e(1)+ (-4.5712*(1.0e+006))-e(2))+0.0000118*\cos(t)* \cos(-4.5712*(1.0e+006))*\sin(-2.7504*(1.0e+006))- 0.0000118*\cos(t)*\cos(-4.5712*(1.0e+006)- e(2))*\sin(-2.7504*(1.0e+006)-e(3))+0.000033*\cos(-2.8935* (1.0e+006))-0.000033*\cos(-2.8935*(1.0e+006)-e(1))+ 0.0000132*\sin(-2.8935*(1.0e+006))*\sin(-2.7504*(1.0e+006))- 0.0000132*\sin(-2.8935*(1.0e+006)-e(1))*\sin(-2.7504* (1.0e+006)-e(3))+0.000354*\cos(-4.5712*(1.0e+006))* \cos(-2.7504*(1.0e+006))-0.000354*\cos(-4.5712*(1.0e+006)- e(2))*\cos(-2.7504*(1.0e+006)-e(3))]','t','e');$$

$$[t,ea] = ode45(f1,[0:0.001:10],[0.00001 \;\; 0.00002 \;\; 0.000001])$$

## *Problem (17) of illustrations (2.5) case(1) to find the solutions of $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$*

```
f=inline('[0.0233x1-0.066x2+0.1321x3+0.0445x4+0.1573x5-0.068x6-0.0933-
0.000008sin(x1+x2)cos(x3+x4)sin(x5)+0.000000072cos(x5+x6)cos(x3+x4)-
0.000016sin(x1+x3)sin(x6+x4)-0.00002799cos(x1+x3)sin(x2)+0.00001041sin(
x1+x5)sin(x3+x2)-0.000000933cos(x4+x6)cos(x3+x5);0.0226x1+0.0182x2+
0.7411x3-0.7065x4+0.0939x5+0.0163x6-0.0902+0.00000047sin(x1+x2)cos(x3+
x4)sin(x5)-0.000007426cos(x5+x6)cos(x3+x4)+0.00000094sin(x1+x3)sin(x6+
x4)+0.00002706cos(x1+x3)sin(x2)+0.0000714sin(x1+x5)sin(x3+x2)+0.0000009
02cos(x4+x6)cos(x3+x5);-0.2203x1-0.1176x2-0.9388x3+0.4747x4-1.6868x5-
0.0987x6+08814+0.00000146sin(x1+x2)cos(x3+x4)sin(x5)+0.000008237cos(x5
+x6)cos(x3+x4)+0.00000292sin(x1+x3)sin(x6+x4)+0.00026442cos(x1+x3)sin(x2
)-0.00006744sin(x1+x5)sin(x3+x2)+0.000008814cos(x4+x6)cos(x3+x5);0.245
4x1+0.2666x2+1.2817x3-0.5556x4+1.3996x5+0.2455x6-0.9817+0.0001193
sin(x1+x2)cos(x3+x4)sin(x5)-0.000009482cos(x5+x6)cos(x3+x4)+0.00002386
sin(x1+x3)sin(x6+x4)-0.00029451cos(x1+x3)sin(x2)+0.00009872sin(x1+x5)cos(
x3+x2)-0.000009817cos(x4+x6)cos(x3+x5);0.1806x1+0.1167x2+0.3808x3+0.18
28x4+0.264x5+0.1012x6-0.7222+0.00000048sin(x1+x2)cos(x3+x4)sin(x5)-0.00
0001061cos(x5+x6)cos(x3+x)4+0.00000096sin(x1+x3)sin(x6+x4)-0.00021666
cos(x1+x3)sin(x2)+0.00001642sin(x1+x5)sin(x3+x2)-0.000007222cos(x4+x6)
cos(x3+x5);0.0037x1+0.0228x2-0.0774x3+0.2116x4-0.8953x5+0.0225x6-0.0147
+0.00000206sin(x1+x2)cos(x3+x4)sin(x5)+0.000002057cos(x5+x6)cos(x3+x4)+
0.00000412sin(x1+x3)sin(x6+x4)-0.00000441cos(x1+x3)sin(x2)-0.00000818sin(
x1+x5)sin(x3+x2)-0.000000147cos(x4+x6)cos(x3+x5)] ' , ' t ',' x ');

[t,xa] = ode45[ f ,[ 0:0.001:10 ],[- 1.5  -1  -0.5  0.5  1  1.5 ]
```

***Problem (18) of illustrations (2.5) case(1) to find the error solutions of $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ and $e_6$***

```
f1=inline('[-9*e(1);- 12.106*e(2)+0.9646*e(3)+0.054*e(4)+ 3*e(5)+4*e(6);
0.9646*e(2)-7.8136*e(3)+0.4119*e(4)+4*e(5)+3*e(6);- 2.045*e(2)+ 1.4119*
e(3) -12.0804*e(4)+2*e(5)+5*e(6);-4.05*e(2)-2.7014*e(3)- 10.8157*e(4)-
2*e(5)+4*e(6)-0.00001997*sin(-15.9522)*cos(-1.796)*sin(3.7295)+
0.00001997*sin(-15.9522-e(1)-e(2))*cos(-1.796-e(3)-e(4))*sin(3.7295-
e(5))+0.000001337*cos(0.0135)*cos(-1.796)-0.000001337*cos(0.0135-e(5)-
e(6))*cos(-1.796-e(3)-e(4))-0.00003994*sin(-3.1365)*sin(-2.2576)+
0.00003994*sin(-3.1365-e(1)-e(3))*sin(-2.2576-e(6)- e(4))+
0.000115232*cos(-3.1365)*sin(-16.0701)-0.000115232*cos(-3.1365-e(1)-
e(3))*sin(-16.0701-e(2))+0.00001237*sin(3.8474)*sin(-19.3245)-
0.00001237*sin(3.8474-e(1)-e(5))*sin(-19.3245-e(3)- e(2))+0.000003844*
cos(-2.2576)*cos(0.4751)-0.000003844*cos(-2.2576-e(4)-e(6))*cos(0.4751-
e(3)-e(5));6.8167*e(2)+4.9595*e(3)+18.3331*e(4)-9*e(6)+0.00002851*
sin(-15.9522)*cos(-1.796)*sin(3.7295)-0.00002851*sin(-15.9522-e(1)- e(2))*
cos(-1.796-e(3)-e(4))*sin(3.7295-e(5))+0.00000145*cos(0.0135)*cos(-1.796)
-0.00000145*cos(0.0135-e(5)-e(6))*cos(-1.796-e(3)-e(4))+0.00005702*
sin(-3.1365)*sin(-2.2576)-0.00005702*sin(-3.1365-e(1)-e(3))*sin(-2.2576-
e(6)-e(4))+0.00012852*cos(-3.1365)*sin(-16.0701)-0.00012852*cos(-3.1365-
e(1)-e(3))*sin(-16.0701-e(2))+0.000001593*sin(3.8474)*sin(-19.3245)-
0.000001593*sin(3.8474-e(1)-e(5))*sin(-19.3245-e(3)-
e(2))+0.000004284*cos(-2.2576)*cos(0.4751)-0.000004284*cos(-2.2576-
e(4)-e(6))*cos(0.4751-e(3)-e(5))] ' ,' t ', ' e ' );

[t,ea]=ode45( f1,[0:0.001:10],[0.00001  0.00004  0.00005  0.00002  0.00006
0.00008] )
```

***Problem (19) of illustrations (2.5) case(2) to find the solutions of $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$***

```
f=inline('[0.0233x1-0.066x2+0.1321x3+0.0445x4+0.1573x5-0.068x6-0.093
cos(t)- 0.000008cos(t)sin(x1+x2)cos(x3+x4)sin(x5)+ 0.000000072cos(t) cos(x5
+x6)cos(x3+x4)- 0.000016cos(t)sin(x1+x3)sin(x6+x4)-0.00002799cos(t)cos(x1+
x3)sin(x2)+0.00001041sin(x1+x5)sin(x3+x2)- 0.000000933cos(x4+x6) cos(x3+
x5); 0.0226x1+0.0182x2+0.7411x3-0.7065x4+0.0939x5+0.0163x6- 0.0902cos(t)
+0.00000047cos(t)sin(x1+x2)cos(x3+ x4)sin(x5)- 0.000007426cos(t)cos(x5+x6)
cos(x3+x4)+ 0.00000094cos(t)sin(x1+x3)sin(x6+ x4)+0.00002706cos(t)cos(x1+
x3)sin(x2)+ 0.0000714sin(x1+x5)sin(x3+x2)+ 0.000000902cos(x4+x6) cos(x3+
x5);-0.2203x1-0.1176x2-0.9388x3+0.4747x4-1.6868x5-0.0987x6+08814cos(t)+
0.00000146cos(t)sin(x1+x2)cos(x3+x4)sin(x5)+0.000008237cos(t)cos(x5+x6)
cos(x3+x4)+0.00000292cos(t)sin(x1+x3)sin(x6+x4)+0.00026442cos(t)cos(x1+x3
)sin(x2)- 0.00006744sin(x1+x5)sin(x3+x2)+0.000008814cos(x4+x6)cos(x3+x5);
0.2454x1+0.2666x2+1.2817x3-0.5556x4+1.3996x5+0.2455x6-0.9817cos(t)+
0.0001193cos(t) sin(x1+x2)cos(x3+x4)sin(x5)- 0.000009482cos(t)cos(x5+x6)
cos(x3+x4)+0.00002386cos(t) sin(x1+x3)sin(x6+x4)-0.00029451cos(t)cos(x1+
x3)sin(x2)+0.00009872sin(x1+x5)cos(x3+x2)- 0.000009817cos(x4+x6)cos(x3+
x5); 0.1806x1+0.1167x2+0.3808x3+0.1828x4+0.264x5+0.1012x6-0.7222cos(t)
+0.00000048cos(t)sin(x1+x2)cos(x3+x4)sin(x5)-0.000001061 cos(t) cos(x5+ x6)
cos(x3+x)4+0.00000096cos(t)sin(x1+x3)sin(x6+x4)-0.00021666cos(t) cos(x1+
x3)sin(x2)+0.00001642sin(x1+x5)sin(x3+x2)-0.000007222cos(x4+x6)cos(x3+ x5
);0.0037x1+0.0228x2-0.0774x3+0.2116x4-0.8953x5+0.0225x6-0.0147cos(t)+
0.00000206cos(t)sin(x1+x2)cos(x3+x4) sin(x5)+ 0.000002057cos(t)cos(x5+x6)
cos(x3+x4)+ 0.00000412cos(t)sin(x1+x3)sin(x6+x4)- cos(x1+x3)sin(x2)-
0.00000818sin(x1+x5)sin(x3+x2)- 0.000000147cos(x4+x6)cos(x3+x5)] ', ' t ',' x
');
[t,xa] = ode45[ f ,[0:0.001:6],[ -1.5   -1   -0.5   0.5   1   1.5 ]
```

***Problem (20) of illustrations (2.5) case(2) to find the error solutions of $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ and $e_6$***

```
f1=inline( ' [-9*e(1);-12.106*e(2)+0.9646*e(3)+0.054*e(4)+3*e(5)+4*e(6);
0.9646*e(2)-7.8136*e(3)+0.4119*e(4)+4*e(5)+3*e(6);-2.045*e(2)+ 1.4119*
e(3)- 12.0804*e(4)+2*e(5)+5*e(6);-4.05*e(2)-2.7014*e(3) -10.8157*e(4)-
2*e(5) +4*e(6)-0.00001997*cos(t)*sin(-2.472)*cos(-0.1824)*sin(-0.8231)+
0.00001997*cos(t)*sin(-2.471-e(1)-e(2))*cos(-0.1824-e(3)-e(4))*sin(-0.8231-
e(5))+0.000001337*cos(t)*cos(0.2654)*cos(-0.1824)- 0.000001337* cos(t)*
cos(0.2654-e(5)-e(6))*cos(-0.1824-e(3)-e(4))-0.00003994*cos(t)*
sin(-0.9855)*sin(0.0602)+0.00003994*cos(t)*sin(-0.9855-e(1)- e(3))*
sin(0.0602-e(6)-e(4))+0.000115232*cos(t)*cos(-0.9855)*sin(-0.6436)-
0.000115232*cos(t)*cos(-0.9855-e(1)-e(3))*sin(-0.6436- e(2))+0.00001237*
sin(-2.6515)*sin(0.1993)-0.00001237*sin(-2.6515-e(1)-e(5))*sin(0.1993-
e(3)-e(2))+0.000003844*cos(0.0602)*cos(0.0198)-0.000003844*cos(0.0602-
e(4)-e(6))*cos(0.0198-e(3)-e(5));6.8167*e(2)+4.9595*e(3)+18.3331*e(4)-
9*e(6)+ 0.00002851*cos(t)*sin(-2.472)*cos(-0.1824)*sin(-0.8231)-
0.00002851*
cos(t)*sin(-2.472-e(1)-e(2))*cos(-0.1824-e(3)-e(4))*sin(-0.8231- e(5))+
0.00000145*cos(t)*cos(0.2654)*cos(-0.1824)- 0.00000145*cos(t)*
cos(0.2654-e(5)-e(6))*cos(-0.1824-e(3)-e(4))+0.00005702*cos(t)*sin(-
0.9855)*sin(0.0602)-0.00005702*cos(t)*sin(-0.9855-e(1)-e(3))*sin(0.0602-
e(6)-e(4))+0.00012852*cos(t)*cos(-0.9855)*sin(-0.6436)-0.00012852*cos(t)*
cos(-0.9855-e(1)-e(3))*sin(-0.6436-e(2))+0.000001593*sin(-2.6515)*
sin(0.1993)-0.000001593*sin(-2.6515-e(1)-e(5))*sin(0.1993-e(3)- e(2))+
0.000004284*cos(0.0602)*cos(0.0198)-0.000004284*cos(0.0602-e(4)- e(6))*
cos(0.0198-e(3)-e(5))] ' , ' t ' , ' e ' );
[t,ea]=ode45( f1,[ 0:0.001:6 ],[ 0.00001  0.00004  0.00005  0.00002  0.00006
0.00008 ] )
```

***Problem (21) 0f illustrations (2.6) case(1) to find the solutions of $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$***

```
f=inline( ' [-0.2131x1-0.5223x2-0.27x3-0.2862x4-0.0935x5-0.4812-0.0365*
sin(x1+x2)sin(x3+x4)+0.002667cos(x4+x1)cos(x3+x5)-0.00199sin(x1)cos(x5)-
0.023635cos(x4+x5)sin(x3+x2)+0.0004224sin(x1+x5)sin(x2+x3)+0.0031115*
cos(x4+x5)sin(x3+x2)-0.0273625sin(x5)cos(x4)-0.0023635cos(x3+x1)sin(x2+
x5);0.0187x1+0.1721x2+0.0289x3+0.0929x4-0.0032x5+0.3078+0.0122125*
sin(x1+x2)sin(x3+x4)-0.005622cos(x4+x1)cos(x3+x5)-0.0000336sin(x1) cos(x5)
+0.010085cos(x4+x5) sin(x3+x2)+0.005775sin(x1+x5)sin(x2+x3)-0.006559*
cos(x4+x5)sin(x3+x2)-0.000462sin(x5) cos(x4)+0.0010085cos(x3+x1)sin(x2+
x5);-0.0903x1+0.1618x2-0.1176x3+0.0583x4-0.128x5+0.5362+0.012625*
sin(x1+x2)sin(x3+x4)-0.0155535cos(x4+x1)cos(x3+x5)-0.0012822sin(x1)cos(x5)
+0.0150725cos(x4+x5)sin(x3+x2)+0.0160347sin(x1+x5)sin(x2+x3)-0.01814575
*cos(x4+x5)sin(x3+x2)-0.01763025sin(x5)cos(x4)+0.00150725cos(x3+x1) sin(x2
+x5);0.0854x1+0.0146x2+0.292x3+0.2747x4+0.5992x5+0.4469-0.00675sin(x1+
x2)sin(x3+x4)+0.010641cos(x4+x)cos(x3+x5)-0.0008196sin(x1)cos(x5)-
0.0107725cos(x4+x5)sin(x3+x2)+0.00878295sin(x1+x5) sin(x2+x3)+0.0124145
*cos(x4+x5)sin(x3+x2)-0.0112695sin(x5)cos(x4)-0.00107725cos(x3+x1) sin(x2+
x5);0.0121x1+0.0685x2-0.0285x3+0.1455x4-0.1999x5+0.6488-0.0118625sin(x1
+x2)sin(x3+x4)-0.01806cos(x4+x1)cos(x3+x5)+0.0000542sin(x1)cos(x5)+
0.0289525cos(x4+x5) sin(x3+x2)+0.0125664sin(x1+x5)sin(x2+x3)-0.02107
cos(x4+x5)sin(x3+x2)+0.00074525sin(x5)cos(x4)+0.00289525cos(x3+x1)sin(x2
+x5)]',' t ', ' x ' );

[ t,xa ] = ode45( f,[0:0.01:6],[ -1.5  -1  -0.5  0.5  1.5 ] )
```

***Problem (22) 0f illustrations (2.6) case(1) to find the solutions of $e_1$, $e_2$, $e_3$, $e_4$ and $e_5$***

```
f1=inline('[-11*e(1);15.2165*e(2)+1.5484*e(3)+1.8205*e(4)+4*e(5);
1.5484*e(2)-11.369*e(3)-1.6417*e(4)+5*e(5);-1.1795*e(2)-6.6417*e(3)-
10.4145*e(4)+3*e(5);6.0623*e(2)+7.3481*e(3)+8.5433*e(4)-9*e(5)+
0.052325*sin(-17.1543)*sin(45.3778)-0.052325*sin(-17.1543-e(1)-e(2))*
sin(45.3778-e(3)-e(4))+0.00392375*cos(12.0366)*cos(18.1221)-
0.00392375*cos(12.0366-e(4)-e(1))*cos(18.1221-e(3)-e(5))+0.0000432*sin(-
25.6059)*cos(10.3868)-0.0000432*sin(-25.6059-e(1))*cos(10.3868-e(5))+
0.0013075*cos(48.0293)*sin(16.1869)-0.0013075*cos(48.0293-e(4)-e(5))*
sin(16.1869-e(3)-e(2))+0.00621555*sin(-15.2191)*sin(16.1869)-0.00621555*
sin(-15.2191-e(1)-e(5))*sin(16.1869-e(2)-e(3))+0.00549325*cos(48.0293)*
sin(16.1869)-0.00549325*cos(48.0293-e(4)-e(5))*sin(16.1869-e(3)-e(2))+
0.000594*sin(10.3868)*cos(37.6425)-0.000594*sin(10.3868-e(5))*
cos(37.6425-e(4))+0.00013075*cos(-17.8706)*sin(18.8384)-0.00013075*
cos(-17.8706-e(3)-e(1))*sin(18.8384-e(2)-e(5))]',' t ',' e ');

[t,ea]=ode45( f1,[0:0.01:6],[ 0.001  0.002  0.005 0. 009  0.008 ] )
```

***Problem (23) 0f illustrations (2.6) case(1) to find the solutions of $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$***

*f=inline*( ' [-0.2131$x_1$-0.5223$x_2$-0.27$x_3$-0.2862$x_4$-0.0935$x_5$-0.4812cos(t)-
0.0365cos(t)sin($x_1$+$x_2$)sin($x_3$+$x_4$)+0.002667cos(t)cos($x_4$+$x_1$)cos($x_3$+$x_5$)-
0.00199cos(t)sin($x_1$)cos($x_5$)-0.023635cos(t)cos($x_4$+$x_5$)sin($x_3$+$x_2$)+0.0004224*
sin($x_1$+$x_5$)sin($x_2$+$x_3$)+0.0031115cos($x_4$+$x_5$)sin($x_3$+$x_2$)0.0273625sin($x_5$)cos($x_4$)-
0.0023635cos($x_3$+$x_1$)sin(x2+x5);0.0187$x_1$+0.1721$x_2$+0.0289$x_3$+0.0929$x_4$-
0.0032$x_5$+0.3078cos(t)+0.0122125cos(t)sin($x_1$+$x_2$)sin($x_3$+$x_4$)-0.005622cos(t)*
cos($x_4$+$x_1$)cos($x_3$+$x_5$)-0.0000336cos(t)sin($x_1$)cos($x_5$)+0.010085cos(t)cos($x_4$+
$x_5$)sin($x_3$+$x_2$)+0.005775sin($x_1$+$x_5$)sin($x_2$+$x_3$)-0.006559cos($x_4$+$x_5$)sin($x_3$+$x_2$)-
0.000462sin($x_5$)cos($x_4$)+0.0010085cos($x_3$+x1)sin($x_2$+$x_5$);-0.0903$x_1$+0.1618$x_2$-
0.1176$x_3$+0.0583$x_4$-0.128$x_5$+0.5362cos(t)+0.012625cos(t)sin($x_1$+$x_2$)sin($x_3$+
$x_4$)-0.0155535cos(t)cos($x_4$+$x_1$)cos($x_3$+$x_5$)-0.0012822cos(t)sin($x_1$)cos($x_5$)+
0.0150725cos(t)cos($x_4$+$x_5$)sin($x_3$+$x_2$)+0.0160347sin($x_1$+$x_5$)sin($x_2$+$x_3$)-
0.01814575cos($x_4$+$x_5$)sin($x_3$+$x_2$)-0.01763025sin($x_5$)cos($x_4$)+0.00150725*
cos($x_3$+$x_1$)sin($x_2$+$x_5$);0.0854$x_1$+0.0146$x_2$+0.292$x_3$+0.2747$x_4$+0.5992$x_5$+
0.4469cos(t)-0.00675cos(t)sin($x_1$+$x_2$)sin($x_3$+$x_4$)+0.010641cos(t)cos($x_4$+$x_1$)*
cos($x_3$+$x_5$)-0.0008196cos(t)sin($x_1$)cos($x_5$)-0.0107725cos(t)cos($x_4$+$x_5$)sin($x_3$+
$x_2$)+0.00878295sin($x_1$+$x_5$) sin($x_2$+$x_3$)+0.0124145cos($x_4$+$x_5$)sin($x_3$+$x_2$)-
0.0112695sin($x_5$)cos($x_4$)-0.00107725cos($x_3$+$x_1$) sin($x_2$+$x_5$);0.0121$x_1$+0.0685$x_2$
-0.0285$x_3$+0.1455$x_4$-0.1999$x_5$+0.6488cos(t)-0.0118625cos(t)sin($x_1$+$x_2$)*
sin($x_3$+$x_4$)-0.01806cos(t)cos($x_4$+$x_1$)cos($x_3$+$x_5$)+0.0000542cos(t)sin($x_1$)cos($x_5$)
+0.0289525cos(t)cos($x_4$+$x_5$) sin($x_3$+$x_2$)+0.0125664sin($x_1$+$x_5$)sin($x_2$+$x_3$)-
0.02107cos($x_4$+$x_5$)sin($x_3$+$x_2$)+0.00074525sin($x_5$)cos($x_4$)+0.00289525cos($x_3$+
$x_1$)sin($x_2$+$x_5$)]',' t ', ' x ' );
[ t,xa ] = ode45( *f*,[0:0.01:10],[ -1.5 -1 -0.5 0.5 1.5 ] )

***Problem (24) 0f illustrations (2.6) case(2) to find the solutions of $e_1, e_2, e_3, e_4$ and $e_5$***

```
f1=inline('[-11*e(1);-15.2165*e(2)+1.5484*e(3)+1.8205*e(4)+4*e(5);
1.5484*e(2)-11.369*e(3)-1.6417*e(4)+5*e(5);-1.1795*e(2)-6.6417*e(3)-
10.4145*e(4)+3*e(5);6.0623*e(2)+7.3481*e(3)+8.5433*e(4)-9*e(5)+
0.052325*cos(t)*sin(-40.1084)*sin(102.2924)-0.052325*cos(t)*sin(-40.1084-
e(1)-e(2))*sin(102.2924-e(3)-e(4))+0.00392375*cos(t)*cos(23.3648)*
cos(37.3742)-0.00392375*cos(t)*cos(23.3648-e(4)-e(1))*cos(37.3742-e(3)-
e(5))+0.0000432*cos(t)*sin(-61.4676)*cos(19.9142)-0.0000432*cos(t)*sin(-
61.4676-e(1))*cos(19.9142-e(5))+0.0013075*cos(t)*cos(104.7466)*
sin(38.8192)-0.0013075*cos(t)*cos(104.7466-e(4)-e(5))*sin(38.8192-e(3)-
e(2))+0.00621555*sin(-41.5534)*sin(38.8192)-0.00621555*sin(-41.5534-
e(1)-e(5))*sin(38.8192-e(2)-e(3))+0.00549325*cos(104.7466)*sin(38.8192)-
0.00549325*cos(104.7466-e(4)-e(5))*sin(38.8192-e(3)-e(2))+0.000594*
sin(19.9142)*cos(21.3592)-0.000594*sin(19.9142-e(5))*cos(21.3592-
e(4))+0.00013075*cos(-44.0076)*sin(41.2734)-0.00013075*cos(-44.0076-
e(3)-e(1))*sin(41.2734-e(2)-e(5))]',' t ', 'e ' );

[t,ea]=ode45( f1,[0:0.01:10],[ 0.0001  0.0002  0.0005  0.0009  0.00075] )
```

## *1.1 INTRODUCTION*

This chapter presents some well-known principles that will be needed latter on, and is divided into seven sections. The first section contains the dynamical control equations. Section two discussed the formulation of control problems, and in the third section discussed the basic concept and definitions. The fourth section discussed the Singular value decomposition. The fifth section discussed the Lyapounov Stability and the sixth section discussed the basic concepts of controllability, where as the last section involve discussion of the concepts of observability.

## *1.2 DYNAMICAL CONTROL EQUATIONS [32]*

Many systems can be described by a set of simultaneous differential equations of the form

$$\dot{x}(t) = f[x(t), u(t), t] \tag{1.1}$$

Here $t$ is the time variable, $x(t)$ is a real $n$-dimensional time-varying column vector which denotes the *state* of the system, and $u(t)$ is a real $k$-dimensional column vector which indicates the *input variable* or *control variable*. The function $f$ is real and vector-valued.
For many systems the choice of the state follows naturally from the physical structure, and (1.1) which will be called the *state differential equation*, usually follows direct from the elementary physical laws that govern the system.

Let $y(t)$ be a real $l$-dimensional system variable that can be observed or through which the system influences its environment. Such a variable we call an *output variable* of the system. It can often be expressed as

$$y(t) = g[x(t), u(t), t] \qquad (1.2)$$

Equation (1.2) is called the *output equation* of the system.

We call a system that is described by (1.1) and (1.2) a *finite dimensional differential system* or for short, a *differential system.* Equations (1.1) and (1.2) together are called the *system equations*. If the vector-valued function $g$ contains $u$ explicitly, we say that the system has a *direct link.*

We are mainly concerned with the case where $f$ and $g$ are linear functions. We then speak of a ( *finite-dimensional* ) *linear differential system*. Its state differential equation has the form

$$\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t) \qquad (1.3)$$

where $\mathrm{A}(t)$ and $\mathrm{B}(t)$ are time-varying matrices of appropriate dimensions. We call the dimension $n$ of $x$ the *dimension* of the system. The output equation for such a system takes the form

$$y(t) = C(t)x(t) + D(t)u(t) \qquad (1.4)$$

If the matrices A, B, C, and D are constants, the system is time-invariant.

## *1.3 THE FORMULATION OF CONTROL PROBLEMS [32]*

We now described in general terms an important class of control problems-*tracking problems*. Given is a system, usually called the *plant*; Which cannot be altered by the designer, with the following variables associated with it ( see Fig. (1.1) )

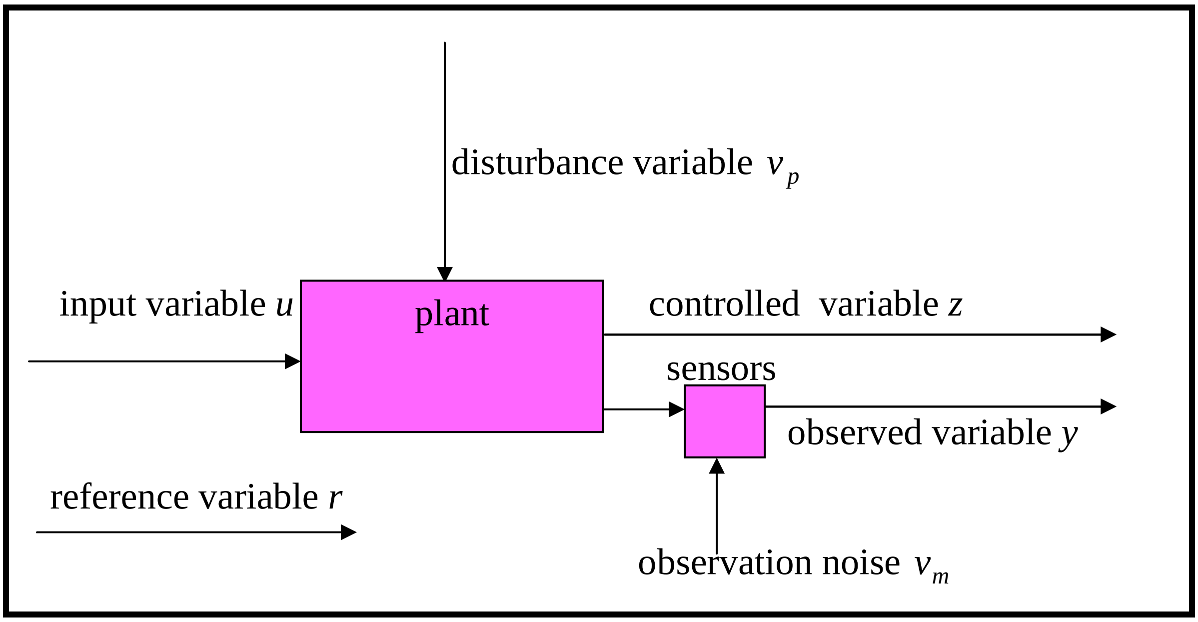


***Fig. (1.1) The plant***

(1) An *input variable u(t)* which influences the plant can be manipulated.

(2) A *disturbance variable* $v_p(t)$ which influences the plant cannot be manipulated.

(3) An *observed variable y(t )* which is measured by means of *sensors* is used to obtain information about the state of the plant; this observed variable is usually contaminated with *observation noise* $v_m(t)$.

(4) A *controlled variable z(t)* which is the variable we wish to control.

(5) A *reference variable r(t)* which represents the prescribed value of the *controlled variable z(t)*.

The input to the plant is to be generated by a piece of equipment that will be called the *controller*. We distinguish between two types of controllers: *open-loop* and *closed-loop* controller. Open-loop controllers generate *u(t)* on the basis of past and present values of the reference variable only (see Fig. (1.2) ), that is

$$u(t) = f_{oL}[r(\tau), t_0 \le \tau \le t], \qquad t \ge t_0 \tag{1.5}$$

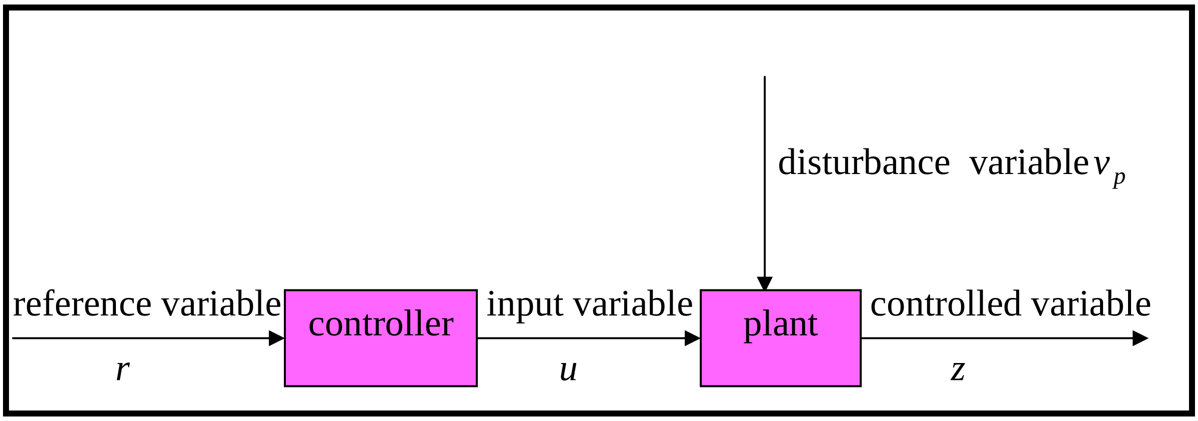


***Fig. (1.2) An open-loop control system***

Closed-loop controllers take advantage of the information about the plant that comes with observed variable; this operation can be represented by (see Fig. (1.3) )

$$u(t) = f_{cL}[r(\tau), t_0 \le \tau \le t; y(\tau), t_0 \le \tau \le t], \quad t \ge t_0 \tag{1.6}$$

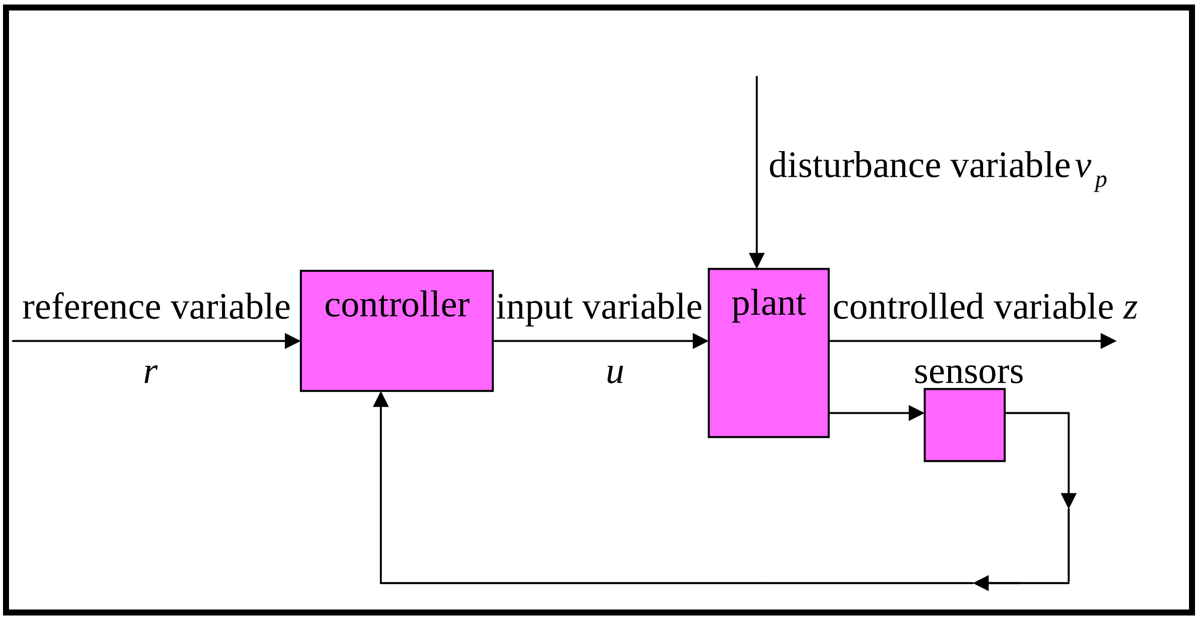


***Fig. (1.3) An closed-loop control system***

Note that neither in (1.5) nor (1.6) are future values of the reference variable or the observed variable used in generating the input variable since they are unknown. The plant and the controller will be referred to as the *control system*.

## 1.4 BASIC CONCPET ANDDEFINITIONS[7],[23],[27],[39],[40]

The following definitions are needed for complete understanding of the subject:

***Definition (1.1): (Controled Variable) [40]***

The controlled variable is the quantity or condition that is measured and controlled.

***Definition (1.2): (Control) [40]***

Control means measuring the value of the controlled variables of the system and applying manipulated variable to the system to correct or limit deviation of the measured value from a desired value.

***Definition (1.3) :( System) [40]***

A system is a combination of components that act together and perform a certain object.

***Definition (1.4):(State Vector) [7]***

The state of a system can be represented by a finite-dimensional column vector X called the state vector. The components of X are called the state variables.

***Definition (1.5): (Dynamical Equations)[40]***

The set of equations that described the unique relations between the input, output and state is called dynamical equations.

***Definition (1.6): (Time-Invariant Control System)[40]***

A time-invariant control system ( constant coefficients control system) is one whose parameters do not vary with time. The response of such a system is independent of the time at which input is applied.

***Definition (1.7): (Time-Varying Control System) [40]***

Time varying control system is a system in which one or more parameters vary with time.

***Definition (1.8): ( Full-Order Observer )[25]***

Full-Order Observer is the state observer observes all state varaibles of the system regardless of whether some state varaibles are available for direct measurements.

***Definition (1.9) :( Reduced-Order Observer)[25]***

Reduced-Order Observer is an observer that estimates fewer than $n$ state variables , where $n$ is the dimension of the state vector , sometimes is called minimum order observer.

***Definition (1.10) (Positive Definite Matrix)[23]***

A real symmetric matrix A is $n\times n$ is called positive definite if $x^{\mathrm{T}}\mathrm{A}x>0$ for every non-zero vector $x\in R^{n}$ .

***Lemma (1.1):[27]***

A symmetric matrix A is positive definite *if and only if* all the eigenvalues of A are positive.

***Lemma (1.2):[39]***

If A is $n\times n$ real symmetric positive definite matrix, then there exists a nonsingular matrix $S$ such that $\mathrm{A}=SS^{\mathrm{T}}$.

***Lemma (1.3):[7]***

Let A be a real symmetric positive definite matrix and $\lambda_{\min}(\mathrm{A})$ and $\lambda_{\max}(\mathrm{A})$ be the smallest and largest eigenvalues of A, respectively, then

$$\lambda_{\min(\mathrm{A})}\|x\|^{2}\le x^{\mathrm{T}}\mathrm{A}x\le\lambda_{\max(\mathrm{A})}\|x\|^{2},\ x\in\Re^{n}$$

Where $\|x\|^{2}=\sum_{i=1}^{n}|x_i|^{2}$ , $x_i$ is the i-th component of x.

***Definition (1.11): (Positive Definite Function)[40]***

A scalar function $V(x)$ is said to be positive definite in a region $\Omega$ ( which includes the origin of the state space ) if $V(x) > 0$ for all non-zero states x in the region $\Omega$ and $V(0) = 0$.

***Remark (1.1): [40]***

A scalar function V(x) is said to be negative definite if –V(x) is positive definite.

***Remarks (1.2): [27]***

**1.** The rank of a symmetric matrix is called the rank of quadratic form $x^{\mathrm{T}}\mathrm{A}x$.

**2.** A quadratic form $x^{\mathrm{T}}\mathrm{A}x$ is said to be singular if the rank of the matrix $\mathrm{A} < n$, i.e., if $|\mathrm{A}| = 0$.

**3.** A quadratic form $x^{\mathrm{T}}\mathrm{A}x$ is said to be nonsingular if the rank of the matrix A is *n*, i.e., $|\mathrm{A}| \neq 0$.

## *1.5 SINGULAR VALUE DECOMPOSITION [4]*

Let A be an $m \times n$ matrix with $m \geq n$. Then there is orthogonal $m \times m$ matrix U, an orthogonal $n \times n$ matrix V and an $m \times n$ matrix $D = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ such that $\mathrm{A} = UDV^{\mathrm{T}}$, rank (A) = *r*.

Consider $\mathrm{AA}^{\mathrm{T}} \sum_i = \sigma_i^2 \sum_i$ and $\mathrm{A}^{\mathrm{T}}\mathrm{A}\eta_i = \lambda_i^2 \eta_i$ and where $\sum_i$ and $\eta_i$ the eigenvectors of $\mathrm{AA}^{\mathrm{T}}$ and $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ respectively. If $\sigma_i = \lambda_i$ then is called *singular value*. Then the eigenvector of $\mathrm{AA}^{\mathrm{T}}$ namely the $\sum_i$ is called the left singular vector of $\mathrm{AA}^{\mathrm{T}}$ and the eigenvector of

$A^T A$ namely the $\eta_i$ is called the right singular vector of $A^T A$. So:

$$A = \begin{bmatrix} \Sigma_1 & \Sigma_2 & \cdots & \Sigma_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_r & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \eta_1^T \\ \eta_2^T \\ \vdots \\ \eta_n^T \end{bmatrix}$$

***Definition ( 1.12) : (Pseudo-Inverse) [33]***

Generalized inverse $A^+$ of matrix A which is not square or singular matrix is called pseudo-inverse which sequence the following properties:

1) $AA^+A = A$
2) $A^+AA^+ = A^+$
3) $(A^+A)^T = A^+A$

***Definition (1.13) :( The Moore-Penrose Pseudo-Inverse) [9]***

The matrix $A^+$ is called penrose pseudo-inverse of a matrix A. Let $A^+ = VD^+U^T$ where $D^+$ be the $n \times m$ matrix whose upper left corner is diagonal matrix that is $D^+ = diag(\sigma_1^{-1},...,\sigma_r^{-1},0,...,0)$ where $A = UDV^T$ be the singular value decomposition ( s.v.d ) of matrix A.

***Definition (1.14) :( The Skinny and Fat matrix)***

The matrix A is called ( skinny ) if A has rank $n$ recall $m \geq n$. And the matrix A is called ( fat ) if A has rank $n$ recall $m \leq n$.

***Definition (1.15):( Pseudo-Inverse for Non-Square Singular Matrix )***

Recalling that not every matrix has inverse . If an arbitrary matrix A is to have an analogous inverse $B = A^{-1}$, then the following must hold:

$BA = AB = I_d$

where $\mathrm{I}_d$ is the identity matrix. Because of can form ability requirement this can never be true if A is not a sequare matrix. In addition A must have a nonzero determinant, i.e., A must be nonsingular.

If A is singular or not square, then there exists which now as pseudo-inverse. We shall use the singular value decomposition (s.v.d) technically. Recording that for any $m\times n$ matrix A has rank $r$, then the (s.v.d) of A is $\mathrm{A}=UD\,V^{\mathrm{T}}$, where V and U are $n\times n$ and $m\times m$ orthogonal matrices and $D=\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ is $m\times n$, here $D_r$ is an $r\times r$ diagonal matrix with $D_r=(\sigma_1,\sigma_2,...,\sigma_r)$, where $\sigma_1\geq\sigma_2\geq...\geq\sigma_r\geq 0$ and $r$ is the rank of A. The numbers $\sigma_i$ is singular value of A $\forall$ i = 1,2,…,$r$ ; so that $\mathrm{A}^+=VD^+U^{\mathrm{T}}$.

***Lemma(1.4):***

Consider $\mathrm{A}^+=VD^+U^{\mathrm{T}}$, $\mathrm{A}^+$ is pseudo-inverse of a matrix A.

***Proof:***

$$\mathrm{AA^+A}=UDV^{\mathrm{T}}(VD^+U^{\mathrm{T}})UDV^{\mathrm{T}}$$

$$=UDV^{\mathrm{T}}VD^+U^{\mathrm{T}}UDV^{\mathrm{T}} \text{ (since V and U are orthogonal)}$$

$$=UDD^+DV^{\mathrm{T}}$$

$$=UDV^{\mathrm{T}}$$

$$=\mathrm{A}.$$

***Lemma(1.5):***

If A is skinny ($m\geq n$) and $\mathrm{A^TA}$ has full rank (see *definition 1.14*), then:

$$\mathrm{A}^+=(\mathrm{A^TA})^{-1}\mathrm{A^T}$$

$$\mathrm{AA^+A}=\mathrm{A}(\mathrm{A^TA})^{-1}\mathrm{A^TA}=\mathrm{A}$$

Similarly, if A is fat ($m \le n$) and $AA^T$ has full rank, then $A^+ = A^T(AA^T)^{-1}$.

***Remark (1.3) :( Computing Pseudo-Inverse Via SVD )***

Consider a matrix A with $m$ rows, $n$ columns and rank(A) = $r$. If $m \ge n$ and rank(A) = $r$, then: (s v d) of $A = UDV^T$

Where

$$U = [\Sigma_1 \quad \Sigma_2 \quad \cdots \quad \Sigma_m],\ D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_r \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ \text{and } V^T = \begin{bmatrix} \eta_1^T \\ \eta_2^T \\ \eta_3^T \\ \vdots \\ \eta_n^T \end{bmatrix}$$

where $\lambda_i^2$ is the eigenvalue of $A^TA$ and $\sigma_i^2$ is the eigenvalue of $AA^T$ and $\eta_i$ is the eigenvector of $A^TA$. $\Sigma_i$ is the eigenvector of $AA^T$ of $\forall$ i = 1, 2,…, $r$.

Then the pseudo-inverse of a matrix A is denoted by $A^+$ which is $A^+ = VD^+U^T$, where $D^+ = {D_r^{-1}}^T$ and 0 otherwise. That is $D_r^{-1} = diag(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_r^{-1})$.

***Remarks (1.4):***

If a matrix A with $m \times n$ one can applied the (s.v.d) technicality or use the following method: Determine the dimension of the matrix A and find the rank(A). Recall that A with $m$ rows and $n$ columns.

**(1)** If $m \le n$ and $AA^T$ invertible, then $A^+ = A^T(AA^T)^{-1}$.

**(2)** If $m \ge n$ and $A^TA$ invertible, then $A^+ = (A^TA)^{-1}A^T$.

**(3)** If $AA^T$ and $A^TA$ are not invertible, then $A^+ = VD^+U^T$, where U, $V$, and $D^+$ are defined in remark (1.3).

## 1.6 LYAPUNOV STABILITY [40]

We present here the Lyapunov methods of stability analysis ( the first method and the second method) which are applicable to both linear and non-linear systems. Our attention will be devoted to the second method of Lyapunov, which provides stability information on linear and non-linear differential equation without solving them, hence the second method is called the direct method of Lyapunov, the direct method is most useful for investigating stability of non-linear systems. It gives sufficient conditions for asymptotic stability of equilibrium states of non-linear systems and gives necessary and sufficient conditions for asymptotic stability of equilibrium states of linear time invariant systems.

**Definition (1.16): (Equilibrium State)[39]**

Consider the system $\dot{x} = f(x,t)$, a state $x_e$, where $f(x_e,t) = 0$, $\forall\ t \in R^+$ is called an equilibrium state of the system.

**Definition (1.17): (Lyapunov Stability) [23]**

An equilibrium state $x_e$ of the dynamical system $\dot{x} = f(x,t)$ is stable (or stable in the sense of the Lyapunov) if for every $\varepsilon > 0$, there exists $\delta > 0(\delta(\varepsilon, t_0))$ such that $\|x_0 - x_e\| \le \delta$ implies $\|x(t, x_0) - x_e\| \le \varepsilon$, for all $t \ge t_0$, where $\|\cdot\|$ denotes the Euclidean norm of a vector in $R^n$.

**Definition (1.18) (Asymptotic Stability) [23]**

An equilibrium state $x_e$ of the system $\dot{x} = f(x,t)$ is asymptotically stable if:

1. It is stable in the sense of Lyapunov.

2. For all $t_0$ , there exists a $\rho(t_0) > 0$ (possible depending on $t_0$ ) such that $\|x_0 - x_e\| < \rho$ implies that $\|x(t, x_0) - x_e\| \to 0$ as $t \to \infty$.

***Definition (1.19): (Asymptotic Stability in the Large) [32]***

The nominal solution $x_0(t)$ of the system $\dot{x} = f(x(t), t)$ is asymptotically stable in the large if:

1. It is stable in the sense of Lyapunov .
2. For any $x_0(t)$ and any $t_0$, $\|x(t) - x_0(t)\| \to 0$ as $t \to \infty$.

A solution that is asymptotically stable in the large has the property that all other solutions eventually approach it.

***Theorem (1.1): (Stability of Time Invariant System) [32]***

The time-invariant linear system

$$\dot{x} = \mathrm{A}x(t)$$

is stable in the sense of Lyapunov if and only if:

**(a)** All the characteristic values (eigenvalues) of A has non-positive real parts, and

**(b)** To any characteristic value on the imaginary axis with multiplicity *m*, there correspond exactly *m* characteristic vectors of the matrix A .

***Theorem (1.2) :( Asymptotically Stable) [32]***

The time-invariant system

$$\dot{x} = \mathrm{A}x(t)$$

Is asymptotically stable *if and only if* of the characteristic values (eigenvalues) of A have strictly negative real part.

***Remark (1.5): [40]***

The second method of Lyapunov attempt to give information on the stability of equilibrium state of linear and non-linear systems without any prior knowledge of their solution.

***Definition (1.20) :( Attraction Domain) [40]***

The largest region of asymptotic stability is called domain of attraction. It is a part of the state space in which asymptotically stable trajectories originate.

***Theorem (1.3): (Lyapunov Main Stability Theorem) [40]***

Consider the system

$$\dot{x} = f(x(t),t)$$

with $f(0,t) = 0$ , $\forall$ $t$.

Suppose also, that there exists a scalar function $V(x,t)$ which has continuous first partial derivatives. If $V(x,t)$ satisfies the following condition:

**1.** $V(x,t)$ is positive definite, namely $V(0,t) = 0$ and $V(x,t) > 0$ $\forall x,t$.

**2.** $V(x,t) \geq \alpha(\|x\|) > 0$, for all $x \neq 0$, and all $t$, where $\alpha$ is continuous, non-decreasing scalar function, such that $\alpha(0) = 0$.

**3.** The total derivative $\dot{V}$ is negative for all $x \neq 0$, and all $t$ or $\dot{V}(x,t) \leq \zeta(\|x\|)<0$, for all $x \neq 0$ and all $t$, where $\zeta$ is continuous, non-decreasing scalar function, such that $\zeta(0) = 0$.

**4.** There exists a continuous non-decreasing function such that $\beta(0) = 0$ for all $t$, $V(x,t) \leq \beta\|x\|$.

**5.** $\alpha(\|x\|)$ approaches infinity as $\|x\|$ increases indefinitely, or $\alpha(\|x\|) \to 0$ as $\|x\| \to \infty$.

Then, the origin $x = 0$ of the above system $\dot{x} = f(x(t),t)$ is uniformly asymptotically stable in the large.

***Theorem (1.4): [39]***

If there exists a scalar function $V(x,t)$ with continuous first partial derivatives satisfying the following conditions:

(a) $V(x,t) > 0$, for all $x \neq 0$ in $\Omega$ and all $t$.

$V(0,t) = 0$, for all $t$.

(b) $\dot{V}(x,t) < 0$, for all $x \neq 0$ in $\Omega$ and all $t$.

$\dot{V}(0,t) = 0$, for all $t$.

$\Omega$ in the region (can be the entire state space), which includes the origin. Then, the origin of the system $\dot{x} = f(x(t),t)$ is uniformly asymptotically stable.

***Theorem(1.5): (Instability Theorem) [39]***

If there exists a scalar function $V(x,t)$ with continuous first partial derivatives satisfying the following conditions:

(a) $V(x,t) > 0$, for all $x \neq 0$ in $\Omega$ and all $t$.

$V(0,t) = 0$, for all $t$.

(b) $\dot{V}(x,t) > 0$, for all $x \neq 0$ in $\Omega$ and all $t$.

$\dot{V}(0,t) = 0$, for all $t$.

Then, the origin of the system $\dot{x} = f(x(t),t)$ is unstable.

***Theorem (1.6): (Lyapunov Equation) [23]***

Consider the system described by

$$\dot{x} = \mathrm{A}x(t)$$

where $x$ is a state vector ($n$-vector) and A is $n \times n$ constant non-singular matrix. A necessary and sufficient condition that the equilibrium state $x = 0$ be asymptotically stable in the large, is that given any positive definite real symmetric matrix $Q$, there exists a positive definite real symmetric matrix P such that:

$$\mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA} = -Q.$$

***Corollary (1.1): [23]***

If the origin of a linear autonomous system $\dot{x} = \mathrm{A}x(t)$ is stable, then there exists a unique Lyapunov function for this system, of the form:

$$V(x) = x^{\mathrm{T}}\mathrm{P}x \text{ where } \mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA} = -Q$$

And $Q$ is any symmetric positive definite matrix, where P is the positive definite solution of $\mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA} = -Q$.

## 1.7 CONTROLLABILITY [25]

Controllability and observability represent two major concepts of modern control system theory. These originally theoretical concepts introduced by R. Kalman in 1960, are particularly important for practical implementations. They play an important role in the design of control system in state space. In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. The solution to this problem may not exists if the system considered is not controllable.

Although most physical system is controllable and observable corresponding mathematical models may not possess the property of controllability and observability. Then it is necessary to know the conditions under which a system is controllable and observable.

***Definition (1.21) : [25]***

A system is said to be ***controllable*** at time $t_0$, if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.

***Theorem (1.7): [7]***

Consider the linear time invariant system of $\dot{x} = \mathrm{A}x + \mathrm{B}u$ where $x \in R^n$ is the state vector, $u \in R^m$ is the control, $\mathrm{A} \in R^{n\times n}$ and $\mathrm{B} \in R^{n\times m}$ are constant matrices .

The necessary and sufficient condition for the complete controllability of the system $\dot{x} = \mathrm{A}x + \mathrm{B}u$ is that $n \times nm$ matrix $\rho(\mathrm{A},\mathrm{B}) = \left[\mathrm{B} \vdots \quad \mathrm{AB} \vdots \quad \cdots \quad \vdots (\mathrm{A})^{n-1}\mathrm{B}\right]$ has rank $n$.

***Example 1.1*** Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ h \end{bmatrix} u$$

since

$$[\mathrm{B} \;\vdots\; \mathrm{AB}] = \begin{bmatrix} e & ae+bh \\ h & ce+dh \end{bmatrix} = \begin{bmatrix} e & f \\ h & g \end{bmatrix}$$

Where $ae+bh=f$ and $ce+dh$

If the $\left|[\mathrm{B} \quad \mathrm{AB}]\right| = eg - fh$ .

If $eg - fh \neq 0$, then the system is completely state controllable .

## *1.8 OBSERVABILITY [25]*

In this section we discuss the observability of linear systems . Consider the unforced system described by the following equations:

$$\dot{x} = \mathrm{A}x \qquad (1.17)$$

$$y = Cx \qquad (1.18)$$

Where x is state vector ($n$-vector)

y is output ($m$-vector)

A is $n \times n$ matrix

C is $m \times n$ matrix

***Definition (1.22) : [25]***

A system is said to be ***observable*** at time $t_0$ if, with the system in state $x(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval.

***Definition (1.23) : [25]***

The system is said to be ***completely observable*** if every state $x(t_0)$ can be determined from the observation of $y(t)$ over a finite time interval, $t_0 \le t \le t_1$. The system is, therefore completely observable if every transition of the state eventually effects every element of the output vector.

The concept of *observability* is useful in solving the problem of reconstructing unmeasureable state variables from measurable variables in the minimum possible length of time.

In this section we treat only linear time-invariant system. Therefore, without loss of generality, we can assume that $t_0 = 0$.

The concept of *observability* is very important because, in practice, the difficulty encountered with state feedback control is that some of the state variables are not accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasureable state variables in order to construct the control signals.

In discussing observability conditions, we consider the unforced system is given by equations (1.17) and (1.18). The reason for this is as follows: If the system is described by

$$\begin{aligned} &\dot{x} = \mathrm{A}x + \mathrm{B}u \\ &y = Cx + Du \\ &x(0) = x_0, \quad t \ge 0 \end{aligned} \tag{1.19}$$

Then

$$x(t) = e^{\mathrm{A}t} x(0) + \int_0^t e^{\mathrm{A}(t-\tau)} \mathrm{B}u(\tau) d\tau \tag{1.20}$$

and $y(t)$ is

$$y(t) = Ce^{\mathrm{A}t} x(0) + C\int_0^t e^{\mathrm{A}(t-\tau)} \mathrm{B}u(\tau) d\tau + Du \tag{1.21}$$

Since the matrices A, B, C, and D are known and $u(t)$ is also known, the last two terms on the right-hand side of this last equation quantities are known quantities.

Therefore, they may be subtracted from the observed value of $y(t)$. Hence, for investigating a necessary and sufficient condition for complete observability, it suffices to consider the system described by equations (1.17) and (1.18).

***Theorem (1.8) :[39]***

The system is described by:

$$\dot{x} = \mathrm{A}x \tag{1.22}$$

$$y = Cx \tag{1.23}$$

where x is state vector ( $n$-vector )

y is output ( $m$-vector )

A is $n \times n$ matrix

C is $m \times n$ matrix

Are constant matrices is the completely observable *if and only if* the matrix

$$\left[ C^{\mathrm{T}} \vdots \quad A^{\mathrm{T}} C^{\mathrm{T}} \vdots \quad \cdots \quad \vdots (A^{\mathrm{T}})^{n-1} C^{\mathrm{T}} \right] \tag{1.24}$$

is of rank $n$, or has $n$ linearly independent column vectors. This matrix is called ***observability matrix***.

**Example 1.2** Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ h \end{bmatrix} u$$

$$y = [c_1 \quad c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is this system controllable and observable ?

Since

$$[\mathrm{B} \ \vdots \ \mathrm{AB}] = \begin{bmatrix} e & ae+bh \\ h & ce+dh \end{bmatrix} = \begin{bmatrix} e & f \\ h & g \end{bmatrix}$$

Where $ae+bh = f$ and $ce+dh$

If the $\left| [\mathrm{B} \quad \mathrm{AB}] \right| = eg - fh$ .

If $eg - fh \neq 0$, then the system is completely state controllable.

Since

$$\left[C^{\mathrm{T}} \ \vdots \ A^{\mathrm{T}} C^{\mathrm{T}}\right] = \begin{bmatrix} c_1 & ac_1 + cc_2 \\ c_2 & bc_1 + dc_2 \end{bmatrix} = \begin{bmatrix} c_1 & m \\ c_2 & n \end{bmatrix} .$$

Where $ac_1 + cc_2 = m$ and $bc_1 + dc_2 = n$

If the $\left| \left[C^{\mathrm{T}} \vdots \ A^{\mathrm{T}} C^{\mathrm{T}}\right] \right| = c_1 n - m c_2$.

If $c_1 n - m c_2 \neq 0$, then the system is completely observable.

**Example 1.3** The following system is not completely observable

$$\dot{x} = \mathrm{A}x + \mathrm{B}u$$
$$y = Cx$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [4 \quad 5 \quad 1]$$

Note that the control function *u* does not affect the complete observability of the system. To examine complete observability, we may simply set *u = 0*.

For this system, we have

$$\left[C^{T} \vdots \;\; A^{T}C^{T} \vdots \;\; (A^{T})^{2}C^{T}\right] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

Note that

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Hence, the rank of the matrix $\left[C^{T} \vdots \;\; A^{T}C^{T} \vdots \;\; (A^{T})^{2}C^{T}\right]$ is less than 3. Therefore, the system is not completely observable.

***Remark (1.6): [25]***

Consider the system described by equations (1.17) and (1.18), rewritten

$$\dot{x} = Ax \qquad (1.25)$$

$$y = Cx \qquad (1.26)$$

Suppose that the transformation matrix P transforms A into a diagonal matrix, or

$$P^{-1}AP = D \qquad (1.27)$$

Where D is a diagonal matrix. Let us define

$$x = Pz \qquad (1.28)$$

Then equations (1.25) and (1.26) can be written

$$\dot{z} = P^{-1}APz = Dz$$
$$y = CPz$$

Hence,

$$y(t) = CPe^{Dt}z(0) \qquad (1.29)$$

or

$$y(t) = C\mathrm{P}\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} z(0) = C\mathrm{P}\begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \vdots \\ e^{\lambda_n t} z_n(0) \end{bmatrix} \qquad (1.30)$$

The system is completely observable if none of the columns m×$n$ matrix CP consists of all zero elements. This is because, if the ith column of CP consists of all zero elements, then the state variable $z_i(0)$ will not appear in the output equation and therefore cannot be determined from observation of $y(t)$.

Thus, x (0) which is related to $z(0)$ by the non-singular matrix P, cannot be determined. (Remember that this test applies only if the matrix $\mathrm{P}^{-1}\mathrm{AP}$ is in diagonal form).

If the matrix A cannot be transformed into a diagonal matrix, then by use of a suitable transformation matrix S, we can transform A into a Jordan canonical form , or

$$S^{-1}\mathrm{A}S = J \qquad (1.31)$$

Where $J$ is in the Jordan canonical form (see in appendix (A)).

Let us define

$$x = Sz \qquad (1.32)$$

Then equations (1.25) and (1.26) can be written

$$\begin{aligned} \dot{z} &= S^{-1}\mathrm{A}Sz = Jz \\ y &= CSz \end{aligned} \qquad (1.33)$$

Hence,

$$y(t) = CSe^{Jt} z(0) \qquad (1.34)$$

The system is ***completely observable*** if:

**(1)** No two Jordan blocks in $J$ are associated with the same eigenvalues.

**(2)** No columns of *CS* that correspond to the first row of each Jordan block consist of zero elements, and

**(3)** No columns of *CS* that correspond to distinct eigenvalues consist Of zero elements.

State observation of non-linear dynamical systems is becoming a growing topic of investigation in the specialised literature ( Tsinias, 1989 [47]), (Walcott, 1987 [50]). The reconstruction of state variables remains a major problem both in control theory and process diagnosis (Magni, 1991 [37]). Researcher attention is being particularly focused on the design of adaptive observers for on-line process state estimation. There is increasing a wareness that to ensure robustness in performance requires simpler and stable adaptive observer schemes. Linear systems have received considerable attention leading to the several stable adaptive observer systems.

Linear observers involving unknown inputs have also been developed and analysed (Chang, 1995 [5]), (Gaddouna , 1996 [15]).

Nevertheless, the design of asymptotically stable observers remains a hard task in the non-linear case, even when the non-linearities are fully known. The nonsingular problem because a new task for design an stabilized observer.

Hence our aim in this work is to design an stabilized full-order observer for some class of non-linear ( singular control system ).

## 2.1 BASIC CONCEPTS AND DEFINITIONS [7], [11] and [25]

The following definitions are needed for complete understanding of the subject:

***Definition (2.1) ( Lipschitz condition ) [11]***

Suppose function $f(t,x)$ has domain $D$ in $(t,x)$-space and suppose there exists $k$, such that if $(t,x_1), (t,x_2) \in D$, then

$$\| f(t,x_2) - f(t,x_1) \| \le k \| x_2 - x_1 \|$$

Then $f$ satisfies a Lipschitz condition with respect to $x$ in $D$, and $k$

is a Lipschitz constant for $f$.

***Definition (2.2): (Full Rank) [7]***

A square matrix has full rank *if and only if* the determinant of the matrix is different from zero. Or correspondingly, a matrix is non-singular *if and only if* the rows or columns of the matrix are linearly independent.

***Remarks (2.1):[7], [25]***

**(1)** The Euclidean norm of a vector $x$ is defined as:

$$\|x\| = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

**(2)** The Euclidean norm of $n \times n$ matrix can be defined a:

$$\|\mathrm{A}\| = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

where $|a_{ij}|$ is the absolute value of the matrix coefficients $a_{ij}$.

**(3)** The Euclidean norm of $n \times n$ matrix is also defined as:

$$\|\mathrm{A}\| = \left(\lambda_{\max}(\mathrm{A}^T \mathrm{A})\right)^{1/2}$$

where $\mathrm{A}^\mathrm{T}$ is the transpose of A, $\lambda_{\max}$ is the maximum eigenvalue of $(\mathrm{A}^\mathrm{T}\mathrm{A})$.

**(4)** It is known from that the properties of a norm of matrices that $\|\mathrm{A}\| = \|\mathrm{A}^\mathrm{T}\|$.

**(5)** Observation is estimation of unmeasurable state variables.

**(6)** State Observer is estimating the state variables based on the measurements of the output and control variables, sometimes is called an observer.

## 2 .2 PROBLEM FORMULATION AND THEOREMS

The following problem formulation has been considered .
The non-linear dynamic control system is assumed to be non-singular because of singularities in the matrix E, and such type of system have take a good attension in applications and researchers see for example [38], [44] and [56].

The following class of nonlinearities have been developed and studied and as follows:

Consider :

$$\mathrm{E}\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + f(x(t),u(t)) + g(x(t)) \qquad (2.1)$$

$$y = Cx(t) \qquad (2.2)$$

where

$\mathrm{E} \in \Re^{q\times n}$, $\mathrm{A} \in \Re^{q\times n}$, $\mathrm{B} \in \Re^{q\times p}$, $C \in \Re^{m\times n}$, $x(t) \in \Re^{n\times 1}$, $u(t) \in \Re^{p\times 1}$,
$y(t) \in \Re^{m\times 1}$, $f : \Re^{n} \times \Re^{p} \to \Re^{q}$, $g : \Re^{n} \to \Re^{q}$

where

**(1)** $f$ is a vector of non-linear functions which may represent a known non-linearity, and notice that range of $u(t)$ is assumed to be in a ball $U$ of $\Re^{p}$.

And $f$ is satisfied Lipschitz condition

$$\forall v \in U, \forall (x_1, x_2) \in \Re^{n} \times \Re^{n}$$

$$\| f(x_2, v) - f(x_1, v) \| \le k \| x_2 - x_1 \| \qquad (2.3)$$

**(2)** $\| g(x_2(t)) - g(x_1(t)) \| \le k_1 \| x_2(t) - x_1(t) \|$, where $k_1$ is Lipschitz constant.

**(3)** Assuming that the matrix E is assumed nonsquare matrix or (singular matrix even for square $q = n$).

**(4)** The matrix $C$ is full row rank. (2.4)

**(5)** $\operatorname{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n\,.$ (2.5)

**(6)** $\operatorname{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n,\ \ \forall s\,.$ (2.6)

***Remarks (2.2):***

**(1)** Hypothesis (2.3) is used for bounding the magnitude of the non-linearity.

**(2)** Hypothesis (2.4) indicates that no redundancy between the measurement devices is considered ( this assumption is not very restrictive and can always be satisfied by redefining the measurement equation).

**(3)** The equation (2.5) may be interpreted as they are enough measurements to compensate the singularity of system (2.1); it will be used further for the design of the observer.

**(4)** Hypothesis (2.6) is the observability condition .

In the following, the proportional observer is used and the proposed technique may be easily extended to proportional-integral observer, and this may be laid to the future work if any.

The following theorem is then developed as follows:

***Theorem (2.1):***

Consider the system defined by (2.1) and (2.2), and let $S(\theta)$ be a symmetric positive definite matrix function of a positive scalar $\theta$ then, if the four hypotheses (2.3), (2.4), (2.5), and (2.6) are hold, the following system is an observer of system (2.1) and (2.2)

$$\frac{dz(t)}{dt} = \mathrm{N}z(t) + Ly(t) + Gu(t) + Rf(\hat{x}(t), u(t)) + Rg(\hat{x}(t)) - S^{-1}(\theta)C^T(C\hat{x}(t) - y(t)) \quad (2.7)$$

with transformation

$$\hat{x}(t) = z(t) + \mathrm{K}y(t) \quad (2.8)$$

and using the Lyapunov function approach $V = \frac{1}{2}e^{\mathrm{T}}(t)S(\theta)e(t)$.

where N, L, R, G, S and K are matrices of proper dimension and satisfied the following conditions:

(1) $-\theta < \mathrm{Re}(\lambda_i(\mathrm{N})) : \theta > 0 \quad \forall i \in \{1, ..., r\}$ (2.9)

(2) RE + KC = I (2.10)

(3) NRE + LC – RA = 0 (2.11)

(4) G – RB = 0 (2.12)

where

$\lambda_i(\mathrm{N})$ represented the $i^{th}$ distinct eigenvalues of N and $\theta$ is a positive parameter.

***Proof :***

Let $e(t)$ be the state reconstruction error defined by:

$e(t) = x(t) - \hat{x}(t)$, from (2.8) we have that

$= x(t) - [z(t) + \mathrm{K}y(t)]$, from (2.2), we have

$= x(t) - z(t) - \mathrm{K}Cx(t)$

From (2.10), one gets

RE + KC = I $\rightarrow$ RE = I – KC, and hence

$$e(t) = REx(t) - z(t) \tag{2.13}$$

Direct Derivation of $e(t)$ yields :

$$\dot{e} = RE\dot{x} - \dot{z} \tag{2.14}$$

Put (2.1) and (2.7) into (2.14), we get

$$\dot{e} = R[Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))] - [Nz(t) + Ly(t) + Gu(t) +$$
$$Rf(\hat{x}(t),u(t)) + Rg(\hat{x}(t)) - S^{-1}(\theta)C^{T}(C\hat{x}(t) - y(t))] \tag{2.15}$$

From (2.13), we have

$$\Rightarrow \quad z(t) = REx(t) - e(t) \tag{2.16}$$

Substitute (2.16) and (2.2) in (2.15), one gets:

$$\dot{e} = R[Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))] - [N(REx(t) - e(t)) + LCx(t) +$$
$$Gu(t) + Rf(\hat{x}(t),u(t)) + Rg(\hat{x}(t)) - S^{-1}(\theta)C^{T}(C\hat{x}(t) - y(t))]$$

$$\dot{e} = RAx(t) + RBu(t) + Rf(x(t),u(t)) + Rg(x(t)) - NREx(t) + Ne(t) - LCx(t) -$$
$$Gu(t) - Rf(\hat{x}(t),u(t)) - Rg(\hat{x}(t)) - S^{-1}(\theta)C^{T}Ce(t)$$

$$\dot{e} = (RA - NRE - LC)x(t) + (RB - G)u(t) + R(g(x(t)) - g(\hat{x}(t))) +$$
$$R(f(x(t),u(t)) - f(\hat{x}(t),u(t))) + (N - S^{-1}(\theta)C^{T}C)e(t) \tag{2.17}$$

From (2.11), we have

$$NRE + LC - RA = 0 \Rightarrow \quad RA - LC - NRE = 0 \tag{2.18}$$

And (2.12), we have

$$G - RB = 0 \Rightarrow \quad RB - G = 0 \tag{2.19}$$

Put (2.19) and (2.20) into (2.18), one gets:

$$\dot{e} = (N - S^{-1}(\theta)C^{T}C)e(t) + R(f - \hat{f}) + R(g - \hat{g}) \tag{2.20}$$

where

$f \equiv f(x(t),u(t))$ and $\hat{f} \equiv f(\hat{x}(t),u(t))$

$g \equiv g(x(t))$ and $\hat{g} \equiv g(\hat{x}(t))$

Let us now examine the stability of $e(t)$ by considering the Lyapunove function :

$$V(t) = \frac{1}{2} e^{\mathrm{T}}(t) S(\theta) e(t) \qquad (2.21)$$

Since $S(\theta)$ is assumed to be positive definite and will be signed later on using this proposition, so it sufficients to ensure the negative definite of total derivatives of $V(t)$ for stabilizing $e(t)$ to zero.

The calculation of the derivation of $V(t)$ with respect to the time $t$ gives:

$$\dot{V} = \frac{1}{2}\left(\dot{e}^{\mathrm{T}}(t) S(\theta) e(t) + e^{\mathrm{T}}(t) S(\theta) \dot{e}(t)\right) \qquad (2.22)$$

From (2.21), taking the transpose by each sides, we get:

$$\dot{e}^{\mathrm{T}}(t) = e^{\mathrm{T}}(t)\left(\mathrm{N}^{\mathrm{T}} - C^{\mathrm{T}} C S^{-1}(\theta)\right) + \left(f - \hat{f}\right)^{\mathrm{T}} R^{\mathrm{T}} + \left(g - \hat{g}\right)^{\mathrm{T}} R^{\mathrm{T}} \qquad (2.23)$$

Put (2.21) and (2.24) into (2.23), we get :

$$\dot{V} = \frac{1}{2}\Big(\Big(e^{\mathrm{T}}(t)(\mathrm{N}^{\mathrm{T}} - C^{\mathrm{T}} C S^{-1}(\theta) + (f - \hat{f})^{\mathrm{T}} R^{\mathrm{T}} + (g - \hat{g})^{\mathrm{T}} R^{\mathrm{T}}\Big) S(\theta) e(t) +$$

$$e^{\mathrm{T}}(t) S(\theta)\Big((\mathrm{N} - S^{-1}(\theta) C^{\mathrm{T}} C) e(t) + R(f - \hat{f}) + R(g - \hat{g})\Big)\Big)$$

$$\dot{V} = \frac{1}{2}[e^{\mathrm{T}}(t)\left(\mathrm{N}^{\mathrm{T}} S(\theta) + S(\theta)\mathrm{N} - 2C^{\mathrm{T}} C\right) e(t) + (f - \hat{f})^{\mathrm{T}} R^{\mathrm{T}} S(\theta) e(t) +$$

$$(g - \hat{g})^{\mathrm{T}} R^{\mathrm{T}} S(\theta) e(t) + e^{\mathrm{T}}(t) S(\theta) R(f - \hat{f}) + e^{\mathrm{T}}(t) S(\theta) R(g - \hat{g})]$$

Since, $S(\theta)$ is symmetric: $\left[(f - \hat{f})^{\mathrm{T}} R^{\mathrm{T}} S(\theta) e(t)\right]^{\mathrm{T}} = e^{\mathrm{T}}(t) S(\theta) R(f - \hat{f})$

$$\dot{V} = \frac{1}{2}[e^{\mathrm{T}}(t)\left(\mathrm{N}^{\mathrm{T}} S(\theta) + S(\theta)\mathrm{N} - 2C^{\mathrm{T}} C\right) e(t) + 2e^{\mathrm{T}}(t) S(\theta) R(f - \hat{f}) +$$

$$2e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g})]$$

$$\Rightarrow\ \dot{V}=\frac{1}{2}e^{\mathrm{T}}(t)(\mathrm{N}^{\mathrm{T}}S(\theta)+S(\theta)\mathrm{N}-2C^{\mathrm{T}}C)e(t)+e^{\mathrm{T}}(t)S(\theta)R(f-\hat{f})+$$

$$e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g}) \qquad (2.24)$$

According to the Lyapunov theory, if the linear part of (2.7) is stable, a symmetric positive definite matrix $Q$ exists such as :

$$\mathrm{N}^{\mathrm{T}}S(\theta)+S(\theta)\mathrm{N}-2C^{\mathrm{T}}C=-Q \qquad (2.25)$$

Let now $Q$ satisfying the following equality :

$$Q=2\theta S(\theta) \qquad (2.26)$$

Then, equation (2.25) may be written as:

$$(\mathrm{N}+\theta\mathrm{I}_n)^{\mathrm{T}}S(\theta)+S(\theta)(\mathrm{N}+\theta\mathrm{I}_n)=2C^{\mathrm{T}}C \qquad (2.27)$$

Moreover, it may be proven that the eigenvalues of the linear part of the observer have a real part equal to $-\theta$ .

$$\mathrm{Re}(\mathrm{N}+\theta\mathrm{I}_n)<0\Rightarrow\ \mathrm{Re}(\mathrm{N})<-\mathrm{Re}(\theta\mathrm{I}_n)\Rightarrow\ \ \mathrm{Re}(\mathrm{N})<-\theta \qquad (2.28)$$

Now examine the condition for which observer (2.7) is stable. Taking (2.28) into account reduces equation (2.25) to:

$$\dot{V}=-\theta e^{\mathrm{T}}(t)S(\theta)e(t)+e^{\mathrm{T}}(t)S(\theta)R(f-\hat{f})+e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g}) \qquad (2.29)$$

Noticed that

$$e^{\mathrm{T}}(t)S(\theta)R(f-\hat{f})=\left\|e^{\mathrm{T}}(t)S(\theta)R(f-\hat{f})\right\|\le\left\|e^{\mathrm{T}}(t)\right\|\left\|S(\theta)R\right\|\left\|f-\hat{f}\right\|,\ \text{from}(2.3)$$

$$\le\left\|e^{\mathrm{T}}(t)\right\|\left\|S(\theta)R\right\|k\left\|x-\hat{x}\right\|$$

$$\le k\left\|e(t)\right\|^2\left\|S(\theta)R\right\|$$

$$\le k\delta_{\max}(S(\theta)R)\left\|e(t)\right\|^2$$

$$\Rightarrow\ \ e^{\mathrm{T}}(t)S(\theta)R(f-\hat{f})\le k\left\|e(t)\right\|^2\delta_{\max}(S(\theta)R) \qquad (2.30)$$

$$e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g})=\left\|e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g})\right\|\le\left\|e^{\mathrm{T}}(t)\right\|\left\|S(\theta)R\right\|\left\|g-\hat{g}\right\|,\ \text{from (2.3)}$$

$$\le\left\|e^{\mathrm{T}}(t)\right\|\left\|S(\theta)R\right\|k_1\left\|x-\hat{x}\right\|$$

$$\leq k_1 \|e(t)\|^2 \|S(\theta)R\|$$

$$\leq k_1 \delta_{\max}(S(\theta)R)\|e(t)\|^2$$

$$\Rightarrow \; e^{\mathrm{T}}(t)S(\theta)R(g-\hat{g}) \leq k_1 \|e(t)\|^2 \delta_{\max}(S(\theta)R) \qquad (2.31)$$

where $\delta_{\max}(.)$ denotes the largest singular eigenvalue.

Substituting (2.30) and (2.31) into (2.29) gives:

$$\frac{dV(t)}{dt} \leq -\theta e^{\mathrm{T}}(t)S(\theta)e(t) + k\|e(t)\|^2 \delta_{\max}(S(\theta)R) + k_1\|e(t)\|^2 \delta_{\max}(S(\theta)R)$$

$$\frac{dV(t)}{dt} \leq -\theta e^{\mathrm{T}}(t)S(\theta)e(t) + (k+k_1)\|e(t)\|^2 \delta_{\max}(S(\theta)R) \qquad (2.32)$$

with

$$e^{\mathrm{T}}(t)S(\theta)e(t) = \left\|e^{\mathrm{T}}(t)S(\theta)e(t)\right\| \leq \left\|e^{\mathrm{T}}(t)\right\|\|S(\theta)\|\|e(t)\| \leq \delta_{\max}(S(\theta))\|e(t)\|^2$$

$$e^{\mathrm{T}}(t)S(\theta)e(t) = \delta_{\max}(S(\theta))\|e(t)\|^2 \geq \delta_{\min}(S(\theta))\|e(t)\|^2$$

$$\Rightarrow \quad e^{\mathrm{T}}(t)S(\theta)e(t) \geq \delta_{\min}(S(\theta))\|e(t)\|^2 \qquad (2.33)$$

Put (2.33) into (2.32), one gets:

$$\frac{dV(t)}{dt} \leq -\theta\delta_{\min}(S(\theta))\|e(t)\|^2 + (k+k_1)\|e(t)\|^2 \delta_{\max}(S(\theta)R)$$

$$\frac{dV(t)}{dt} \leq \left(-\theta\delta_{\min}(S(\theta)) + (k+k_1)\delta_{\max}(S(\theta)R)\right)\|e(t)\|^2 \qquad (2.34)$$

Since

$$-\theta\delta_{\min}(S(\theta)) + (k+k_1)\delta_{\max}(S(\theta)R) < 0 \qquad (2.35)$$

**Remarks (2.3):**

**Case ( I ):** In the case of $(k+k_1)$ is known, the requirement inequality (2.35) is as follows:

$$k+k_1<\frac{\theta\delta_{\min}(S(\theta))}{\delta_{\max}(S(\theta)R)} \qquad (2.36)$$

**Case ( II ):** In the case of $\theta$ is known, the r equirement inequality (2.35) is as follows:

$$-\theta<\frac{-(k+k)\delta_{\max}(S(\theta)R)}{\delta_{\min}(S(\theta))} \qquad (2.37)$$

Since, then $\dot{V}<0$ and $S(\theta)>0$, hence the system (2.1) is stable and this completes the proof.

For practical application and to overcome the difficulties of selection and computing the positive definite symmetric matrix $S(\theta)$, the following lemma is very necessary.

***Lemma (2.1):***

Consider the dynamical system

$$\mathrm{E}\dot{x} = \mathrm{A}x(t) + \mathrm{B}u(t) + f(x(t),u(t)) + g(x(t)) \qquad (2.38)$$

$$y(t) = Cx(t) \qquad (2.39)$$

where

$\mathrm{E} \in \Re^{q\times n}$, $\mathrm{A} \in \Re^{q\times n}$, $\mathrm{B} \in \Re^{q\times p}$, $C \in \Re^{m\times n}$, $x(t) \in \Re^{n\times 1}$, $u(t) \in \Re^{p\times 1}$,
$y(t) \in \Re^{m\times 1}$, $f : \Re^{n} \times \Re^{p} \to \Re^{q}$, $g : \Re^{n} \to \Re^{q}$

and satisfied the conditions of theorem (2.1), and using the Lyapunov function $V = \frac{1}{2} e^{\mathrm{T}}(t)\mathrm{P}e(t)$, where P is positive definite solution of $\mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{N} = -Q$.

And then, the following system is an observer of system (2.39) and (2.40)

$$\dot{z} = \mathrm{N}z(t) + Ly(t) + Gu(t) + Rf(\hat{x}(t),u(t)) + Rg(\hat{x}(t)) - \mathrm{P}^{-1}(\theta)C^{T}(C\hat{x}(t) - y(t)) \qquad (2.40)$$

with transformation

$$\hat{x}(t) = z(t) + \mathrm{K}y(t) \qquad (2.41)$$

where N, L, R, G, P and K are matrices of proper dimension, and satisfied the following conditions of theorems (2.1).

***Proof :***

Let $e(t)$ be the state error defined by :

$$e(t) = x(t) - \hat{x}(t) \qquad (2.42)$$

From (2.39), (2.10) and (2.41), one gets:

$$\Rightarrow \quad e(t) = REx(t) - z(t) \qquad (2.43)$$

Derivation of $e(t)$ yields:

$$\Rightarrow \quad \dot{e} = R\mathrm{E}\dot{x} - \dot{z} \tag{2.44}$$

From (2.15), (2.16), (2.17) and (2.18), one gets:

$$\dot{e} = (R\mathrm{A} - \mathrm{N}R\mathrm{E} - LC)x(t) + (R\mathrm{B} - G)u(t) + R(g(x(t)) - g(\hat{x}(t))) +$$

$$R(f(x(t),u(t)) - f(\hat{x}(t),u(t))) + (\mathrm{N} - \mathrm{P}^{-1}(\theta)C^{\mathrm{T}}C)e(t) \tag{2.45}$$

From (2.11) and (2.12), the equation (2.46) , one gets:

$$\Rightarrow \quad \dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g}) \tag{2.46}$$

where

$f \equiv f(x(t),u(t))$ and $\hat{f} \equiv f(\hat{x}(t),u(t))$

$g \equiv g(x(t))$ and $\hat{g} \equiv g(\hat{x}(t))$

Let us now examine the stability of $e(t)$ by considering the Lyapunove function :

$$V(t) = \frac{1}{2}e^{\mathrm{T}}(t)\mathrm{P}e(t) \tag{2.47}$$

Derivation of $V(t)$ yields:

$$\dot{V} = \frac{1}{2}\left(\dot{e}^{\mathrm{T}}(t)\mathrm{P}e(t) + e^{\mathrm{T}}(t)\mathrm{P}\dot{e}(t)\right) \tag{2.48}$$

$$\Rightarrow \quad \dot{V} = \frac{1}{2}e^{\mathrm{T}}(t)\left(\mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{N} - 2C^{\mathrm{T}}C\right)e(t) + e^{\mathrm{T}}(t)\mathrm{P}R(f - \hat{f}) + e^{\mathrm{T}}(t)\mathrm{P}R(g - \hat{g}) \tag{2.49}$$

According to the Lyapunov theory, if the linear part of (2.40) is stable, a symmetric positive definite matrix $Q$ exists such as:

$$\mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{N} - 2C^{\mathrm{T}}C = -Q \tag{2.50}$$

Let now $Q$ satisfying the following equality:

$$Q = 2\mathrm{P} \tag{2.51}$$

Then, the equation may be written as:

$$\mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{N} - 2C^{\mathrm{T}}C = -2\mathrm{P} \tag{2.52}$$

$$\Rightarrow \quad \mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{N} + 2\mathrm{P} = 2C^{\mathrm{T}}C$$

$$\Rightarrow \quad N^T P + PN + P + P = 2C^T C$$

$$\Rightarrow \quad (N + I_n)^T P + P(N + I_n) = 2C^T C \tag{2.53}$$

Put (2.52) into (2.49), we get:

$$\dot{V} = -e^T(t)Pe(t) + e^T(t)PR(f - \hat{f}) + e^T(t)PR(g - \hat{g}) \tag{2.54}$$

Now,

$$\Rightarrow \quad e^T(t)Pe(t) \geq \delta_{\min}(P)\|e(t)\|^2 \tag{2.55}$$

$$\Rightarrow \quad e^T(t)PR(f - \hat{f}) \leq k\delta_{\max}(PR)\|e(t)\|^2 \tag{2.56}$$

$$\Rightarrow \quad e^T(t)PR(g - \hat{g}) \leq k_1\delta_{\max}(PR)\|e(t)\|^2 \tag{2.57}$$

Put (2.55), (2.56) and (2.57) into (2.54), we get:

$$\dot{V} \leq -\delta_{\min}(P)\|e(t)\|^2 + k\delta_{\max}(PR)\|e(t)\|^2 + k_1\delta_{\max}(PR)\|e(t)\|^2$$

$$\dot{V} \leq \left[-\delta_{\min}(P) + (k + k_1)\delta_{\max}(PR)\right]\|e(t)\|^2 \tag{2.58}$$

According to the Lyapunov point of view, the stability is ensured if:

$$-\delta_{\min}(P) + (k + k_1)\delta_{\max}(PR) < 0 \tag{2.59}$$

$$\Rightarrow \quad (k + k_1)\delta_{\max}(PR) < \delta_{\min}(P)$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(P)}{\delta_{\max}(PR)} \tag{2.60}$$

Put (2.60) into (2.58), we get:

$$\Rightarrow \quad \dot{V} \leq \left(-\delta_{\min}(P) + \left(\frac{\delta_{\min}(P)}{\delta_{\max}(PR)}\right)\delta_{\max}(PR)\right)\|e(t)\|^2 \tag{2.61}$$

and this complete the proof .

***Remark(2.4):***

Since $P > 0$ and $\dot{V} < 0$ , Then $e \to 0$ as $t \to \infty$ and hence $x(t) \cong \hat{x}(t)$.

In order to solve the main problem (2.1) – (2.12), numerically based on the results that obtained from theorems (2.1), lemmas (2.1) the following computational algorithm has been presented and developed.

***Computationl Algorithm (2.1)***

***STEP 1 :***

Consider the dynamical system discussed in (2.1), where

$$\mathrm{E}\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + f(x(t),u(t)) + g(x(t))$$

E is non-singular matrix .

***STEP 2 :***

Check the rank$\left(\begin{bmatrix} \mathrm{E} \\ C \end{bmatrix}\right) = n$.

***STEP 3 :***

Check the rank$\left(\begin{bmatrix} s\mathrm{E} - \mathrm{A} \\ C \end{bmatrix}\right) = n,\ \forall s$.

***STEP 4 :***

Find the pseudo-inverse $\mathrm{E}^{+}$ of the non-singular matrix E as follows ( For more detailes see *remarks (1.4)* ).

***STEP 5 :***

Premultiplying $\mathrm{E}^{+}$ the dynamical system (2.38), we get

$$\Rightarrow \quad \mathrm{E}^{+}\mathrm{E}\dot{x} = \mathrm{E}^{+}\mathrm{A}x + \mathrm{E}^{+}\mathrm{B}u + \mathrm{E}^{+}f(x,u) + \mathrm{E}^{+}g(x)$$

$$\Rightarrow \quad \dot{x} = \overline{\mathrm{A}}x + \overline{\mathrm{B}}u + \mathrm{E}^{+}f(x,u) + \mathrm{E}^{+}g(x)$$

where $\overline{\mathrm{A}} = \mathrm{E}^{+}\mathrm{A}$ and $\overline{\mathrm{B}} = \mathrm{E}^{+}\mathrm{B}$.

**STEP 6 :**

Partitioned the matrix A, $C$, E and I, as the following:

(1) $E = [E_1 \;\vdots\; E_2]$, where $E_1 \in \Re^{q \times m}$ and $E_2 \in \Re^{q \times n-m}$

(2) $A = [A_1 \;\vdots\; A_2]$, where $A_1 \in \Re^{q \times m}$ and $A_2 \in \Re^{q \times n-m}$

(3) $I = [J_1 \;\vdots\; J_2]$, where $J_1 \in \Re^{n \times m}$ and $J_2 \in \Re^{n \times n-m}$ and

$$J_1 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$$

(4) $C = [I_m \quad 0]$

**STEP 7 :**

Compute of the matrix $R$ and K from the conditions $RE + KC = I$, we get :

$R[E_1 \;\vdots\; E_2] + K[I_m \quad 0] = [J_1 \;\vdots\; J_2]$

$RE_1 + RE_2 + KI_m = [J_1 \;\vdots\; J_2]$

$\Rightarrow \; RE_2 = J_2$ , $\because E_2$ with rank $(n-m)$ is skinny

$$\Rightarrow \; R = J_2 E^+ \tag{2.62}$$

$\Rightarrow \; RE_1 + KI_m = J_1$

$$\Rightarrow \; K = J_1 - RE_1 \tag{2.63}$$

**STEP 8 :**

Computatonal of the matrix $L$ from the conditions

NRE + LC – RA = 0

$\Rightarrow \; NR[E_1 \;\vdots\; E_2] + L[I_m \quad 0] = R[A_1 \;\vdots\; A_2]$

$\Rightarrow \; NRE_1 + NRE_2 + L = RA_1 + RA_2$

$\Rightarrow \; NRE_1 + L = RA_1$

$$\Rightarrow \; L = RA_1 - NRE_1 \tag{2.64}$$

***STEP 9 :***

Computational the matrix $G$ from the conditions $G - R\mathrm{B} = 0$

$$\Rightarrow G = R\mathrm{B} \qquad (2.65)$$

***STEP 10 :***

Solve the following algebraic Ricatti equation

$\mathrm{N}^{\mathrm{T}}\mathrm{P} + \mathrm{PN} - 2C^{\mathrm{T}}C = -2\mathrm{P}$, we simplified the last equation the form:

$$(\mathrm{N} + \mathrm{I}_n)^{\mathrm{T}}\mathrm{P} + \mathrm{P}(\mathrm{N} + \mathrm{I}_n) = 2C^{\mathrm{T}}C \qquad (2.66)$$

***STEP 11 :***

Find the Lipschitz constant of $f(x(t), u(t))$ and $g(x(t))$ to be $\| f_2(x,u) - f_1(x,u) \| \le k \| x_2 - x_1 \|$ and $\| g(x_2) - g(x_1) \| \le k_1 \| x_2 - x_1 \|$

and then check the Lipschitz condition $k + k_1 < \dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$.

***STEP 12 :***

The observer is shown as the following :

$$\dot{z} = \mathrm{N}z(t) + Ly(t) + Gu(t) + Rf(\hat{x}(t), u(t)) + Rg(\hat{x}(t)) - P^{-1}C^{\mathrm{T}}(C\hat{x}(t) - y(t)).$$

***STEP 13 :***

Find the solutions of the dynamical system (2.38)

$x_1(t), x_2(t), \ldots, x_n(t)$.

***STEP 14 :***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, where $e = x - \hat{x}$.

Based on the algorithm steps, the following illustrations have been implemented.

***Illustrations (2.1): E is square invertible matrix***

***STEP (1)*** *( Problem Formulatons )*

Consider the dynamical system

$$\mathrm{E}\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + f(x(t),u(t)) + g(x(t))$$

where

$$\mathrm{E} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix},\ \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.6 & -1.5 & -0.9 \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f(x,u) = \begin{bmatrix} 0.01u\cos(x_1 + x_2) \\ 0.002u\sin(x_2) \\ 0.001u\sin(x_3) \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) \\ 0.001\cos(x_3) \end{bmatrix},\ |\mathrm{E}| \neq 0$$

***Case ( I ):*** *( When control u = 1 )*

If u =1, Then, the dynamical system is written as the following:

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.6 & -1.5 & -0.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.01\cos(x_1 + x_2) \\ 0.002\sin(x_2) \\ 0.001\sin(x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) \\ 0.001\cos(x_3) \end{bmatrix}$$

***STEP (2)***

Check the rank $\left(\begin{bmatrix} \mathrm{E} \\ C \end{bmatrix}\right) = n$.

$$\mathrm{M} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the rank(M) = 3.

***STEP (3)***

Check the rank$\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s$.

where $s = 100$

$$M' = \begin{bmatrix} 400 & 199 & 0 \\ 0 & -200 & -499 \\ 0.6 & 1.5 & 200.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the rank($M'$) = 3.

***STEP (4)***

Find the inverse of the matrix E.

Since E is square and invertible, then

$$\Rightarrow E^{-1} = \begin{bmatrix} 0.25 & 0.25 & -0.4167 \\ 0 & -0.5 & 0.8333 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

***STEP (5)***

Multiplying the system by $E^{-1}$, we get:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.875 & 0.625 \\ -0.5 & -1.25 & -1.25 \\ -0.2 & -0.5 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -0.4167 \\ 0.8333 \\ 0.3333 \end{bmatrix} +$$

$$\begin{bmatrix} 0.00025\cos(x_1 + x_2) + 0.0005\sin(x_2) - 0.0004167\sin(x_3) \\ -0.001\sin(x_2) + 0.0008333\sin(x_3) \\ 0.0003333\sin(x_3) \end{bmatrix} +$$

$$\begin{bmatrix} 0.00075\sin(x_1)\cos(x_2) - 0.0004167\cos(x_3) \\ -0.0015\sin(x_1)\cos(x_2) + 0.0008333\cos(x_3) \\ 0.0003333\cos(x_3) \end{bmatrix}$$

***STEP (6)***

Partitioned the matrix $A, E$ and $I$, as the following

(1) $E = [E_1 \;\vdots\; E_2]$, where $E_1 \in \Re^{q\times m}$ and $E_2 \in \Re^{q\times n-m}$

$$E_1 = \begin{bmatrix} 4 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

where $q = 3$, $m = 2$, and $n - m = 1$

(2) $A = [A_1 \;\vdots\; A_2]$, where $A_1 \in \Re^{q\times m}$ and $A_2 \in \Re^{q\times n-m}$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -0.6 & -1.5 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 \\ 1 \\ -0.9 \end{bmatrix}$$

where $q = 3$, $m = 2$, and $n - m = 1$

(3) $I = [J_1 \;\vdots\; J_2]$, where $J_1 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$

$$J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

***STEPS (7 , 8 , and 9)***

Computational the matrix $R$, from the conditions

$RE + KC = I$ , from the eq. (2.62), we get:

$$\Rightarrow \; R = J_2 E^{+} \Rightarrow \; R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1471 & 0.0882 \end{bmatrix}$$

from eq. (2.63), we get:

$$\Rightarrow \; K = J_1 - RE_1 \Rightarrow \; K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0.2941 \end{bmatrix}$$

from eq. (2.64), we get:

$$\Rightarrow\ L = RA_1 - NRE_1 \Rightarrow\ L = \begin{bmatrix} 0 & 0 \\ 0 & 0.5882 \\ -0.0529 & -3.0735 \end{bmatrix}$$

from eq. (2.65), we get:

$$\Rightarrow\ G = RB \Rightarrow\ G = \begin{bmatrix} 0 \\ 0 \\ 0.0882 \end{bmatrix}$$

***STEP (10)***

Solve algebraic Ricatti equation , from eq. (2.66), we get :

$$(N + I_n)^T P + P(N + I_n) = 2C^T C$$

where

$$N = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -10 \end{bmatrix},$$

Find the eigenvalues of N, from MATLAB , *Program ( 1 ) (Computation matrices of observer when E is invertible matrix)* see ( *appendix (B)* ), we get:

$$\text{eig}(N) = \begin{bmatrix} -10.6056 \\ -5 \\ -3.3944 \end{bmatrix},$$ hence the matrix N is stable .

And then the positive definite matrix solution is obtained as follows:

$$\left(\begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -10 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^T \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} +$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}\left(\begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -10 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)$$

$$\Rightarrow P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.3768 & 0.0652 \\ 0 & 0.0652 & 0.0145 \end{bmatrix},$$

We test this matrix is symmetric positive definite, we find the eigenvalues of P, from MATLAB, *Program ( 2 ), (Ricatti Algabraic equations when E is invertible matrix),* see ( *appendix ( B )* ), we get:

$$\text{eig}(P) = \begin{bmatrix} 0.0031 \\ 0.25 \\ 0.3882 \end{bmatrix}, \text{ hence the matrix P is positive definite.}$$

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(P)}{\delta_{\max}(PR)}$

$$(1)\ f(x(t),u(t)) = \begin{bmatrix} 0.01\cos(x_1 + x_2) \\ 0.002\sin(x_2) \\ 0.001\sin(x_3) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1 = \begin{bmatrix} -0.01\sin(x_1 + x_2) & -0.01\sin(x_1 + x_2) & 0 \\ 0 & 0.002\cos(x_2) & 0 \\ 0 & 0 & 0.001\cos(x_3) \end{bmatrix}$$

$$\|J_1\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_1\| \le 0.0143$$

$$\Rightarrow \ \| f(x(t),u(t)) - f(\hat{x}(t),u(t)) \| \le 0.0143 \| x(t) - \hat{x}(t) \|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.0143$.

$$(2)\, g(x(t)) = \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) \\ 0.001\cos(x_3) \end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.003\cos(x_1)\cos(x_2) & -0.003\sin(x_1)\sin(x_2) & 0 \\ 0 & 0 & -0.001\sin(x_3) \end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.00435$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.00435\|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.00435$.

$$k + k_1 = 0.0143 + 0.00435 = 0.01865, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0031}{0.0109} = 0.2844$$

$$\Rightarrow k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \Rightarrow 0.01865 < 0.2844$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -5z_1 + 4e_1$$

$$\dot{z}_2 = -4z_2 + 2z_3 + 0.5882x_2 + 12e_2$$

$$\dot{z}_3 = 2z_1 - 10z_3 - 0.0529\,x_1 - 3.0735\,x_2 + 0.0882 + 0.0002942\,\cos(\hat{x}_2) + 0.0000882\,\sin(\hat{x}_3) + 0.0004413\,\sin(\hat{x}_1)\cos(\hat{x}_2) + 0.0000882\,\cos(\hat{x}_3) - 54e_2$$

**STEP (13)**

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (1) of illustrations (2.1) case(1) to find the solutions of $x_1$, $x_2$ and $x_3$* see ( *appendix ( C )* ).

**STEP (14)**

Find the solution of the error by the equation

$$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g}), \text{ we get}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -16 & 2 \\ 0 & 56 & -10 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1471 & 0.0882 \end{bmatrix} \begin{bmatrix} 0.01\cos(x_1 + x_2) - 0.01\cos(\hat{x}_1 + \hat{x}_2) \\ 0.002\sin(x_2) - 0.002\sin(\hat{x}_2) \\ 0.001\sin(x_3) - 0.001\sin(\hat{x}_3) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1471 & 0.0882 \end{bmatrix} \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) - 0.005\sin(\hat{x}_1)\cos(\hat{x}_2) \\ 0.001\cos(x_3) - 0.001\cos(\hat{x}_3) \end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -9e_1$$

$$\dot{e}_2 = -16e_2 + 2e_3$$

$$\begin{aligned} \dot{e}_3 = {} & 56e_2 - 10e_3 + 0.0002942\ \sin(x_2) - 0.0002942\ \sin(\hat{x}_2) + \\ & 0.0000882\sin(x_3) - 0.0000882\sin(\hat{x}_3) + 0.0004413\sin(x_1)\cos(x_2) \\ & - 0.0004413\sin(\hat{x}_1)\cos(\hat{x}_2) + 0.0000882\cos(x_3) - 0.0000882\cos(\hat{x}_3) \end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. *Problem (2) of illustrations (2.1) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$ see* ( *appendix (C )* ). The numerical solution of error can be found in Figure (2.1) .

**Case (II ):** *( When control* $u = \sin(t)$ *)*

If u = sin(t), Then the dynamical system is written as the following:

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.6 & -1.5 & -1.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix} + \begin{bmatrix} 0.01\sin(t)\cos(x_1+x_2) \\ 0.002\sin(t)\sin(x_2) \\ 0.001\sin(t)\sin(x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) \\ 0.001\cos(x_3) \end{bmatrix}$$

***STEP (2), STEP (3), and STEP(4)***

Are the same as steps of *case ( I )*.

***STEPS (6) , (7) , (8) , (9) and (10)***

Are the same as steps of *case ( I )* .

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1)\ f(x(t),u(t)) = \begin{bmatrix} 0.01\sin(t)\cos(x_1+x_2) \\ 0.002\sin(t)\sin(x_2) \\ 0.001\sin(t)\sin(x_3) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1 = \begin{bmatrix} -0.01\sin(t)\sin(x_1+x_2) & -0.01\sin(t)\sin(x_1+x_2) & 0 \\ 0 & 0.002\sin(t)\cos(x_2) & 0 \\ 0 & 0 & 0.001\sin(t)\cos(x_3) \end{bmatrix}$$

$$\|J_1\| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_1\| \le 0.0143$$

$$\Rightarrow \ \|f(x(t),u(t)) - f(\hat{x}(t),u(t))\| \le 0.0143\|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.0143$.

$$(2)\ g(x(t)) = \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) \\ 0.001\cos(x_3) \end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.003\cos(x_1)\cos(x_2) & -0.003\sin(x_1)\sin(x_2) & 0 \\ 0 & 0 & -0.001\sin(x_3) \end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.00435$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.00435 \|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.00435$.

$$k + k_1 = 0.0143 + 0.00435 = 0.01865, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0031}{0.0109} = 0.2844$$

$$\Rightarrow k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \Rightarrow 0.01865 < 0.2844$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -5z_1 + 4e_1$$

$$\dot{z}_2 = -4z_2 + 2z_3 + 0.5882x_2 + 12e_2$$

$$\begin{aligned} \dot{z}_3 = {} & 2z_1 - 10z_3 - 0.0529\,x_1 - 3.0735\,x_2 + 0.0882\sin(t) + \\ & 0.0002942\ \sin(t)\sin(\hat{x}_2) + 0.0000882\ \sin(t)\sin(\hat{x}_3) + \\ & 0.0004413\ \sin(\hat{x}_1)\cos(\hat{x}_2) + 0.0000882\ \cos(\hat{x}_3) - 54e_2 \end{aligned}$$

**STEP (13)**

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (3) of illustrations (2.1) case(2) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

**STEP (14)**

Find the solution of the error by the equation

$$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g}),$$ we get:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -16 & 2 \\ 0 & 56 & -10 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1471 & 0.0882 \end{bmatrix} \begin{bmatrix} 0.01\sin(t)\cos(x_1 + x_2) - 0.01\sin(t)\cos(\hat{x}_1 + \hat{x}_2) \\ 0.002\sin(t)\sin(x_2) - 0.002\sin(t)\sin(\hat{x}_2) \\ 0.001\sin(t)\sin(x_3) - 0.001\sin(t)\sin(\hat{x}_3) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1471 & 0.0882 \end{bmatrix} \begin{bmatrix} 0 \\ 0.003\sin(x_1)\cos(x_2) - 0.003\sin(\hat{x}_1)\cos(\hat{x}_2) \\ 0.001\cos(x_3) - 0.001\cos(\hat{x}_3) \end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -9e_1$$

$$\dot{e}_2 = -16e_2 + 2e_3$$

$$\begin{aligned} \dot{e}_3 = {} & 56e_2 - 10e_3 + 0.0002942\ \sin(x_2) - 0.0002942\ \sin(\hat{x}_2) + \\ & 0.0000882\sin(x_3) - 0.0000882\sin(\hat{x}_3) + 0.0004413\sin(x_1)\cos(x_2) \\ & - 0.0004413\sin(\hat{x}_1)\cos(\hat{x}_2) + 0.0000882\cos(x_3) - 0.0000882\cos(\hat{x}_3) \end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.2). *Problem (4) of illustrations (2.1) case(2) to find the error solutions of* $e_1$, $e_2$ *and* $e_3$ see ( *appendix (C )* ). The results of error solution (error solutions ) are ploted in the fllowing graph

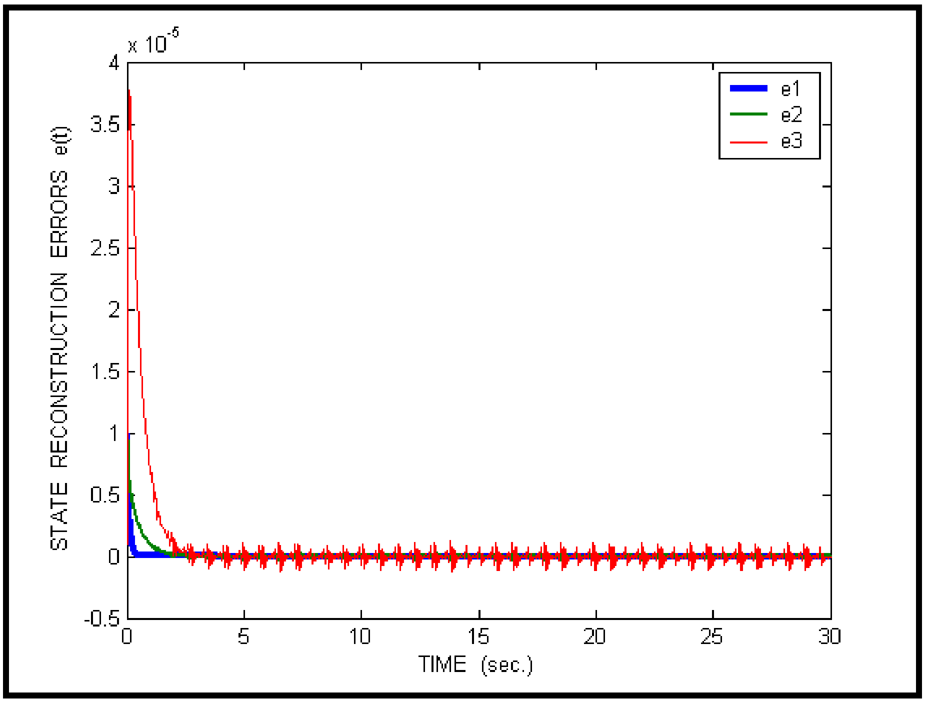


***Figure (2.1) The non-linear system of illustrations (2.1)***

***If u =1 represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

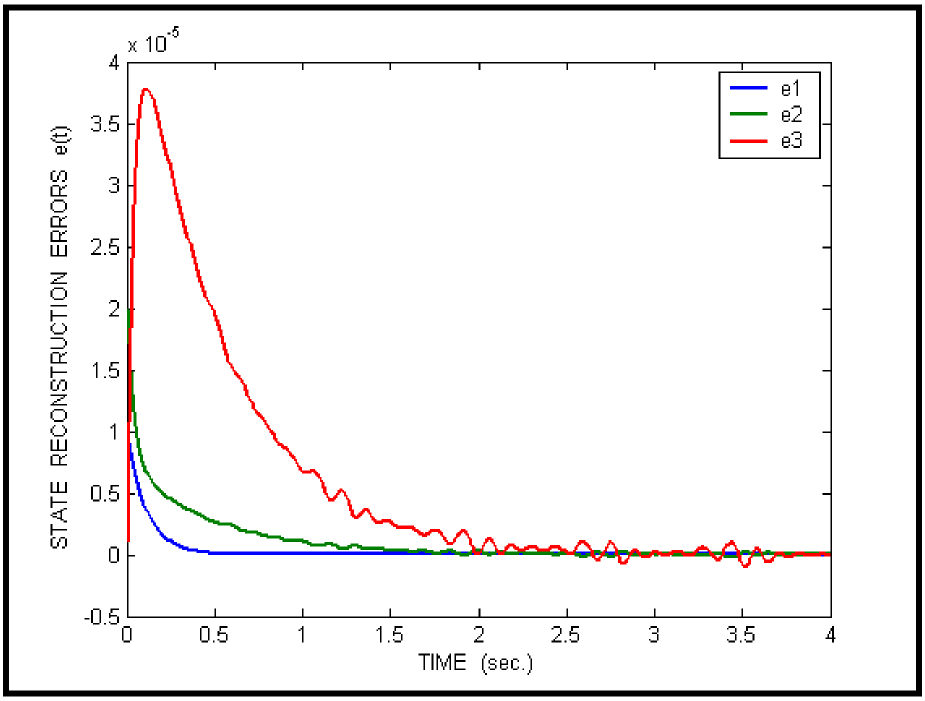


***Figure (2.2) The non-linear system of illustrations (2.1)***

***If u =sin(t) represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

***Illustrations (2.2): E is square and not invertible matrix ( S.V.D )***

***STEP (1)*** *( Problem Formulation )*

Consider the dynamical system

$$\mathrm{E}\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + f(x(t),u(t)) + g(x(t))$$

where

$$\mathrm{E} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{bmatrix},\ \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & -0.6 & 0.4 \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f(x,u) = \begin{bmatrix} 0.05u\sin(x_1) \\ 0 \\ 0.003u\cos(x_3) \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 0.004\cos(x_1) \\ 0.005\sin(x_1)\cos(x_2) \\ 0 \end{bmatrix},\ |\mathrm{E}| = 0$$

***Case ( I ):*** *( When control u=1 )*

If u =1, Then, the dynamical system is written as the following

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & -0.6 & 0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.05\sin(x_1) \\ 0 \\ 0.003\cos(x_3) \end{bmatrix} + \begin{bmatrix} 0.004\cos(x_1) \\ 0.005\sin(x_1)\cos(x_2) \\ 0 \end{bmatrix}$$

***STEP (2)***

Check the rank$\left(\begin{bmatrix} \mathrm{E} \\ C \end{bmatrix}\right) = n$.

$$\mathrm{M}'' = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the rank$(\mathrm{M}'') = 3$.

***STEP (3)***

Check the $\text{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s.$

where $s = 10$

$$M''' = \begin{bmatrix} 10 & 19 & 30 \\ -10 & 0 & 69 \\ 10 & 20.6 & 29.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(M''') = 3$.

***STEP (4)***

Find the pseudo-inverse of the matrix E.

Since, E is singular value decomposition, then

$$EE^T = \begin{bmatrix} 14 & 20 & 14 \\ 20 & 50 & 20 \\ 14 & 20 & 14 \end{bmatrix}, \quad |EE^T| = 0$$

$\Rightarrow \quad (EE^T)^{-1}$ not exists

$$E^T E = \begin{bmatrix} 3 & 4 & -1 \\ 4 & 8 & 12 \\ -1 & 12 & 67 \end{bmatrix}, \quad |E^T E| = 0$$

$\Rightarrow \quad (E^T E)^{-1}$ not exists

And we see that, the eigenvalues of $EE^T$ and $E^T E$ are equals. From MATLAB, *program ( 3 )* see ( *appendix ( B )* ).

$$\text{eig}(EE^T) = \begin{bmatrix} 0 \\ 8.652 \\ 69.348 \end{bmatrix} \quad \text{and} \quad \text{eig}(E^T E) = \begin{bmatrix} 0 \\ 8.652 \\ 69.348 \end{bmatrix}$$

The pseudo-inverse of the matrix E is give by the following commands:

$[U,S,V]$ = svd(E), from MATLAB, we get:

$$U=\begin{bmatrix}-0.3992 & 0.5836 & 0.7071\\ -0.8254 & -0.5646 & 0\\ -0.3992 & 0.5836 & -0.7071\end{bmatrix},$$

The matrix $U$ is represented the eigenvectors of $EE^{T}$.

$$S=\begin{bmatrix}8.3275 & 0 & 0\\ 0 & 2.9414 & 0\\ 0 & 0 & 0\end{bmatrix},$$

The matrix $S$ is represented the diagonal of $D_r$.

$$V=\begin{bmatrix}0.0032 & 0.5888 & -0.8083\\ -0.1918 & 0.7937 & 0.5774\\ -0.9814 & -0.1531 & -0.1155\end{bmatrix},$$

The matrix $V$ is represented the eigenvectors of $E^{T}E$ and also is represented the $V^{T}$.

The pseudo-inverse of the matrix E is

$E^{+}=VD^{+}U^{T}$, where $D^{+}=(D_r^{-1})^{T}$

$$=\begin{bmatrix}-0.0382 & 0.0365 & -0.0382\\ 0.1293 & -0.2107 & 0.1293\\ 0.1533 & -0.0307 & 0.1533\end{bmatrix}$$

**STEP (5)**

Multiplying the system by $E^{+}$, we get:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0.0115 & -0.0153 & 0.0212\\ -0.0388 & 0.0517 & -0.159\\ -0.046 & 0.0613 & 0.0306\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}-0.0382\\ 0.1293\\ 0.1533\end{bmatrix}+$$
$$\begin{bmatrix}-0.00191\sin(x_1)-0.0001146\cos(x_3)\\ 0.006465\sin(x_1)+0.0003879\cos(x_3)\\ 0.007665\sin(x_1)+0.0004599\cos(x_3)\end{bmatrix}+$$
$$\begin{bmatrix}-0.0001529\cos(x_1)+0.0001825\sin(x_1)\cos(x_2)\\ 0.0005172\cos(x_1)-0.0010535\sin(x_1)\cos(x_2)\\ 0.0006132\cos(x_1)-0.0001535\sin(x_1)\cos(x_2)\end{bmatrix}$$

**STEP (6)**

Partitioned the matrix $A, E$ and $I$, as the following

(1 ) $E=[E_1 \;\vdots\; E_2]$, where $E_1 \in \Re^{q\times m}$ and $E_2 \in \Re^{q\times n-m}$

$$E_1=\begin{bmatrix}1 & 2\\ -1 & 0\\ 1 & 2\end{bmatrix} \quad \text{and} \quad E_2=\begin{bmatrix}3\\ 7\\ 3\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(2) $A=[A_1 \;\vdots\; A_2]$, where $A_1 \in \Re^{q\times m}$ and $A_2 \in \Re^{q\times n-m}$

$$A_1=\begin{bmatrix}0 & 1\\ 0 & 0\\ -0.3 & -0.6\end{bmatrix} \quad \text{and} \quad A_2=\begin{bmatrix}0\\ 1\\ 0.4\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(3) $I=[J_1 \;\vdots\; J_2]$, where $J_1=\begin{bmatrix}I_m\\ 0\end{bmatrix}$ and $J_2=\begin{bmatrix}0\\ I_{n-m}\end{bmatrix}$

$$J_1=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \quad \text{and} \quad J_2=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

***STEPS (7, 8, and 9)***

Computational the matrix $R$, from the conditions $R\mathrm{E}+\mathrm{K}C=\mathrm{I}$, from the eq. (2.62), we get:

$$\Rightarrow\ R=J_2\mathrm{E}^{+}$$

$$\Rightarrow\ R=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0.0448 & 0.1045 & 0.0448\end{bmatrix}$$

From eq. (2.63), we get:

$$\Rightarrow\ \mathrm{K}=J_1-R\mathrm{E}_1$$

$$\Rightarrow\ \mathrm{K}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0.0448 & -0.1791\end{bmatrix}$$

From eq. (2.64), we get:

$$\Rightarrow\ L=R\mathrm{A}_1-\mathrm{N}R\mathrm{E}_1$$

$$\Rightarrow\ L=\begin{bmatrix}0 & 0\\ 0.0149 & -0.1791\\ -0.0881 & 0.9134\end{bmatrix}$$

From eq. (2.65), we get:

$$\Rightarrow\ G=R\mathrm{B}$$

$$\Rightarrow\ G=\begin{bmatrix}0\\ 0\\ 0.0448\end{bmatrix}$$

***STEP (10)***

Solve algebraic Ricatti equation, from eq. (2.66), we get:

$$(\mathrm{N}+\mathrm{I}_n)^{\mathrm{T}}\mathrm{P}+\mathrm{P}(\mathrm{N}+\mathrm{I}_n)=2C^{\mathrm{T}}C$$

where

$$\mathrm{N}=\begin{bmatrix}-3 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -5\end{bmatrix},$$

Find the eigenvalues of N, from MATLAB, *Program ( 4 ) (Computation matrices of observer when E is singular value decomposition)* see ( *appendix (B)* ), we get:

$$\text{eig(N)} = \begin{bmatrix} -5.3028 \\ -3 \\ -1.6972 \end{bmatrix}$$, hence the matrix N is stable.

And then the positive definite matrix solution is obtained as follows:

$$\left(\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{\mathrm{T}} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} +$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \left(\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)$$

$$\Rightarrow \mathrm{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.2667 & 0.2667 \\ 0 & 0.2667 & 0.0667 \end{bmatrix},$$

We test this matrix is symmetric positive definite , we find the eigenvalues of P, from MATLAB, *Program ( 5 ), (Ricatti Algabraic equations when E is singular value decomposition),* see ( *appendix ( B )* ), we get:

$$\text{eig(P)} = \begin{bmatrix} 0.0101 \\ 0.55 \\ 1.3233 \end{bmatrix}$$, hence the matrix P is positive definite .

**STEP (11)**

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions: $k+k_1<\dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1)\quad f(x(t),u(t))=\begin{bmatrix}0.05\sin(x_1)\\0\\0.003\cos(x_3)\end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1=\begin{bmatrix}0.05\cos(x_1) & 0 & 0\\0 & 0 & 0\\0 & 0 & -0.003\cos(x_3)\end{bmatrix}$$

$$\|J_1\|=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_1\|\le 0.05$$

$$\Rightarrow\quad \|f(x(t),u(t))-f(\hat{x}(t),u(t))\|\le 0.05\|x(t)-\hat{x}(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k=0.05$.

$$(2)\; g(x(t))=\begin{bmatrix}0.004\cos(x_1)\\0.005\sin(x_1)\cos(x_2)\\0\end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2=\begin{bmatrix}-0.004\sin(x_1) & 0 & 0\\0.005\cos(x_1)\cos(x_2) & -0.005\sin(x_1)\sin(x_2) & 0\\0 & 0 & 0)\end{bmatrix}$$

$$\|J_2\|=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_2\|\le 0.0081$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.0081\|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.0081$.

$$k + k_1 = 0.05 + 0.0081 = 0.0581, \quad \frac{\delta_{\min}(\text{P})}{\delta_{\max}(\text{P}R)} = \frac{0.0101}{0.0309} = 0.3268$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\text{P})}{\delta_{\max}(\text{P}R)} \quad \Rightarrow \quad 0.0581 < 0.3268$$

Hence, Lipschitz constant is satisfied.

***<u>STEP (12)</u>***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -3z_1 + 2e_1$$

$$\dot{z}_2 = -2z_2 + z_3 + 0.0149x_1 - 0.1791x_2 + 5e_2$$

$$\dot{z}_3 = z_2 - 5z_3 - 0.0881x_1 + 0.9134x_2 + 0.0448 + 0.00224\sin(\hat{x}_1) + 0.0001344\cos(\hat{x}_3) + 0.0001792\cos(\hat{x}_1) + 0.0005225\sin(\hat{x}_1)\cos(\hat{x}_2) - 20e_2$$

***<u>STEP (13)</u>***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (5) of illustrations (2.2) case(1) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

***<u>STEP (14)</u>***

Find the solution of the error by the equation

$\dot{e} = (\text{N} - \text{P}^{-1}C^{\text{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

$$\begin{bmatrix}\dot e_1\\ \dot e_2\\ \dot e_3\end{bmatrix}=\begin{bmatrix}-5&0&0\\0&-7&1\\0&21&-5\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}+$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0.0448&0.1045&0.0448\end{bmatrix}\begin{bmatrix}0.05\sin(x_1)-0.05\sin(\hat x_1)\\0\\0.003\cos(x_3)-0.003\cos(\hat x_3)\end{bmatrix}+$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0.0448&0.1045&0.0448\end{bmatrix}\begin{bmatrix}0.004\cos(x_1)-0.004\cos(\hat x_1)\\0.005\sin(x_1)\cos(x_2)-0.005\sin(\hat x_1)\cos(\hat x_2)\\0\end{bmatrix}$$

and simplified the system, we get:

$$\dot e_1=-5e_1$$

$$\dot e_2=-7e_2+e_3$$

$$\begin{aligned}\dot e_3 &= 21e_2-5e_3+0.00224\sin(x_1)-0.00224\sin(\hat x_1)+\\ &\quad 0.0001344\cos(x_3)-0.0001344\cos(\hat x_3)+\\ &\quad 0.0001792\cos(x_1)-0.0001792\cos(\hat x_1)+\\ &\quad 0.0005225\sin(x_1)\cos(x_2)-0.0005225\sin(\hat x_1)\cos(\hat x_2)\end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot e$. *Problem ( 6 ) of illustrations (2.2) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$* see ( *appendix (C* ) ). The numerical solution of error can be found in Figure (2.3).

***<u>Case (II)</u> :*** *( When control $u=\sin(t)$ )*

If u = sin( t), Then, the dynamical system is written as the following:

$$\begin{bmatrix}1&2&3\\-1&0&7\\1&2&3\end{bmatrix}\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-0.3&-0.6&0.4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\\sin(t)\end{bmatrix}+$$

$$\begin{bmatrix}0.05\sin(t)\sin(x_1)\\0\\0.003\sin(t)\cos(x_3)\end{bmatrix}+\begin{bmatrix}0.004\cos(x_1)\\0.005\sin(x_1)\cos(x_2)\\0\end{bmatrix}$$

***STEP (2),STEP (3), and STEP(4)***

Are the same as steps of *case ( I ).*

***STEPS (6) ,(7) ,(8) ,(9) and (10)***

Are the same as steps of *case ( I ) .*

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions: $k+k_1<\dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1)\quad f(x(t),u(t))=\begin{bmatrix}0.05\sin(t)\sin(x_1)\\0\\0.003\sin(t)\cos(x_3)\end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1=\begin{bmatrix}0.05\sin(t)\cos(x_1) & 0 & 0\\0 & 0 & 0\\0 & 0 & -0.003\sin(t)\cos(x_3)\end{bmatrix}$$

$$\|J_1\|=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_1\|\le 0.05$$

$$\Rightarrow\quad \|f(x(t),u(t))-f(\hat{x}(t),u(t))\|\le 0.05\|x(t)-\hat{x}(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k=0.05$.

$$(2)\, g(x(t))=\begin{bmatrix}0.004\cos(x_1)\\0.005\sin(x_1)\cos(x_2)\\0\end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2=\begin{bmatrix}-0.004\sin(x_1) & 0 & 0\\0.005\cos(x_1)\cos(x_2) & -0.005\sin(x_1)\sin(x_2) & 0\\0 & 0 & 0)\end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.0081$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.0081 \|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.008124$.

$$k + k_1 = 0.05 + 0.0081 = 0.0581, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0101}{0.0309} = 0.3268$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.0581 < 0.3268$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -3z_1 + 2e_1$$

$$\dot{z}_2 = -2z_2 + z_3 + 0.0149x_1 - 0.1791x_2 + 5e_2$$

$$\dot{z}_3 = z_2 - 5z_3 - 0.0881\, x_1 + 0.9134\, x_2 + 0.0448\, \sin(t) + 0.00224\, \sin(t)\sin(\hat{x}_1) + 0.0001344\, \sin(t)\cos(\hat{x}_3) + 0.0001792\, \cos(\hat{x}_1) 0.0005225\, \sin(\hat{x}_1)\cos(\hat{x}_2) - 20\, e_2$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (7) of illustrations (2.2) case(2) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

and simplified the system, we get:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}=\begin{bmatrix}-5 & 0 & 0\\ 0 & -7 & 1\\ 0 & 21 & -5\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix}+$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0.0448 & 0.1045 & 0.0448\end{bmatrix}\begin{bmatrix}0.05\sin(t)\sin(x_1)-0.05\sin(t)\sin(\hat{x}_1)\\ 0\\ 0.003\sin(t)\cos(x_3)-0.003\sin(t)\cos(\hat{x}_3)\end{bmatrix}+$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0.0448 & 0.1045 & 0.0448\end{bmatrix}\begin{bmatrix}0.004\cos(x_1)-0.004\cos(\hat{x}_1)\\ 0.005\sin(x_1)\cos(x_2)-0.005\sin(\hat{x}_1)\cos(\hat{x}_2)\\ 0\end{bmatrix}$$

$$\dot{e}_1=-5e_1$$

$$\dot{e}_2=-7e_2+e_3$$

$$\begin{aligned}\dot{e}_3 = {} & 21e_2-5e_3+0.00224\sin(t)\sin(x_1)- \\ & 0.00224\sin(t)\sin(\hat{x}_1)+0.0001344\sin(t)\cos(x_3)- \\ & 0.0001344\sin(t)\cos\hat{x}_3+0.0001792\cos(x_1)- \\ & 0.0001792\cos(\hat{x}_1)+0.0005225\sin(x_1)\cos(x_2)- \\ & 0.0005225\sin(\hat{x}_1)\cos(\hat{x}_2)\end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.4). *Problem (8) of illustrations (2. 2) case(2) to find the error solutions of* $e_1$, $e_2$ *and* $e_3$ see ( *appendix (C )* ). The results of error solution ( error solutions ) are ploted in the fllowing graphs.

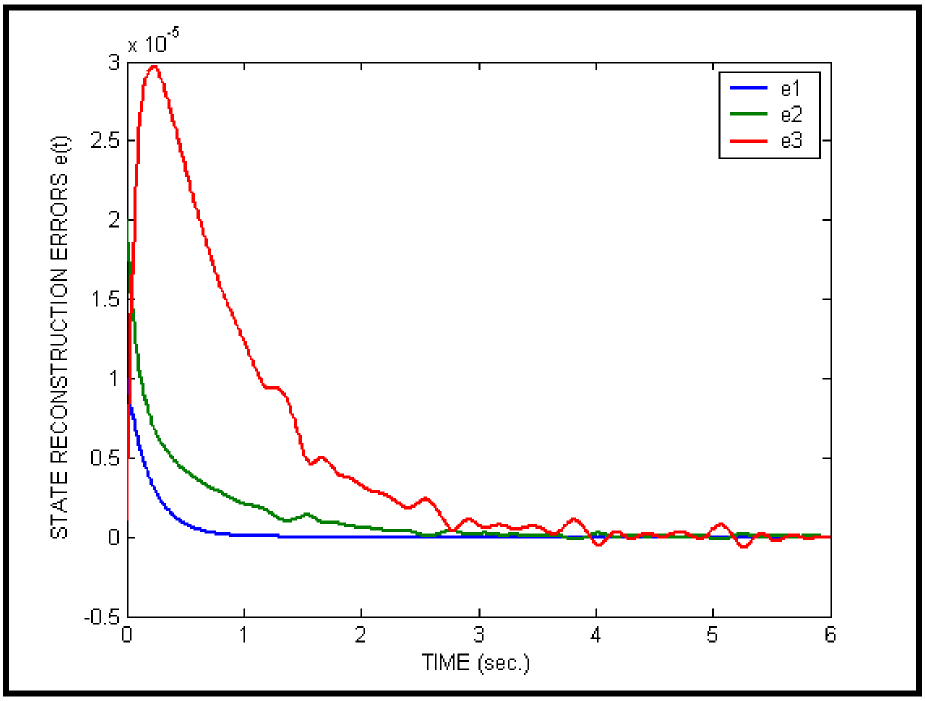


***Figure (2.3) The non-linear system of illustrations (2.2) If u =1 represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

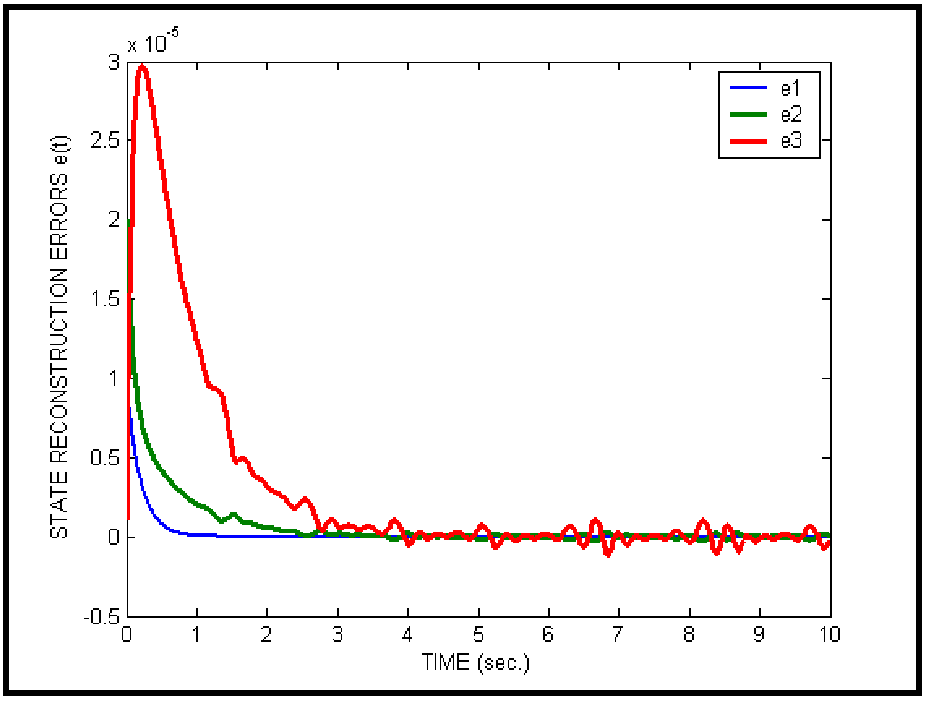


***Figure (2.4) The non-linear system of illustrations (2.2) If u =sin(t) represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

***Illustrations (2.3): E is square and not invertible matrix ( S.V.D )***

***STEP (1)*** *( Problem Formulation )*

Consider the dynamical system

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))$$

where

$$E = \begin{bmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & -1.2 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f(x,u) = \begin{bmatrix} 0.001u\sin(x_1)\cos(x_2) \\ 0.01u\cos(x_2)\sin(x_3) \\ 0.02u\sin(x_2)\cos(x_1) \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 0.003\cos(x_3) \\ 0 \\ 0.01\cos(x_1 + x_2) \end{bmatrix},\ |E| = 0$$

***Case ( I ):*** *( When control u=1 )*

If u =1, Then, the dynamical system is written as the following

$$\begin{bmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & -1.2 & -0.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.001\sin(x_1)\cos(x_2) \\ 0.01\cos(x_2)\sin(x_3) \\ 0.02\sin(x_2)\cos(x_1) \end{bmatrix} + \begin{bmatrix} 0.003\cos(x_3) \\ 0 \\ 0.01\cos(x_1 + x_2) \end{bmatrix}$$

***STEP (2)***

Check the rank $\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n.$

$$T' = \begin{bmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(T') = 3$.

**STEP (3)**

Check the $\text{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s$.

where $s = 10$

$$T'' = \begin{bmatrix} 10 & 29 & -40 \\ -20 & 10 & 19 \\ -89.7 & 151.2 & 0.9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(T'') = 3$.

**STEP (4)**

Find the pseudo-inverse of the matrix E .

Since, E is singular value decomposition, then

$$EE^T = \begin{bmatrix} 26 & -7 & 36 \\ -7 & 9 & 33 \\ 36 & 33 & 306 \end{bmatrix}, \ |EE^T| = 0$$

$\Rightarrow \quad (EE^T)^{-1}$ not exists

$$E^T E = \begin{bmatrix} 86 & -134 & -8 \\ -134 & 235 & -10 \\ -8 & -10 & 20 \end{bmatrix}, \ |E^T E| = 0$$

$\Rightarrow \quad (E^T E)^{-1}$ not exists

And we see that, the eigenvalues of $EE^T$ and $E^TE$ are equals. From MATLAB, *program ( 6 )* see ( *appendix ( B )* ).

$$\text{eig}(EE^T) = \begin{bmatrix} 0 \\ 27.1115 \\ 313.8885 \end{bmatrix} \quad \text{and} \quad \text{eig}(E^TE) = \begin{bmatrix} 0 \\ 27.1115 \\ 313.8885 \end{bmatrix}$$

The pseudo-inverse of the matrix E is give by the following commands:

$[U,S,V] = \text{svd}(E)$, from MATLAB, we get :

$$U = \begin{bmatrix} 0.1209 & 0.8887 & 0.4423 \\ 0.1041 & -0.4545 & 0.8847 \\ 0.9872 & -0.0609 & -0.1474 \end{bmatrix},$$

The matrix $U$ is represented the eigenvectors of $EE^T$.

$$S = \begin{bmatrix} 17.7169 & 0 & 0 \\ 0 & 5.2069 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The matrix $S$ is represented the diagonal of $D_r$.

$$V = \begin{bmatrix} -0.0564 & 0.4506 & 0.7352 \\ 0.8622 & 0.2492 & 0.4411 \\ -0.0156 & -0.8573 & 0.5147 \end{bmatrix},$$

The matrix $V$ is represented the eigenvectors of $E^TE$ and also is represented the $V^T$.

The pseudo-inverse of the matrix E is

$E^+ = VD^+U^T$, where $D^+ = (D_r^{-1})^T$

$$= \begin{bmatrix} 0.1437 & -0.0782 & -0.0383 \\ 0.0456 & -0.0191 & 0.0222 \\ 0.0803 & -0.0342 & 0.0358 \end{bmatrix}$$

***STEP (5)***

Multiplying the system by $E^{+}$, we get:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0.0115 & 0.1897 & -0.0438\\ -0.0067 & 0.019 & -0.0391\\ -0.0107 & 0.0374 & -0.0664\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}-0.0383\\ 0.0222\\ 0.0358\end{bmatrix}+$$

$$\begin{bmatrix}0.0001437\sin(x_1)\cos(x_2)-0.000782\cos(x_2)\sin(x_3)-\\ 0.000766\sin(x_2)\cos(x_1)\\ 0.0000456\sin(x_1)\cos(x_2)-0.000191\cos(x_2)\sin(x_3)+\\ 0.000444\sin(x_2)\cos(x_1)\\ 0.0000456\sin(x_1)\cos(x_2)-0.000342\cos(x_2)\sin(x_3)+\\ 0.000716\sin(x_2)\cos(x_1)\end{bmatrix}+$$

$$\begin{bmatrix}0.0004311\cos(x_3)-0.000383\cos(x_1+x_2)\\ 0.0001368\cos(x_3)+0.000222\cos(x_1+x_2)\\ 0.0002409\cos(x_3)+0.000358\cos(x_1+x_2)\end{bmatrix}$$

***STEP (6)***

Partitioned the matrix $A, E$ and $I$, as the following

(1) $E=[E_1 \;\vdots\; E_2]$, where $E_1\in\Re^{q\times m}$ and $E_2\in\Re^{q\times n-m}$

$$E_1=\begin{bmatrix}1 & 3\\ -2 & 1\\ -9 & 15\end{bmatrix} \quad\text{and}\quad E_2=\begin{bmatrix}-4\\ 2\\ 0\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(2) $A=[A_1 \;\vdots\; A_2]$, where $A_1\in\Re^{q\times m}$ and $A_2\in\Re^{q\times n-m}$

$$A_1=\begin{bmatrix}0 & 1\\ 0 & 0\\ -0.3 & -1.2\end{bmatrix} \quad\text{and}\quad A_2=\begin{bmatrix}0\\ 1\\ -0.9\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(3) $I=[J_1 \;\vdots\; J_2]$, where $J_1=\begin{bmatrix}I_m\\ 0\end{bmatrix}$ and $J_2=\begin{bmatrix}0\\ I_{n-m}\end{bmatrix}$

$$J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

***STEPS (7, 8, and 9)***

Computational the matrix $R$ , from the conditions

$R\mathrm{E} + \mathrm{K}C = \mathrm{I}$, from the eq. (2.62), we get:

$$\Rightarrow R = J_2 \mathrm{E}^{+}$$

$$\Rightarrow R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.2 & 0.1 & 0 \end{bmatrix}$$

From eq. (2.63), we get:

$$\Rightarrow \mathrm{K} = J_1 - R\mathrm{E}_1$$

$$\Rightarrow \mathrm{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.4 & 0.5 \end{bmatrix}$$

From eq. (2.64), we get:

$$\Rightarrow L = R\mathrm{A}_1 - \mathrm{N}R\mathrm{E}_1$$

$$\Rightarrow L = \begin{bmatrix} 0 & 0 \\ 1.2 & 1.5 \\ -1.2 & -1.7 \end{bmatrix}$$

From eq. (2.65), we get :

$$\Rightarrow G = R\mathrm{B}$$

$$\Rightarrow G = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**STEP (10)**

Solve algebraic Ricatti equation, from eq. (2.66), we get:

$$(\mathrm{N}+\mathrm{I}_n)^{\mathrm{T}}\mathrm{P}+\mathrm{P}(\mathrm{N}+\mathrm{I}_n)=2C^{\mathrm{T}}C$$

where

$$\mathrm{N}=\begin{bmatrix}-3 & 0 & 0\\ 0 & -6 & 3\\ 0 & 3 & -3\end{bmatrix},$$

Find the eigenvalues of N, from MATLAB, *Program ( 7 ) (Computation matrices of observer when E is singular value decomposition)* see ( *appendix (B)* ), we get:

$$\mathrm{eig(N)}=\begin{bmatrix}-7.8541\\ -3\\ -1.1459\end{bmatrix}$$, hence the matrix N is stable.

And then the positive definite matrix solution is obtained as follows:

$$\left(\begin{bmatrix}-3 & 0 & 0\\ 0 & -6 & 3\\ 0 & 3 & -3\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}+$$

$$\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}\left(\begin{bmatrix}-3 & 0 & 0\\ 0 & -6 & 3\\ 0 & 3 & -3\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=2\left(\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\right)$$

$$\Rightarrow \mathrm{P}=\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}=\begin{bmatrix}0.5 & 0 & 0\\ 0 & 0.7143 & 0.8571\\ 0 & 0.8571 & 1.2857\end{bmatrix},$$

We test this matrix is symmetric positive definite , we find the eigenvalues of P, from MATLAB, *Program ( 8 ), (Ricatti Algabraic equations when E is singular value decomposition)*, see ( *appendix ( B )* ), we get:

$$\text{eig(P)} = \begin{bmatrix} 0.0965 \\ 0.5 \\ 1.9035 \end{bmatrix}, \text{ hence the matrix P is positive definite .}$$

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(\text{P})}{\delta_{\max}(\text{P}R)}$

$$(1) \quad f(x(t),u(t)) = \begin{bmatrix} 0.001\sin(x_1)\cos(x_2) \\ 0.01\cos(x_2)\sin(x_3) \\ 0.02\sin(x_2)\cos(x_1) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1 = \begin{bmatrix} 0.001\cos(x_1)\cos(x_2) & -0.001\sin(x_1)\sin(x_2) & 0 \\ 0 & -0.01\sin(x_2)\sin(x_3) & 0.01\cos(x_2)\cos(x_3) \\ -0.02\sin(x_2)\sin(x_1) & 0.02\cos(x_2)\cos(x_1) & 0 \end{bmatrix}$$

$$\|J_1\| = \left( \sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_1\| \le 0.0316$$

$$\Rightarrow \quad \| f(x,u) - f(\hat{x},u) \| \le 0.0316 \|x - \hat{x}\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.0316$.

$$(2)\, g(x(t)) = \begin{bmatrix} 0.003\cos(x_3) \\ 0 \\ 0.01\cos(x_1 + x_2) \end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2 = \begin{bmatrix} 0 & 0 & -0.003\sin(x_3) \\ 0 & 0 & 0 \\ -0.01\sin(x_1 + x_2) & -0.01\sin(x_1 + x_2) & 0 \end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.0104$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.0104 \|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.0104$.

$$k + k_1 = 0.0316 + 0.0104 = 0.042 \ , \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0965}{0.0857} = 1.126$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.042 < 1.126$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -3z_1 + 2e_1$$

$$\dot{z}_2 = -6z_2 + 3z_3 + 1.2x_1 + 1.5x_2 + 7e_2$$

$$\dot{z}_3 = 3z_2 - 3z_3 - 1.2x_1 - 1.7x_2 - 0.0002 \sin(\hat{x}_1)\cos(\hat{x}_2) + 0.001 \cos(\hat{x}_2)\sin(\hat{x}_3) - 0.0006 \cos(\hat{x}_3) - 4.6667 e_2$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (9) of illustrations (2.3) case(1) to find the solutions of $x_1$, $x_2$ and $x_3$* see ( *appendix ( C )* ).

***<u>STEP (14)</u>***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -13 & 3 \\ 0 & 7.6667 & -3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.2 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.001[\sin(x_1)\cos(x_2) - \sin(\hat{x}_1)\cos(\hat{x}_2)] \\ 0.01[\cos(x_2)\sin(x_3) - \cos(\hat{x}_2)\sin(\hat{x}_3)] \\ 0.02[\sin(x_2)\cos(x_1) - \sin(\hat{x}_2)\cos(\hat{x}_1)] \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.2 & 0.11 & 00 \end{bmatrix} \begin{bmatrix} 0.003[\cos(x_3) - \cos(\hat{x}_3)] \\ 0 \\ 0.01[\cos(x_1 + x_2) - \sin(\hat{x}_1 + \hat{x}_2)] \end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -5e_1$$

$$\dot{e}_2 = -13e_2 + 3e_3$$

$$\begin{aligned} \dot{e}_3 = {} & 7.6667\ e_2 - 3e_3 - 0.0002\ \sin(x_1)\cos(x_2) + \\ & 0.0002\ \sin(\hat{x}_1)\cos(\hat{x}_2) + 0.001\ \cos(x_2)\sin(x_3) - \\ & 0.001\ \cos(\hat{x}_2)\sin(\hat{x}_3) - 0.0006\ \cos(x_3) + \\ & 0.0006\ \cos(\hat{x}_3) \end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. *Problem ( 10 ) of illustrations (2.3) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$* see ( *appendix (C )* ). The numerical solution of error can be found in Figure (2.5).

***Case (II) :*** *( When control* $u = \cos(t)$ *)*

If u = cos( t), Then, the dynamical system is written as the following:

$$\begin{bmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3 & -1.2 & -0.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cos(t) + \begin{bmatrix} 0.001\cos(t)\sin(x_1)\cos(x_2) \\ 0.01\cos(t)\cos(x_2)\sin(x_3) \\ 0.02\cos(t)\sin(x_2)\cos(x_1) \end{bmatrix} + \begin{bmatrix} 0.003\cos(x_3) \\ 0 \\ 0.01\cos(x_1 + x_2) \end{bmatrix}$$

***STEP (2), STEP (3), and STEP(4)***

Are the same as steps of *case ( I ).*

***STEPS (6), (7) ,(8) ,(9) and (10)***

Are the same as steps of *case ( I )* .

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the

conditions : $k + k_1 < \dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1) \quad f(x(t),u(t)) = \begin{bmatrix} 0.001\cos(t)\sin(x_1)\cos(x_2) \\ 0.01\cos(t)\cos(x_2)\sin(x_3) \\ 0.02\cos(t)\sin(x_2)\cos(x_1) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1 = \begin{bmatrix} 0.001\cos(t)\cos(x_1) * \cos(x_2) & -0.001\cos(t)\sin(x_1) * \sin(x_2) & 0 \\ 0 & -0.01\cos(t)\sin(x_2) * \sin(x_3) & 0.01\cos(t)\cos(x_2) * \cos(x_3) \\ -0.02\cos(t)\sin(x_2) * \sin(x_1) & 0.02\cos(t)\cos(x_2) * \cos(x_1) & 0 \end{bmatrix}$$

$$\|J_1\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_1\| \leq 0.0316$$

$$\Rightarrow \quad \| f(x,u) - f(\hat{x},u) \| \leq 0.0316 \|x - \hat{x}\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.0316$.

$$(2)\; g(x(t)) = \begin{bmatrix} 0.003\cos(x_3) \\ 0 \\ 0.01\cos(x_1 + x_2) \end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2 = \begin{bmatrix} 0 & 0 & -0.003\sin(x_3) \\ 0 & 0 & 0 \\ -0.01\sin(x_1 + x_2) & -0.01\sin(x_1 + x_2) & 0 \end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \leq 0.0104$$

$$\Rightarrow \quad \| g(x(t)) - g(\hat{x}(t)) \| \leq 0.0104 \|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.0104$.

$$k + k_1 = 0.0316 + 0.0104 = 0.042 \;, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0965}{0.0857} = 1.126$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.042 < 1.126$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -3z_1 + 2e_1$$

$$\dot{z}_2 = -6z_2 + 3z_3 + 1.2x_1 + 1.5x_2 + 7e_2$$

$$\dot{z}_3 = 3z_2 - 3z_3 - 1.2x_1 - 1.7x_2 - 0.0002\cos(t)\sin(\hat{x}_1)\cos(\hat{x}_2) + 0.001\cos(t)\cos(\hat{x}_2)\sin(\hat{x}_3) - 0.0006\cos(\hat{x}_3) - 4.6667e_2$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (11) of illustrations (2.3) case(2) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\ 0 & -13 & 3\\ 0 & 7.6667 & -3\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix} +$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -0.2 & 0.1 & 0\end{bmatrix}\begin{bmatrix}0.001\cos(t)[\sin(x_1)\cos(x_2) - \sin(\hat{x}_1)\cos(\hat{x}_2)]\\ 0.01\cos(t)[\cos(x_2)\sin(x_3) - \cos(\hat{x}_2)\sin(\hat{x}_3)]\\ 0.02\cos(t)[\sin(x_2)\cos(x_1) - \sin(\hat{x}_2)\cos(\hat{x}_1)]\end{bmatrix} +$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -0.2 & 0.1 & 00\end{bmatrix}\begin{bmatrix}0.003[\cos(x_3) - \cos(\hat{x}_3)]\\ 0\\ 0.01[\cos(x_1 + x_2) - \sin(\hat{x}_1 + \hat{x}_2)]\end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -5e_1$$

$$\dot{e}_2 = -13e_2 + 3e_3$$

$$\dot{e}_3 = 7.6667\, e_2 - 3e_3 - 0.0002\, \cos(t)\sin(x_1)\cos(x_2) + 0.0002\, \cos(t)\sin(\hat{x}_1)\cos(\hat{x}_2) + 0.001\cos(t)\cos(x_2)\sin(x_3) - 0.001\cos(t)\cos(\hat{x}_2)\sin(\hat{x}_3) - 0.0006\, \cos(x_3) + 0.0006\, \cos(\hat{x}_3)$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.6). *Problem (12) of illustrations (2.3) case(2) to find the error solutions of* $e_1$, $e_2$ *and* $e_3$ see ( *appendix (C )* ).
The results of error solution ( error solutions ) are ploted in the fllowing graphs.

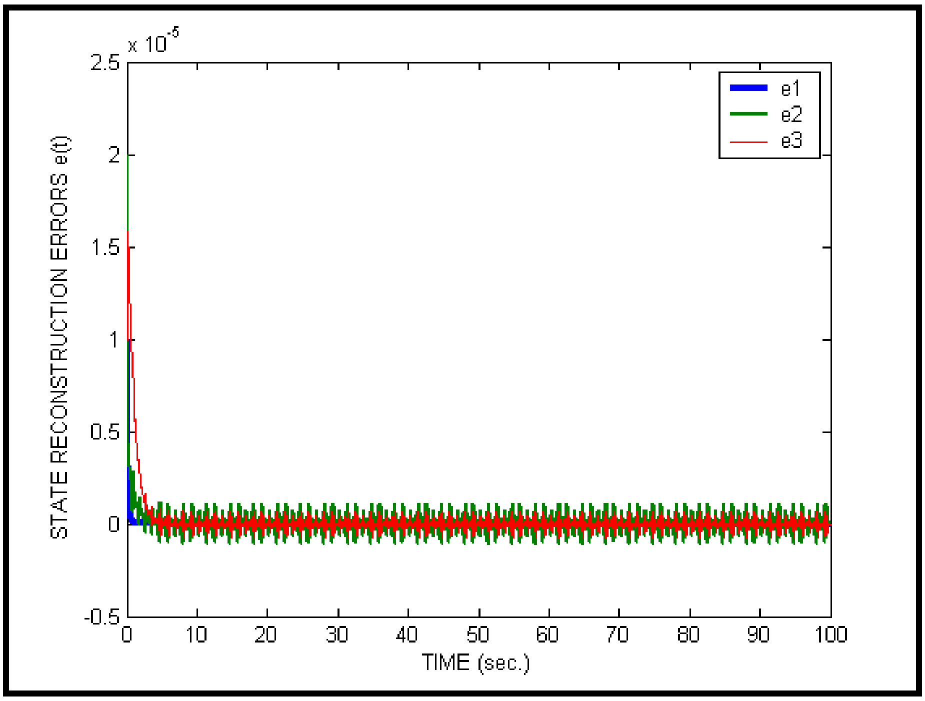


***Figure (2.5) The non-linear system of illustrations (2.3) If u =1 represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

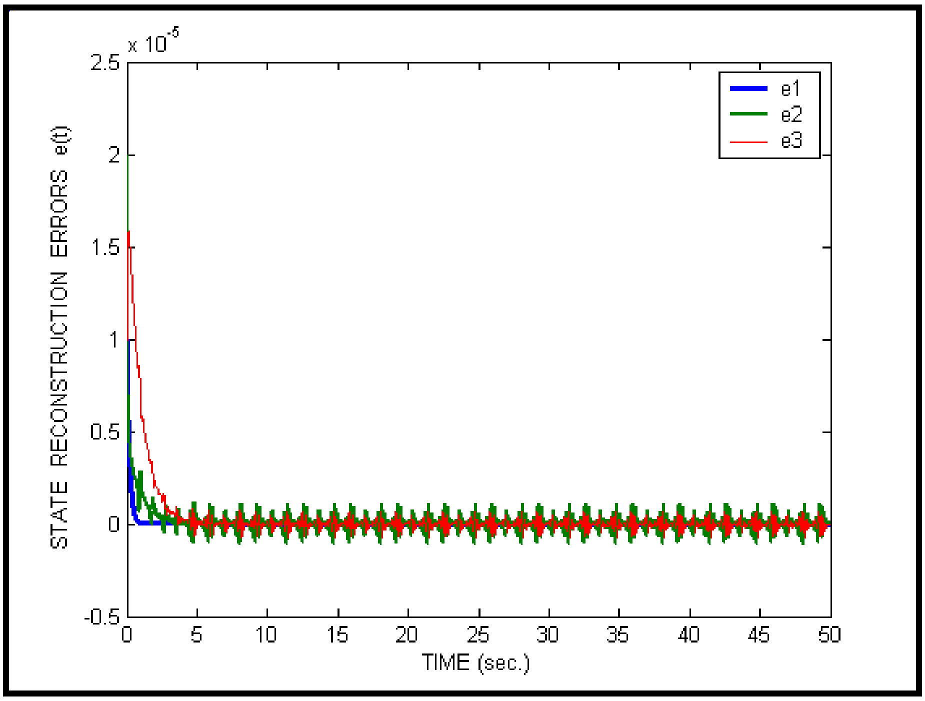


***Figure (2.6) The non-linear system of illustrations (2.3) If u = cos(t) represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

***Illustrations (2.4): E is square and not invertible matrix ( S.V.D )***

***STEP (1)*** *( Problem Formulation )*

Consider the dynamical system

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))$$

where

$$E = \begin{bmatrix} 5 & 10 & 17 \\ 10 & 20 & 34 \\ 17 & 34 & 61 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.4 & -1.6 & -0.2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f(x,u) = \begin{bmatrix} 0.002u\cos(x_3)\sin(x_2) \\ 0.003u\sin(x_1+x_2) \\ 0.001u\cos(x_2)\sin(x_3) \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 0.01\cos(x_1) \\ 0.002\sin(x_1)\sin(x_3) \\ 0.03\cos(x_2)\cos(x_3) \end{bmatrix},\ |E| = 0$$

***Case ( I ):*** *( When control u=1 )*

If u =1, Then, the dynamical system is written as the following

$$\begin{bmatrix} 5 & 10 & 17 \\ 10 & 20 & 34 \\ 17 & 34 & 61 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.4 & -1.6 & -0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0.002\cos(x_3)\sin(x_2) \\ 0.003\sin(x_1+x_2) \\ 0.001\cos(x_2)\sin(x_3) \end{bmatrix} + \begin{bmatrix} 0.01\cos(x_1) \\ 0.002\sin(x_1)\sin(x_3) \\ 0.03\cos(x_2)\cos(x_3) \end{bmatrix}$$

***STEP (2)***

Check the rank $\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n$.

$$Y' = \begin{bmatrix} 5 & 10 & 17 \\ 10 & 20 & 34 \\ 17 & 34 & 61 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(Y') = 3$.

***STEP (3)***

Check the $\text{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s$.

where $s = 10$

$$Y'' = \begin{bmatrix} 50 & 99 & 170 \\ 100 & 200 & 339 \\ 170.4 & 341.6 & 612 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(Y'') = 3$.

***STEP (4)***

Find the pseudo-inverse of the matrix E .

Since, E is singular value decomposition, then

$$EE^T = \begin{bmatrix} 414 & 828 & 1462 \\ 828 & 1656 & 2924 \\ 1462 & 2924 & 5166 \end{bmatrix}, \quad |EE^T| = 0$$

$\Rightarrow \quad (EE^T)^{-1}$ not exists

$$E^T E = \begin{bmatrix} 414 & 828 & 1426 \\ 828 & 1656 & 2924 \\ 1462 & 2924 & 5166 \end{bmatrix}, \quad |E^T E| = 0$$

$\Rightarrow \quad (E^T E)^{-1}$ not exists

And we see that , the eigenvalues of $EE^T$ and $E^T E$ are equals. From MATLAB, *program ( 9 )* see ( *appendix ( B )* ).

$$\text{eig}(EE^T) = \begin{bmatrix} 0 \\ 0.0009 \\ 7.2351 \end{bmatrix} \quad \text{and} \quad \text{eig}(E^T E) = \begin{bmatrix} 0 \\ 0.0009 \\ 7.2351 \end{bmatrix}$$

The pseudo-inverse of the matrix E is give by the following commands:

$[U,S,V]$ = svd(E), from MATLAB, we get :

$$U = \begin{bmatrix} -0.2392 & -0.3779 & -0.8944 \\ -0.4783 & -0.7558 & 0.4472 \\ -0.845 & 0.5348 & 0 \end{bmatrix},$$

The matrix $U$ is represented the eigenvectors of $EE^T$.

$$S = \begin{bmatrix} 85.0595 & 0 & 0 \\ 0 & 0.9405 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The matrix $S$ is represented the diagonal of $D_r$.

$$V = \begin{bmatrix} -0.2392 & -0.3779 & 0.8944 \\ -0.4783 & -0.7558 & -0.4472 \\ -0.845 & 0.5348 & 0 \end{bmatrix},$$

The matrix $V$ is represented the eigenvectors of $E^TE$ and also is represented the $V^T$.

The pseudo-inverse of the matrix E is

$E^+ = VD^+U^T$, where $D^+ = (D_r^{-1})^T$

$$= \begin{bmatrix} 0.1929 & 0.3858 & -0.2696 \\ 0.3047 & 0.6095 & -0.426 \\ 0.1772 & 0.3543 & -0.2632 \end{bmatrix}$$

**STEP (5)**

Multiplying the system by $E^{+}$, we get:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0.1079 & 0.6243 & 0.4397\\ 0.1704 & 0.9863 & 0.6947\\ 0.1053 & 0.5983 & 0.407\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}-0.2696\\ -0.426\\ -0.2632\end{bmatrix}+$$

$$\begin{bmatrix}0.0003858\ \cos(x_3)\sin(x_2)+0.0011574\ \sin(x_1+x_2)\\ 0.00002696\ \cos(x_2)\sin(x_3)\\ 0.0006094\ \cos(x_3)\sin(x_2)+0.0018285\ \sin(x_1+x_2)-\\ 0.000426\ \cos(x_2)\sin(x_3)\\ 0.0003544\ \cos(x_3)\sin(x_2)+0.0010629\ \sin(x_1+x_2)\\ 0.0002632\ \cos(x_2)\sin(x_3)\end{bmatrix}+$$

$$\begin{bmatrix}0.001929\ \cos(x_1)+0.0007716\ \sin(x_1)\sin(x_3)-\\ 0.008088\ \cos(x_2)\cos(x_3)\\ 0.003047\ \cos(x_1)+0.001219\ \sin(x_1)\sin(x_3)-\\ 0.01278\ \cos(x_2)\cos(x_3)\\ 0.001772\ \cos(x_1)+0.000786\ \sin(x_1)\sin(x_3)-\\ 0.007896\ \cos(x_2)\cos(x_3)\end{bmatrix}$$

**STEP (6)**

Partitioned the matrix $A, E$ and $I$, as the following

(1) $E=[E_1 \ \vdots \ E_2]$, where $E_1 \in \Re^{q\times m}$ and $E_2 \in \Re^{q\times n-m}$

$$E_1=\begin{bmatrix}5 & 10\\ 10 & 20\\ 17 & 34\end{bmatrix} \quad \text{and} \quad E_2=\begin{bmatrix}17\\ 34\\ 61\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(2) $A=[A_1 \ \vdots \ A_2]$, where $A_1 \in \Re^{q\times m}$ and $A_2 \in \Re^{q\times n-m}$

$$A_1=\begin{bmatrix}0 & 1\\ 0 & 0\\ -0.4 & -1.6\end{bmatrix} \quad \text{and} \quad A_2=\begin{bmatrix}0\\ 1\\ -0.2\end{bmatrix}$$

where $q=3$, $m=2$, and $n-m=1$

(3) $\mathrm{I} = [J_1 \;\vdots\; J_2]$, where $J_1 = \begin{bmatrix} \mathrm{I}_m \\ 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 \\ \mathrm{I}_{n-m} \end{bmatrix}$

$$J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

***<u>STEPS (7, 8, and 9)</u>***

Computational the matrix $R$ , from the conditions $R\mathrm{E} + \mathrm{K}C = \mathrm{I}$, from the eq. (2.62), we get:

$$\Rightarrow R = J_2 \mathrm{E}^+$$

$$\Rightarrow R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0033 & 0.0066 & 0.0118 \end{bmatrix}$$

From eq. (2.63), we get:

$$\Rightarrow \mathrm{K} = J_1 - R\mathrm{E}_1$$

$$\Rightarrow \mathrm{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.283 & -0.566 \end{bmatrix}$$

From eq. (2.64), we get:

$$\Rightarrow L = R\mathrm{A}_1 - \mathrm{N}R\mathrm{E}_1$$

$$\Rightarrow L = \begin{bmatrix} 0 & 0 \\ -0.566 & -1.132 \\ 1.9763 & 3.9465 \end{bmatrix}$$

From eq. (2.65), we get :

$$\Rightarrow G = R\mathrm{B}$$

$$\Rightarrow G = \begin{bmatrix} 0 \\ 0 \\ 0.0118 \end{bmatrix}$$

***STEP (10)***

Solve algebraic Ricatti equation, from eq. (2.66), we get:

$$(\mathrm{N}+\mathrm{I}_n)^{\mathrm{T}}\mathrm{P}+\mathrm{P}(\mathrm{N}+\mathrm{I}_n)=2C^{\mathrm{T}}C$$

where

$$\mathrm{N}=\begin{bmatrix}-5 & 0 & 0\\ 0 & -3 & 2\\ 0 & 2 & -7\end{bmatrix},$$

Find the eigenvalues of N, from MATLAB, *Program ( 10 ) (Computation matrices of observer when E is singular value decomposition)* see ( *appendix (B)* ), we get:

$$\mathrm{eig(N)}=\begin{bmatrix}-7.8284\\ -5\\ -2.1716\end{bmatrix}$$ , hence the matrix N is stable.

And then the positive definite matrix solution is obtained as follows:

$$\left(\begin{bmatrix}-5 & 0 & 0\\ 0 & -3 & 2\\ 0 & 2 & -7\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}+$$

$$\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}\left(\begin{bmatrix}-5 & 0 & 0\\ 0 & -3 & 2\\ 0 & 2 & -7\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=2\left(\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\right)$$

$$\Rightarrow \mathrm{P}=\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}=\begin{bmatrix}0.25 & 0 & 0\\ 0 & 0.6875 & 0.1875\\ 0 & 0.1875 & 0.0625\end{bmatrix},$$

We test this matrix is symmetric positive definite , we find the eigenvalues of P, from MATLAB, *Program ( 11 ), (Ricatti Algabraic equations when E is singular value decomposition)*, see ( *appendix ( B )* ), we get:

$$\text{eig(P)} = \begin{bmatrix} 0.0106 \\ 0.25 \\ 0.7394 \end{bmatrix}, \text{ hence the matrix P is positive definite .}$$

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(\text{P})}{\delta_{\max}(\text{P}R)}$

$$(1) \quad f(x(t),u(t)) = \begin{bmatrix} 0.002\cos(x_3)\sin(x_2) \\ 0.003\sin(x_1 + x_2) \\ 0.001\cos(x_2)\sin(x_3) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1 = \begin{bmatrix} 0 & 0.002\cos(x_3)\cos(x_2) & -0.002\sin(x_3)\sin(x_2) \\ 0.003\cos(x_1 + x_2) & 0.003\cos(x_1 + x_2) & 0 \\ 0 & -0.001\sin(x_2)\sin(x_3) & 0.001\cos(x_2)\cos(x_3) \end{bmatrix}$$

$$\|J_1\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_1\| \le 0.0052$$

$$\Rightarrow \quad \| f(x,u) - f(\hat{x},u) \| \le 0.0052 \|x - \hat{x}\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.0052$.

$$(2)\, g(x(t)) = \begin{bmatrix} 0.01\cos(x_1) \\ 0.002\sin(x_1)\sin(x_3) \\ 0.03\cos(x_2)\cos(x_3) \end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2 = \begin{bmatrix} -0.01\sin(x_1) & 0 & 0 \\ 0.002\cos(x_1)\sin(x_3) & 0 & 0.002\sin(x_1)\cos(x_3) \\ 0 & -0.03\sin(x_2)\cos(x_3) & -0.03\cos(x_2)\sin(x_3) \end{bmatrix}$$

$$\|J_2\| = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.0436$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.0436 \|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.0436$.

$$k + k_1 = 0.0052 + 0.0436 = 0.0488 \;, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0106}{0.002} = 5.3$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.0488 < 5.3$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -5z_1 + 4e_1$$

$$\dot{z}_2 = -3z_2 + 2z_3 - 0.566x_1 - 1.132x_2 + 8e_2$$

$$\begin{aligned}\dot{z}_3 = {} & 2z_2 - 7z_3 + 1.9763\, x_1 + 3.9465\, x_2 + 0.0118 + \\ & 0.0000066 \cos(\hat{x}_3)\sin(\hat{x}_2) + 0.0000198 \sin(\hat{x}_1 + x_2) + \\ & 0.0000118 \cos(\hat{x}_2)\sin(\hat{x}_3) + 0.000033 \cos(\hat{x}_1) + \\ & 0.0000132 \sin(\hat{x}_1)\sin(\hat{x}_3) + 0.000354 \cos(\hat{x}_2)\cos(\hat{x}_3)\end{aligned}$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (13) of illustrations (2.4) case(1) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -11 & 2 \\ 0 & 26 & -7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0033 & 0.0066 & 0.0118 \end{bmatrix} \begin{bmatrix} 0.002[\cos(x_3)\sin(x_2) - \cos(\hat{x}_3)\sin(\hat{x}_2)] \\ 0.003[\sin(x_1 + x_2) - \sin(\hat{x}_1 + \hat{x}_2)] \\ 0.001[\cos(x_2)\sin(x_3) - \cos(\hat{x}_2)\sin(\hat{x}_3)] \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0033 & 0.0066 & 0.0118 \end{bmatrix} \begin{bmatrix} 0.01[\cos(x_1) - \cos(\hat{x}_1)] \\ 0.002[\sin(x_1)\sin(x_3) - \sin(\hat{x}_1)\sin(\hat{x}_3)] \\ 0.03[\cos(x_2)\cos(x_3) - \cos(\hat{x}_2)\cos(\hat{x}_3)] \end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -9e_1$$

$$\dot{e}_2 = -11e_2 + 2e_3$$

$$\begin{aligned} \dot{e}_3 = {} & 26e_2 - 7e_3 + 0.0000066\cos(x_3)\sin(x_2) - \\ & 0.0000066\cos(\hat{x}_3)\sin(\hat{x}_2) + 0.0000198\sin(x_1 + x_2) - \\ & 0.0000198\sin(\hat{x}_1 + \hat{x}_2) + 0.0000118\cos(x_2)\sin(x_3) - \\ & 0.0000118\cos(\hat{x}_2)\sin(\hat{x}_3) + 0.000033\cos(x_1) - \\ & 0.000033\cos(\hat{x}_1) + 0.0000132\sin(x_1)\sin(x_3) - \\ & 0.0000132\sin(\hat{x}_1)\sin(\hat{x}_3) + 0.000354\cos(x_2)\cos(x_3) - \\ & 0.000354\cos(\hat{x}_2)\cos(\hat{x}_3) \end{aligned}$$

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. *Problem (14) of illustrations (2.4) case(1) to find the error solutions of $e_1$, $e_2$ and $e_3$* see ( *appendix (C )* ). The numerical solution of error can be found in Figure (2.5).

**Case (II) :** *( When control $u = \cos(t)$ )*

If u = cos( t), Then , the dynamical system is written as the following:

$$\begin{bmatrix} 5 & 10 & 17 \\ 10 & 20 & 34 \\ 17 & 34 & 61 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.4 & -1.6 & -0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cos(t) + \begin{bmatrix} 0.002\cos(t)\cos(x_3)\sin(x_2) \\ 0.003\cos(t)\sin(x_1 + x_2) \\ 0.001\cos(t)\cos(x_2)\sin(x_3) \end{bmatrix} + \begin{bmatrix} 0.01\cos(x_1) \\ 0.002\sin(x_1)\sin(x_3) \\ 0.03\cos(x_2)\cos(x_3) \end{bmatrix}$$

***STEP (2), STEP (3), and STEP(4)***

Are the same as steps of *case ( I ).*

***STEPS (6) , (7), (8) , (9) and (10)***

Are the same as steps of *case ( I )* .

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1) \quad f(x(t),u(t)) = \begin{bmatrix} 0.002\cos(t)\cos(x_3)\sin(x_2) \\ 0.003\cos(t)\sin(x_1 + x_2) \\ 0.001\cos(t)\cos(x_2)\sin(x_3) \end{bmatrix}$$

The Jacobian matrix for $f(x(t),u(t))$ is found

$$J_1=\begin{bmatrix} 0 & \begin{matrix}0.002\cos(t)\cos(x_3)*\\ \cos(x_2)\end{matrix} & \begin{matrix}-0.002\cos(t)\sin(x_3)*\\ \sin(x_2)\end{matrix} \\ \begin{matrix}0.003\cos(t)*\\ \cos(x_1+x_2)\end{matrix} & \begin{matrix}0.003\cos(t)*\\ \cos(x_1+x_2)\end{matrix} & 0 \\ 0 & \begin{matrix}-0.001\cos(t)\sin(x_2)*\\ \sin(x_3)\end{matrix} & \begin{matrix}0.001\cos(t)\cos(x_2)*\\ \cos(x_3)\end{matrix}\end{bmatrix}$$

$$\|J_1\|=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_1\|\le 0.0052$$

$$\Rightarrow \quad \|f(x,u)-f(\hat{x},u)\|\le 0.0052\|x-\hat{x}\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k=0.0052$.

$$(2)\; g(x(t))=\begin{bmatrix}0.01\cos(x_1)\\ 0.002\sin(x_1)\sin(x_3)\\ 0.03\cos(x_2)\cos(x_3)\end{bmatrix}$$

The Jacobian matrix for the function $g(x(t))$ is

$$J_2=\begin{bmatrix}-0.01\sin(x_1) & 0 & 0\\ 0.002\cos(x_1)\sin(x_3) & 0 & 0.002\sin(x_1)\cos(x_3)\\ 0 & -0.03\sin(x_2)\cos(x_3) & -0.03\cos(x_2)\sin(x_3)\end{bmatrix}$$

$$\|J_2\|=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$$

$$\|J_2\|\le 0.0436$$

$$\Rightarrow \quad \|g(x(t))-g(\hat{x}(t))\|\le 0.0436\|x(t)-\hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1=0.0436$.

$$k+k_1=0.0052+0.0436=0.0488\ , \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}=\frac{0.0106}{0.002}=5.3$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.0488 < 5.3$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -5z_1 + 4e_1$$

$$\dot{z}_2 = -3z_2 + 2z_3 - 0.566x_1 - 1.132x_2 + 8e_2$$

$$\begin{aligned}\dot{z}_3 = {} & 2z_2 - 7z_3 + 1.9763x_1 + 3.9465x_2 + 0.0118\cos(t) + \\ & 0.0000066\cos(t)\cos(\hat{x}_3)\sin(\hat{x}_2) + 0.0000198\cos(t)\sin(\hat{x}_1 + x_2) + \\ & 0.0000118\cos(t)\cos(\hat{x}_2)\sin(\hat{x}_3) + 0.000033\cos(\hat{x}_1) + \\ & 0.0000132\sin(\hat{x}_1)\sin(\hat{x}_3) + 0.000354\cos(\hat{x}_2)\cos(\hat{x}_3)\end{aligned}$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (15) of illustrations (2.4) case(2) to find the solutions of* $x_1$, $x_2$ *and* $x_3$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation

$\dot{e} = (\mathrm{N} - \mathrm{P}^{-1}C^{\mathrm{T}}C)e(t) + R(f - \hat{f}) + R(g - \hat{g})$, we get:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -11 & 2 \\ 0 & 26 & -7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0033 & 0.0066 & 0.0118 \end{bmatrix} \begin{bmatrix} 0.002\cos(t)[\cos(x_3)\sin(x_2) - \cos(\hat{x}_3)\sin(\hat{x}_2)] \\ 0.003\cos(t)[\sin(x_1 + x_2) - \sin(\hat{x}_1 + \hat{x}_2)] \\ 0.001\cos(t)[\cos(x_2)\sin(x_3) - \cos(\hat{x}_2)\sin(\hat{x}_3)] \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0033 & 0.0066 & 0.0118 \end{bmatrix} \begin{bmatrix} 0.01[\cos(x_1) - \cos(\hat{x}_1)] \\ 0.002[\sin(x_1)\sin(x_3) - \sin(\hat{x}_1)\sin(\hat{x}_3)] \\ 0.03[\cos(x_2)\cos(x_3) - \cos(\hat{x}_2)\cos(\hat{x}_3)] \end{bmatrix}$$

and simplified the system, we get:

$$\dot{e}_1 = -9e_1$$

$$\dot{e}_2 = -11e_2 + 2e_3$$

$$\begin{aligned} \dot{e}_3 = {} & 26e_2 - 7e_3 + 0.0000066\cos(t)\cos(x_3)\sin(x_2) - \\ & 0.0000066\cos(t)\cos(\hat{x}_3)\sin(\hat{x}_2) + 0.0000198\cos(t)\sin(x_1 + x_2) - \\ & 0.0000198\cos(t)\sin(\hat{x}_1 + \hat{x}_2) + 0.0000118\cos(t)\cos(x_2)\sin(x_3) - \\ & 0.0000118\cos(t)\cos(\hat{x}_2)\sin(\hat{x}_3) + 0.000033\cos(x_1) - \\ & 0.000033\cos(\hat{x}_1) + 0.0000132\sin(x_1)\sin(x_3) - \\ & 0.0000132\sin(\hat{x}_1)\sin(\hat{x}_3) + 0.000354\cos(x_2)\cos(x_3) - \\ & 0.000354\cos(\hat{x}_2)\cos(\hat{x}_3) \end{aligned}$$

By MATLAB, we used the fourth-order (Runge-Kutta) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.8). *Problem (16) of illustrations (2.4) case(2) to find the error solutions of $e_1$, $e_2$ and $e_3$* see ( *appendix (C )* ). The results of error solution ( error solutions ) are ploted in the fllowing graphs.

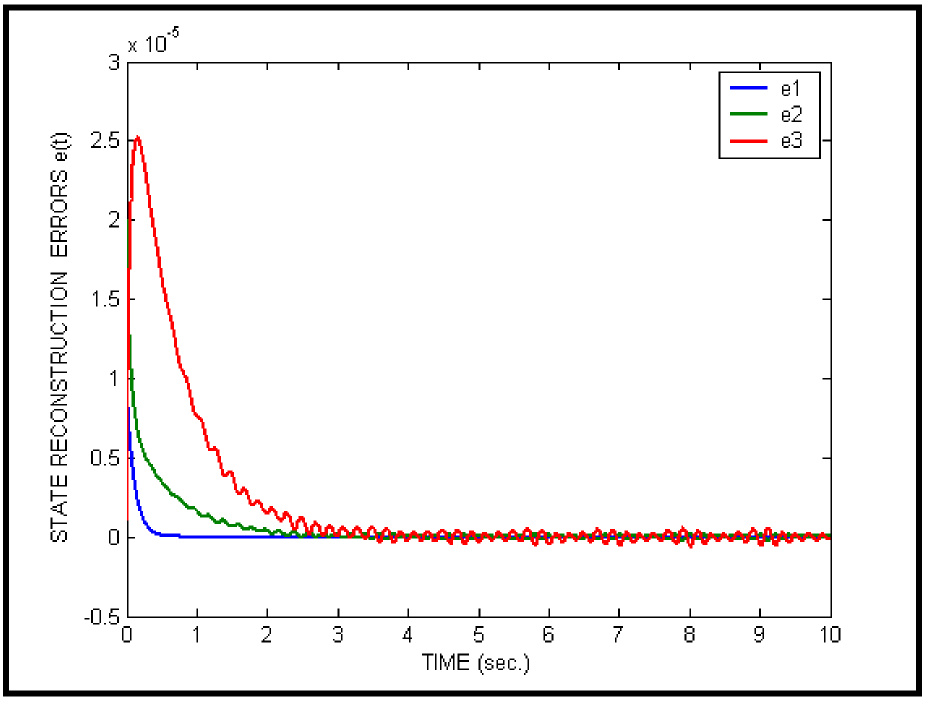


***Figure (2.7) The non-linear system of illustrations (2.4) If u =1 represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

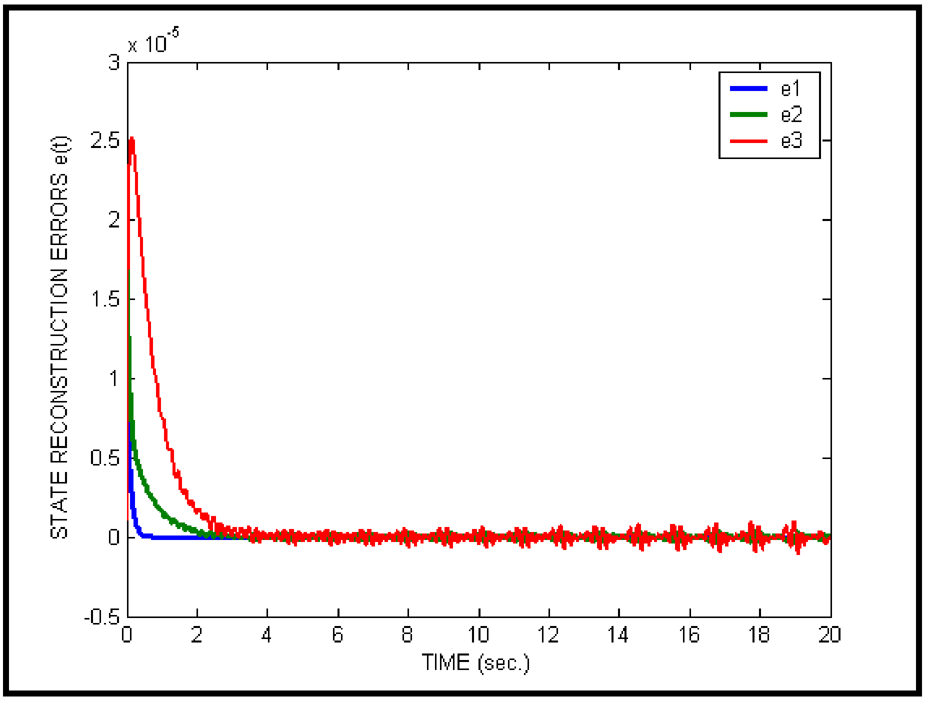


***Figure (2.8) The non-linear system of illustrations (2.4) If u = cos(t) represented the states [$e_1$, $e_2$ and $e_3$] with initial conditions [$e_1(0)$, $e_2(0)$ and $e_3(0)$]=[0.00001 0.00002 0.000001]***

***Illustrations (2.5): E is square and not invertible matrix ( S.V.D )***

***STEP (1)*** *( Problem Formulation )*

Consider the dynamical system

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))$$

where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.25 & -0.15 & -0.3 & -0.4 & -0.75 & -0.1285 \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$E=\begin{bmatrix} 1 & -1 & 2 & 3 & -1 & 1 \\ 4 & 5 & 0.1 & 0.2 & 0.25 & 0.3 \\ 5 & 4 & 0.2 & 0.1 & 0.3 & 0.25 \\ 0.9 & 0.3 & 0.4 & 0.125 & 0.75 & 0.15 \\ 1 & -1 & 2 & 3 & -1 & 1 \\ 0.3 & 0.9 & 0.125 & 0.65 & 0.85 & 0.75 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ |E| = 0$$

$$f(x,u)=\begin{bmatrix} 0.0001u\sin(x_1+x_2)\cos(x_3+x_4)\sin(x_5) \\ 0 \\ 0.00001u\cos(x_5+x_6)\cos(x_3+x_4) \\ 0 \\ 0.0002u\sin(x_1+x_3)\sin(x_6+x_4) \\ 0.0003u\cos(x_1+x_3)\sin(x_2) \end{bmatrix} \quad \text{and}$$

$$g(x)=\begin{bmatrix} 0 \\ 0.0001\sin(x_1+x_5)\sin(x_3+x_2) \\ 0 \\ 0 \\ 0 \\ 0.00001\cos(x_4+x_6)\cos(x_3+x_5) \end{bmatrix}$$

**Case ( I ):** *( When control u=1 )*

**STEP (2)**

Check the $\operatorname{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n$.

$$Z' = \begin{bmatrix} 1 & -1 & 2 & 3 & -1 & 1 \\ 4 & 5 & 0.1 & 0.2 & 0.25 & 0.3 \\ 5 & 4 & 0.2 & 0.1 & 0.3 & 0.25 \\ 0.9 & 0.3 & 0.4 & 0.125 & 0.75 & 0.15 \\ 1 & -1 & 2 & 3 & -1 & 1 \\ 0.3 & 0.9 & 0.125 & 0.65 & 0.85 & 0.75 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\operatorname{rank}(Z') = 6$.

**STEP (3)**

Check the $\operatorname{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s$.

where $s = 10$

$$Z'' = \begin{bmatrix} 10 & -11 & 20 & 30 & -10 & 10 \\ 40 & 50 & 0 & 2 & 2.5 & 3 \\ 50 & 40 & 2 & 0 & 3 & 2.5 \\ 9 & 3 & 4 & 1.25 & 6.5 & 1.5 \\ 10 & -10 & 20 & 30 & -10 & 9 \\ 3.25 & 9.15 & 1.55 & 6.9 & 8.675 & 7.6285 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(Z'') = 6$ .

***STEP (4)***

Find the pseudo-inverse of the matrix E .

Since, E is singular value decomposition, then

$$EE^T = \begin{bmatrix} 17 & -0.15 & 1.65 & 1.175 & 17 & 1.5 \\ -0.15 & 41.2029 & 40.19 & 5.3975 & -0.15 & 6.28 \\ 1.65 & 40.19 & 41.2025 & 6.055 & 1.65 & 5.6325 \\ 1.175 & 5.3975 & 6.055 & 1.6606 & 1.175 & 1.4213 \\ 17 & -0.15 & 1.65 & 1.175 & 17 & 1.5 \\ 1.5 & 6.28 & 5.6325 & 1.4213 & 1.5 & 2.6231 \end{bmatrix}, \ \left|EE^T\right| = 0$$

$\Rightarrow \ (EE^T)^{-1}$ not exists

$$E^TE = \begin{bmatrix} 43.9 & 38.54 & 5.7975 & 7.6075 & 1.43 & 4.81 \\ 38.54 & 43.9 & -2.4675 & -3.9775 & 5.44 & 1.22 \\ 5.7975 & -2.4675 & 8.2256 & 12.1713 & -3.0588 & 4.2337 \\ 7.6075 & -3.9775 & 12.1713 & 18.4881 & -5.2737 & 6.5912 \\ 1.43 & 5.44 & -3.5088 & -5.2737 & 3.3475 & -1.1 \\ 4.81 & 1.22 & 4.2337 & 6.5912 & -1.1 & 2.7375 \end{bmatrix}, \ \left|E^TE\right| = 0$$

$\Rightarrow \ (E^TE)^{-1}$ not exists

And we see that , the eigenvalues of $EE^T$ and $E^TE$ are equals. From MATLAB, *program ( 12 )* see ( *appendix ( B )* ).

$$\text{eig}(EE^T) = \begin{bmatrix} 0 \\ 0.1017 \\ 1.2169 \\ 1.9782 \\ 34.2111 \\ 83.1809 \end{bmatrix} \quad \text{and} \quad \text{eig}(E^TE) = \begin{bmatrix} 0 \\ 0.1017 \\ 1.2169 \\ 1.9782 \\ 34.2111 \\ 83.1809 \end{bmatrix}$$

The pseudo-inverse of the matrix E is give by the following commands:

$[U,S,V] = \text{svd}(E)$, from MATLAB, we get :

$$U=\begin{bmatrix} -0.027 & 0.7036 & -0.03 & 0.0546 & -0.0201 & -0.7071 \\ -0.6984 & -0.0724 & 0.1963 & 0.5421 & -0.4178 & 0 \\ -0.6995 & 0.0036 & -0.3595 & -0.4257 & 0.4474 & 0 \\ -0.1008 & 0.0419 & 0.2191 & -0.7114 & -0.6587 & 0 \\ -0.027 & 0.7036 & -0.03 & 0.0546 & -0.0201 & 0.7071 \\ -0.1061 & 0.055 & 0.0045 & -0.1134 & 0.4364 & 0 \end{bmatrix},$$

The matrix $U$ is represented the eigenvectors of $\mathrm{EE}^{\mathrm{T}}$ .

$$S=\begin{bmatrix} 9.1204 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.849 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.4056 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.1031 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3189 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The matrix $S$ is represented the diagonal of $D_r$ .

$$V=\begin{bmatrix} -0.7092 & 0.2034 & -0.4334 & -0.476 & 0.1996 & 0.0381 \\ -0.6976 & -0.2894 & 0.3311 & 0.5286 & -0.201 & -0.0157 \\ -0.0407 & 0.4841 & 0.0184 & -0.1009 & -0.7578 & -0.4233 \\ -0.0497 & 0.7263 & 0.3025 & 0.2092 & 0.1313 & 0.5635 \\ -0.0544 & -0.2301 & 0.6523 & -0.6629 & -0.1666 & 0.2266 \\ -0.0585 & 0.2451 & 0.4303 & 0.4303 & 0.5482 & -0.671 \end{bmatrix},$$

The matrix $V$ is represented the eigenvectors of $\mathrm{E}^{\mathrm{T}}\mathrm{E}$ and also is represented the $V^{\mathrm{T}}$.

The pseudo-inverse of the matrix E is

$\mathrm{E}^{+}=VD^{+}U^{\mathrm{T}}$, where $D^{+}=(D_r^{-1})^{\mathrm{T}}$

$$=\begin{bmatrix} -0.08 & 0.1041 & 0.0072 & 0.141 & -0.08 & -0.0933 \\ 0.0047 & 0.714 & -0.7426 & 0.0781 & 0.0047 & 0.0902 \\ 0.0146 & -0.6744 & 0.8273 & -1.5326 & 0.0146 & 0.8814 \\ 0.1193 & 0.9872 & -0.9482 & 1.2278 & 0.1193 & -0.9817 \\ 0.0084 & 0.1642 & -0.1061 & 0.1376 & 0.0084 & -0.7222 \\ 0.0206 & -0.0818 & 0.2057 & -0.8979 & 0.0206 & -0.0147 \end{bmatrix}$$

***STEP (5)***

Multiplying the system by $E^{+}$ and simplified this system in *Problem (17) of illustrations (2.5) case (1) to find the solutions of* $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ *and* $x_6$ see ( *appendix ( C )* ).

***STEP (6)***

Partitioned the matrix $A, E$ and $I$, as the following

(1 ) $E=[E_1 \;\vdots\; E_2]$, where $E_1 \in \Re^{q\times m}$ and $E_2 \in \Re^{q\times n-m}$

$$E_1=\begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 5 & 0.1 & 0.2 \\ 5 & 4 & 0.2 & 1.0 \\ 0.9 & 0.3 & 0.4 & 0.125 \\ 1 & -1 & 2 & 2 \\ 0.3 & 0.9 & 0.125 & 0.65 \end{bmatrix} \quad \text{and} \quad E_2=\begin{bmatrix} -1 & 1 \\ 0.25 & 0.3 \\ 0.3 & 0.25 \\ 0.75 & 0.15 \\ -1 & 1 \\ 0.85 & 0.75 \end{bmatrix}$$

where $q=6$, $m=4$, and $n-m=2$

(2) $A=[A_1 \;\vdots\; A_2]$ , where $A_1 \in \Re^{q\times m}$ and $A_2 \in \Re^{q\times n-m}$

$$A_1=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.25 & -0.15 & -0.3 & -0.4 \end{bmatrix} \quad \text{and} \quad A_2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -0.175 & -0.1285 \end{bmatrix}$$

where $q=6$, $m=4$, and $n-m=2$

(3) $I=[J_1 \;\vdots\; J_2]$, where $J_1=\begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $J_2=\begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

***STEPS (7, 8, and 9)***

Computational the matrix $R$, from the conditions

$R\mathrm{E} + \mathrm{K}C = \mathrm{I}$, from the eq. (2.62), we get:

$$\Rightarrow R = J_2 \mathrm{E}^+$$

$$\Rightarrow R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1997 & 0.1237 & 0.1337 & 0.2705 & -0.1997 & 0.3844 \\ 0.2851 & 0.1593 & 0.145 & 0.1635 & 0.2851 & 0.4284 \end{bmatrix}$$

From eq. (2.63), we get:

$$\Rightarrow \mathrm{K} = J_1 - R\mathrm{E}_1$$

$$\Rightarrow \mathrm{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.1226 & -1.9797 & 0.6034 & 0.8764 \\ 2.2082 & -1.2412 & -1.3041 & -2.0556 \end{bmatrix}$$

From eq. (2.64), we get:

$\Rightarrow \; L = R\mathrm{A}_1 - \mathrm{N}R\mathrm{E}_1$

$$\Rightarrow \; L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -12.2005 & -10.9038 & -3.4062 & -5.5933 \\ -11.115 & -11.6424 & -1.4987 & -2.6613 \\ -13.2861 & -10.1635 & -5.3137 & -8.5253 \\ -6.6836 & -1.2626 & -6.4149 & -9.9953 \\ 19.7665 & 11.3913 & 11.7677 & 18.4741 \end{bmatrix}$$

From eq. (2.65), we get :

$\Rightarrow \; G = R\mathrm{B}$

$$\Rightarrow \; G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.3844 \\ 0.4284 \end{bmatrix}$$

***<u>STEP (10)</u>***

Solve algebraic Ricatti equation, from eq. (2.66), we get:

$$(\mathrm{N} + \mathrm{I}_n)^{\mathrm{T}}\mathrm{P} + \mathrm{P}(\mathrm{N} + \mathrm{I}_n) = 2C^{\mathrm{T}}C$$

where

$$\mathrm{N} = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 1 & 2 & 3 & 4 \\ 0 & 1 & -4 & 1 & 4 & 3 \\ 0 & 0 & 2 & -3 & 2 & 5 \\ 0 & 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix},$$

Find the eigenvalues of N, from MATLAB, *Program (13) (Computation matrices of observer when E is singular value decomposition)* see ( *appendix ( B* ) ), we get:

$$\text{eig(N)} = \begin{bmatrix} -1.6348 \\ -5.6826+0.3583i \\ -5.6826-0.3583i \\ -5 \\ -2 \\ -9 \end{bmatrix}$$, hence the matrix N is stable.

And then the positive definite matrix solution is obtained as follows:

$$\Rightarrow \text{P} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2374 & 0.1869 & 0.1955 & 0.8652 & 0.6398 \\ 0 & 0.1869 & 0.8668 & 0.7068 & 3.3403 & 2.3694 \\ 0 & 0.1955 & 0.7068 & 1.0489 & 3.5273 & 2.5905 \\ 0 & 0.8652 & 3.3403 & 3.5273 & 23.0114 & 15.5269 \\ 0 & 0.6398 & 2.3694 & 2.5905 & 15.5269 & 10.591 \end{bmatrix},$$

We test this matrix is symmetric positive definite, we find the eigenvalues of P, from MATLAB, *Program ( 14 ), (Ricatti Algabraic equations when E is singular value decomposition),* see ( *appendix ( B )* ), we get:

$$\text{eig(P)} = \begin{bmatrix} 0.0129 \\ 0.1869 \\ 0.2459 \\ 0.25 \\ 0.6608 \\ 34.6489 \end{bmatrix}$$, hence the matrix P is positive definite .

***STEP (11)***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k+k_1<\dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1)\quad f(x,u)=\begin{bmatrix} 0.0001\sin(x_1+x_2)\cos(x_3+x_4)\sin(x_5) \\ 0 \\ 0.00001\cos(x_5+x_6)\cos(x_3+x_4) \\ 0 \\ 0.0002\sin(x_1+x_3)\sin(x_6+x_4) \\ 0.0003\cos(x_1+x_3)\sin(x_2) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $f(x(t),u(t))$ is found

$$\Rightarrow \quad \|J_1\|\le 0.0006$$

$$\Rightarrow \quad \|f(x_2(t),u(t))-f(x_1(t),u(t))\|\le 0.0006\|x_2(t)-x_1(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k=0.0006$.

$$(2)\quad g(x)=\begin{bmatrix} 0 \\ 0.0001\sin(x_1+x_5)\sin(x_3+x_2) \\ 0 \\ 0 \\ 0 \\ 0.00001\cos(x_4+x_6)\cos(x_3+x_5) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $g(x(t))$ is found

$$\Rightarrow \quad \|J_2\|\le 0.0002$$

$$\Rightarrow \quad \|g(x(t))-g(\hat{x}(t))\|\le 0.0002\|x(t)-\hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1=0.0002$.

$$k+k_1=0.0006+0.0002=0.0008\,,\quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}=\frac{0.0129}{12.8279}=0.001$$

$$\Rightarrow\quad k+k_1<\frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}\quad\Rightarrow\quad 0.0008<0.001$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1=-5z_1+4e_1$$

$$\dot{z}_2=-6z_1+z_3+2z_4+3z_5+4z_6-12.2005x_1-10.9038x_2-3.4062x_3-5.5933x_4+6.106e_2+0.0354e_3+2.054e_4$$

$$\dot{z}_3=z_2-4z_3+z_4+4z_5+3z_6-11.115x_1-11.6424x_2-1.4987x_3-2.6613x_4+0.0354e_2+3.8136e_3+0.5881e_4$$

$$\dot{z}_4=2z_3-3z_4+2z_5+5z_6-13.2861x_1-10.1653x_2-5.3135x_3-8.5253x_4+2.054e_2+0.5881e_3+9.0804e_4$$

$$\dot{z}_5=-2z_5+4z_6-6.6386x_1-1.2626x_2-6.4149x_3-9.9953x_4+0.3844-0.00001997\sin(\hat{x}_1+\hat{x}_2)\cos(\hat{x}_3+\hat{x}_4)\sin(\hat{x}_5)+0.000001337\cos(\hat{x}_5+\hat{x}_6)\cos(\hat{x}_3+\hat{x}_4)-0.00003994*\sin(\hat{x}_1+\hat{x}_3)\sin(\hat{x}_6+\hat{x}_4)+0.000115232\cos(\hat{x}_1+\hat{x}_3)*\sin(\hat{x}_2)+0.00001237\sin(\hat{x}_1+\hat{x}_5)\sin(\hat{x}_3+\hat{x}_2)+0.000003844\cos(\hat{x}_4+\hat{x}_6)\cos(\hat{x}_3+\hat{x}_5)+4.05e_2+2.7014e_3+10.8157e_4$$

$$\dot{z}_6=-9z_6+19.7665x_1+11.3913x_2+11.7677x_3+18.471x_4+0.4284+0.00002851\sin(\hat{x}_1+\hat{x}_2)\cos(\hat{x}_3+\hat{x}_4)\sin(\hat{x}_5)+0.00000145\cos(\hat{x}_5+\hat{x}_6)\cos(\hat{x}_3+\hat{x}_4)+0.00005702*\sin(\hat{x}_1+\hat{x}_3)\sin(\hat{x}_6+\hat{x}_4)+0.00012825\cos(\hat{x}_1+\hat{x}_3)*\sin(\hat{x}_2)+0.000001593\sin(\hat{x}_1+\hat{x}_5)\sin(\hat{x}_3+\hat{x}_2)+0.000002484\cos(\hat{x}_4+\hat{x}_6)\cos(\hat{x}_3+\hat{x}_5)-6.8167e_2-4.9595e_3-18.3331e_4$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (17) of illustrations (2.5) case(1) to find the solutions of* $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ *and* $x_6$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation (2.21) .

By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. *Problem (18) of illustrations (2.5) case(1) to find the error solutions of* $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ *and* $e_6$ see appendix (C ). The numerical solution of error can be found in Figure (2.9).

***Case (II) :*** *( When control* $u = \cos(t)$ *)*

***STEP (2), STEP (3), and STEP(4)***

Are the same as steps of *case ( I )*.

***STEPS (6) , (7) , (8), (9) and (10)***

Are the same as steps of *case ( I )* .

**STEP (11)**

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k+k_1<\dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1)\quad f(x,u)=\begin{bmatrix} 0.0001\cos(t)\sin(x_1+x_2)\cos(x_3+x_4)\sin(x_5) \\ 0 \\ 0.00001\cos(t)\cos(x_5+x_6)\cos(x_3+x_4) \\ 0 \\ 0.0002\cos(t)\sin(x_1+x_3)\sin(x_6+x_4) \\ 0.0003\cos(t)\cos(x_1+x_3)\sin(x_2) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $f(x(t),u(t))$ is found

$$\Rightarrow \quad \|J_1\|\le 0.0006$$

$$\Rightarrow \quad \|f(x_2(t),u(t))-f(x_1(t),u(t))\|\le 0.0006\|x_2(t)-x_1(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k=0.0006$.

$$(2)\quad g(x)=\begin{bmatrix} 0 \\ 0.0001\sin(x_1+x_5)\sin(x_3+x_2) \\ 0 \\ 0 \\ 0 \\ 0.00001\cos(x_4+x_6)\cos(x_3+x_5) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $g(x(t))$ is found

$$\Rightarrow \quad \|J_2\|\le 0.0002$$

$$\Rightarrow \quad \|g(x(t))-g(\hat{x}(t))\|\le 0.0002\|x(t)-\hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.0002$.

$$k + k_1 = 0.0006 + 0.0002 = 0.0008, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0129}{12.8279} = 0.001$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.0008 < 0.001$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -5z_1 + 4e_1$$

$$\dot{z}_2 = -6z_1 + z_3 + 2z_4 + 3z_5 + 4z_6 - 12.2005x_1 - 10.9038x_2 - 3.4062x_3 - 5.5933x_4 + 6.106e_2 + 0.0354e_3 + 2.054e_4$$

$$\dot{z}_3 = z_2 - 4z_3 + z_4 + 4z_5 + 3z_6 - 11.115x_1 - 11.6424x_2 - 1.4987x_3 - 2.6613x_4 + 0.0354e_2 + 3.8136e_3 + 0.5881e_4$$

$$\dot{z}_4 = 2z_3 - 3z_4 + 2z_5 + 5z_6 - 13.2861x_1 - 10.1653x_2 - 5.3135x_3 - 8.5253x_4 + 2.054e_2 + 0.5881e_3 + 9.0804e_4$$

$$\begin{aligned}\dot{z}_5 = {} & -2z_5 + 4z_6 - 6.6386x_1 - 1.2626x_2 - 6.4149x_3 - 9.9953x_4 + \\ & 0.3844\cos(t) - 0.00001997\cos(t)\sin(\hat{x}_1 + \hat{x}_2) * \\ & \cos(\hat{x}_3 + \hat{x}_4)\sin(\hat{x}_5) + 0.000001337\cos(t)\cos(\hat{x}_5 + \hat{x}_6) * \\ & \cos(\hat{x}_3 + \hat{x}_4) - 0.00003994\cos(t)\sin(\hat{x}_1 + \hat{x}_3) * \\ & \sin(x_6 + x_4) + 0.000115232\cos(t)\cos(x_1 + x_3)\sin(x_2) + \\ & 0.00001237\sin(\hat{x}_1 + \hat{x}_5)\sin(\hat{x}_3 + \hat{x}_2) + 0.000003844 * \\ & \cos(\hat{x}_4 + \hat{x}_6)\cos(\hat{x}_3 + \hat{x}_5) + 4.05e_2 + 2.7014e_3 + 10.8157e_4\end{aligned}$$

$$\dot{z}_6 = -9z_6 + 19.7665x_1 + 11.3913x_2 + 11.7677x_3 + 18.471x_4 + 0.4284\cos(t) + 0.00002851\cos(t)\sin(\hat{x}_1 + \hat{x}_2) * \cos(\hat{x}_3 + \hat{x}_4)\sin(\hat{x}_5) + 0.00000145\cos(t)\cos(\hat{x}_5 + \hat{x}_6) * \cos(\hat{x}_3 + \hat{x}_4) + 0.00005702\cos(t)\sin(\hat{x}_1 + \hat{x}_3)\sin(\hat{x}_6 + \hat{x}_4) + 0.00012825\cos(t)\cos(\hat{x}_1 + \hat{x}_3)\sin(\hat{x}_2) + 0.000001593 * \sin(\hat{x}_1 + \hat{x}_5)\sin(\hat{x}_3 + \hat{x}_2) + 0.000002484\cos(\hat{x}_4 + \hat{x}_6) * \cos(\hat{x}_3 + \hat{x}_5) - 6.8167e_2 - 4.9595e_3 - 18.3331e_4$$

***STEP (13)***

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (19) of illustrations (2.5) case (2) to find the solutions of* $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ *and* $x_6$ see ( *appendix ( C )* ).

***STEP (14)***

Find the solution of the error by the equation (2.21) .

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.10). *Problem (20) of illustrations (2.5) case(2) to find the error solutions of* $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ *and* $e_6$ see ( *appendix (C )* ). The results of error solution ( error solutions ) are ploted in the fllowing graphs.

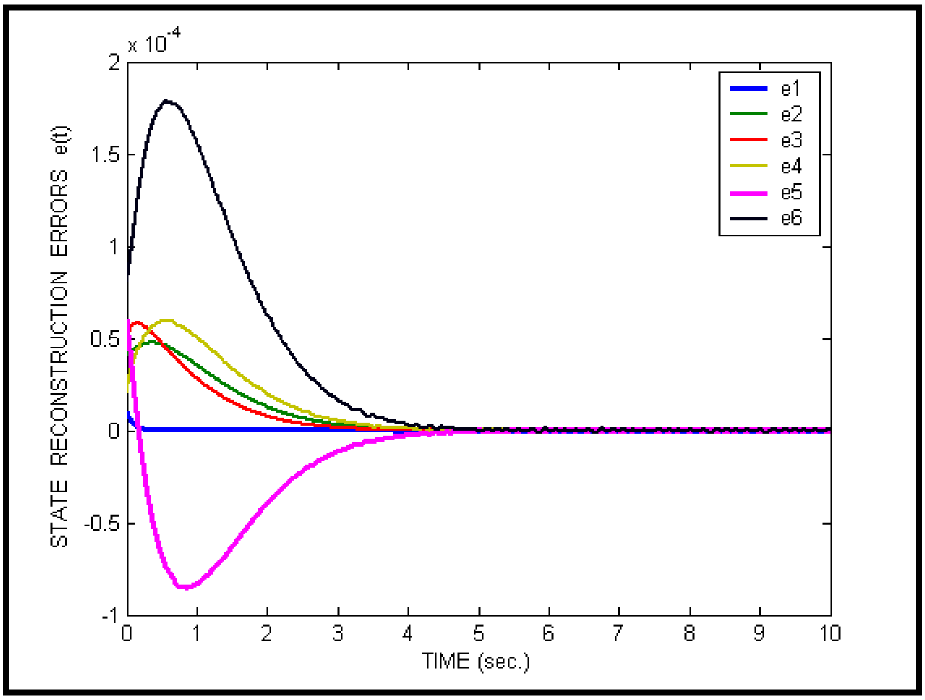


***Figure (2.9) The non-linear system of illustrations (2.5) If u =1 represented the states [$e_1$, $e_2$, $e_3$, $e_4$, $e_5$ and $e_6$] with initial conditions [$e_1(0)$, $e_2(0)$, $e_3(0)$, $e_4(0)$, $e_5(0)$ and $e_6(0)$]=[0.00001 0.00004 0.00005 0.00002 0.00006 0.00008]***

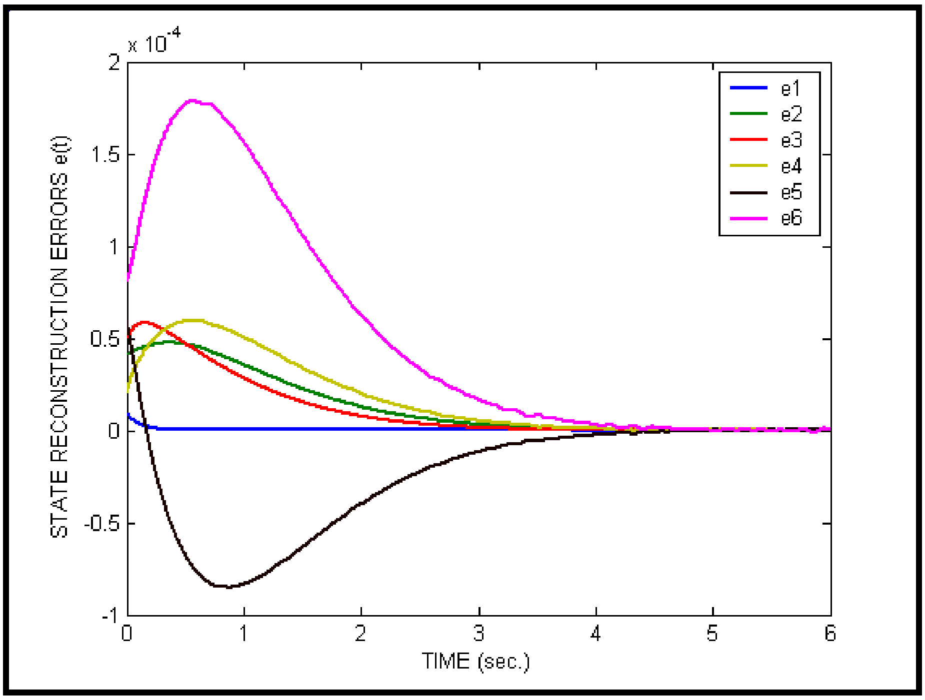


***Figure (2.10) The non-linear system of illustrations (2.5) If u = cos(t) represented the states [$e_1$, $e_2$, $e_3$, $e_4$, $e_5$ and $e_6$] with initial conditions [$e_1(0)$, $e_2(0)$, $e_3(0)$, $e_4(0)$, $e_5(0)$ and $e_6(0)$]=[0.00001 0.00004 0.00005 0.00002 0.00006 0.00008]***

***Illustrations (2.6): E is square and not invertible matrix ( S.V.D )***

***STEP (1)*** *( Problem Formulation )*

Consider the dynamical system

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(x(t),u(t)) + g(x(t))$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ 0.2 & 0.1 & 0.4 & 0.3 & 0.6 \\ 0.11 & 0.15 & 0.17 & 0.19 & 0.1 \\ 0.15 & 0.11 & 0.19 & 0.17 & 0.12 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 0.2 \\ 0.45 \\ 0.1 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & -2 & -3 & -1 & 1 \\ 0.1 & 0.4 & 0.25 & 0.1 & 0.9 \\ 0.4 & 0.1 & 0.35 & 0.4 & 0.75 \\ 0.5 & 0.3 & 0.11 & 0.5 & 0.03 \\ 0.3 & 0.5 & 0.13 & 0.3 & 0.125 \end{bmatrix}, \ f(x,u) = \begin{bmatrix} 0.125u\sin(x_1+x_2)\sin(x_3+x_4) \\ 0 \\ 0.015u\cos(x_4+x_1)\cos(x_3+x_5) \\ 0.002u\sin(x_1)\cos(x_5) \\ 0.025u\cos(x_4+x_5)\sin(x_3+x_2) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0.016\sin(x_1+x_5)\sin(x_2+x_3) \\ 0.0175\cos(x_4+x_5)\sin(x_3+x_2) \\ 0.0275\sin(x_5)\cos(x_4) \\ 0.0025\cos(x_3+x_1)\sin(x_2+x_5) \end{bmatrix} \quad \text{and} \quad |E| = 0$$

***Case ( I ):*** *( When control u=1 )*

**STEP (2)**

Check the $\text{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n$.

$$H'' = \begin{bmatrix} -1 & -2 & -3 & -1 & 1 \\ 0.1 & 0.4 & 0.25 & 0.1 & 0.9 \\ 0.4 & 0.1 & 0.35 & 0.4 & 0.75 \\ 0.5 & 0.3 & 0.11 & 0.5 & 0.03 \\ 0.3 & 0.5 & 0.13 & 0.3 & 0.125 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(H'') = 5$.

**STEP (3)**

Check the $\text{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \ \forall s$.

where $s = 10$

$$H''' = \begin{bmatrix} -10 & -21 & -30 & -10 & 10 \\ 0.9 & 3.8 & 2.2 & 0.6 & 8.5 \\ 3.8 & 0.9 & 3.1 & 3.7 & 6.9 \\ 4.89 & 2.85 & 0.93 & 4.81 & 0.2 \\ 2.85 & 4.89 & 1.11 & 2.38 & 1.13 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

By using Gause -Elimenation technique, we conclude that the $\text{rank}(H''') = 5$.

***STEP (4)***

Find the pseudo-inverse of the matrix E .

Since, E is singular value decomposition, then

$$EE^T = \begin{bmatrix} 16 & -0.85 & -1.3 & -1.9 & -1.865 \\ -0.85 & 1.0525 & 0.8825 & 0.2745 & 0.405 \\ -1.3 & 0.8825 & 1.015 & 0.491 & 0.4292 \\ -1.9 & 0.2745 & 0.491 & 0.603 & 0.468 \\ -1.865 & 0.405 & 0.4292 & 0.468 & 0.4625 \end{bmatrix}, \ \left|EE^T\right| = 0$$

$\Rightarrow \quad (EE^T)^{-1}$ not exists

$$E^TE = \begin{bmatrix} 1.51 & 2.38 & 3.259 & 1.51 & -0.5575 \\ 2.38 & 4.51 & 6.233 & 2.38 & -1.4935 \\ 3.259 & 6.233 & 9.214 & 3.259 & -2.493 \\ 1.51 & 2.38 & 3.259 & 1.51 & -0.5575 \\ -0.5575 & -1.4935 & -2.493 & -0.5575 & 2.389 \end{bmatrix}, \ \left|E^TE\right| = 0$$

$\Rightarrow \quad (E^TE)^{-1}$ not exists

And we see that, the eigenvalues of $EE^T$ and $E^TE$ are equals. From MATLAB, *program ( 15 )* see ( *appendix ( B )* ).

$$\text{eig}(EE^T) = \begin{bmatrix} 0 \\ 0.1492 \\ 0.4021 \\ 1.935 \\ 16.6467 \end{bmatrix} \quad \text{and} \quad \text{eig}(E^TE) = \begin{bmatrix} 0 \\ 0.1492 \\ 0.4201 \\ 1.935 \\ 16.6467 \end{bmatrix}$$

The pseudo-inverse of the matrix E is give by the following commands:

$[U,S,V] = \text{svd}(E)$, from MATLAB, we get :

$$U = \begin{bmatrix} -0.9786 & -1.1626 & -0.1179 & -0.0408 & 0.0167 \\ 0.0639 & -0.6694 & 0.4799 & -0.41 & -0.3866 \\ 0.0922 & -0.6454 & -0.0552 & 0.6665 & 0.3573 \\ 0.1233 & -0.2279 & -0.7875 & -0.0031 & -0.5592 \\ 0.1204 & -0.2388 & -0.3642 & -0.6213 & 0.6402 \end{bmatrix},$$

The matrix $U$ is represented the eigenvectors of $EE^T$ .

$$S=\begin{bmatrix} 4.08 & 0 & 0 & 0 & 0\\ 0 & 1.391 & 0 & 0 & 0\\ 0 & 0 & 0.6341 & 0 & 0\\ 0 & 0 & 0 & 0.3863 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The matrix $S$ is represented the diagonal of $D_r$ .

$$V=\begin{bmatrix} 0.2744 & -0.2503 & -0.5664 & 0.2031 & 0.7071\\ 0.5121 & -0.1401 & 0.0062 & -0.8474 & 0\\ 0.7386 & 0.0276 & 0.5053 & 0.4454 & 0\\ 0.2744 & -0.2503 & -0.5664 & 0.2031 & -0.7071\\ -0.2042 & -0.9243 & 0.3209 & 0.0318 & 0 \end{bmatrix},$$

The matrix $V$ is represented the eigenvectors of $E^TE$ and also is represented the $V^T$.

The pseudo-inverse of the matrix E is

$E^+ = VD^+U^T$, where $D^+ = (D_r^{-1})^T$

$$=\begin{bmatrix} -0.292 & 0.0256 & 0.1778 & -0.995 & -0.9454\\ 0.0977 & 0.35 & -0.3748 & -0.0168 & 0.4034\\ 0.101 & 0.9718 & -1.0369 & -0.6411 & 0.6029\\ -0.054 & 0.5325 & 0.7094 & -0.4098 & -0.4309\\ -0.0949 & 0.7616 & -1.204 & 0.0271 & 1.1581 \end{bmatrix}$$

***STEP (5)***

Multiplying the system by $E^+$ and simplified this system in *Problem (21) of illustrations (2.6) case (1) to find the solutions of* $x_1$, $x_2$, $x_3$, $x_4$ *and* $x_5$ see ( *appendix ( C )* ).

***STEP (6)***

Partitioned the matrix $A, E$ and $I$, as the following

(1 ) $E = [E_1 \;\vdots\; E_2]$, where $E_1 \in \Re^{q\times m}$ and $E_2 \in \Re^{q\times n-m}$

$$E_1 = \begin{bmatrix} -1 & -2 & -3 & -1 \\ 0.1 & 0.4 & 0.25 & 0.1 \\ 0.4 & 0.1 & 0.35 & 0.4 \\ 0.5 & 0.3 & 0.11 & 0.5 \\ 0.3 & 0.5 & 0.13 & 0.3 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 1 \\ 0.9 \\ 0.75 \\ 0.03 \\ 0.125 \end{bmatrix}$$

where $q = 5$, $m = 4$, and $n - m = 1$

(2) $A = [A_1 \;\vdots\; A_2]$, where $A_1 \in \Re^{q\times m}$ and $A_2 \in \Re^{q\times n-m}$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.11 & 0.11 & 0.17 & 0.19 \\ 0.15 & 0.15 & 0.19 & 0.17 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 \\ 0.5 \\ 0.6 \\ 0.1 \\ 0.12 \end{bmatrix}$$

where $q = 5$, $m = 4$, and $n - m = 1$

(3) $I = [J_1 \;\vdots\; J_2]$, where $J_1 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } J_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

***STEPS (7, 8, and 9)***

Computational the matrix $R$ , from the conditions $RE + KC = I$, from the eq. (2.62), we get:

$$\Rightarrow\ R = J_2E^{+}$$

$$\Rightarrow\ R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.4186 & 0.3767 & 0.3139 & 0.0126 & 0.0523 \end{bmatrix}$$

From eq. (2.63), we get:

$$\Rightarrow\ K = J_1 - RE_1$$

$$\Rightarrow\ K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2334 & 0.6252 & 1.0435 & 0.2334 \end{bmatrix}$$

From eq. (2.64), we get:

$$\Rightarrow\ L = RA_1 - NRE_1$$

$$\Rightarrow\ L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.9334 & 2.5006 & 4.174 & 0.9334 \\ 1.1668 & 3.1258 & 5.2175 & 1.1668 \\ 0.7001 & 1.8755 & 3.1305 & 0.7001 \\ -1.9905 & -5.0934 & -9.1408 & -1.8441 \end{bmatrix}$$

From eq. (2.65), we get :

$$\Rightarrow\ G = RB$$

$$\Rightarrow\ G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4504 \end{bmatrix}$$

**STEP (10)**

Solve algebraic Ricatti equation, from eq. (2.66), we get:

$$(\mathrm{N}+\mathrm{I}_n)^{\mathrm{T}}\mathrm{P}+\mathrm{P}(\mathrm{N}+\mathrm{I}_n)=2C^{\mathrm{T}}C$$

where

$$\mathrm{N}=\begin{bmatrix} -6 & 0 & 0 & 0 & 0 \\ 0 & -7 & 2 & 3 & 4 \\ 0 & 2 & -5 & 2 & 5 \\ 0 & 0 & -3 & -4 & 3 \\ 0 & 0 & 0 & 0 & -9 \end{bmatrix},$$

Find the eigenvalues of N, from MATLAB, *Program ( 16 ) (Computation matrices of observer when E is singular value decomposition)* see ( *appendix (B)* ), we get:

$$\mathrm{eig(N)}=\begin{bmatrix} -3.6952+2.7779i \\ -3.6952-2.7779i \\ -8.6096 \\ -6 \\ -9 \end{bmatrix}$$ , hence the matrix N is stable.

And then the positive definite matrix solution is obtained as follows:

$$\Rightarrow \mathrm{P}=\begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.1924 & 0.0773 & 0.0757 & 0.1164 \\ 0 & 0.0773 & 0.3077 & -0.0254 & 0.1228 \\ 0 & 0.0757 & -0.0254 & 0.3921 & 0.177 \\ 0 & 0.1164 & 0.1228 & 0.177 & 0.2013 \end{bmatrix},$$

We test this matrix is symmetric positive definite , we find the eigenvalues of P, from MATLAB, *Program ( 17 ), (Ricatti Algabraic equations when E is singular value decomposition),* see ( *appendix ( B )* ), we get:

$$\text{eig}(P) = \begin{bmatrix} 0.0342 \\ 0.1298 \\ 0.2 \\ 0.3635 \\ 0.566 \end{bmatrix}, \text{ hence the matrix P is positive definite.}$$

***<u>STEP (11)</u>***

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(P)}{\delta_{\max}(PR)}$

$$(1) \quad f(x,u) = \begin{bmatrix} 0.125\sin(x_1 + x_2)\sin(x_3 + x_4) \\ 0 \\ 0.015\cos(x_4 + x_1)\cos(x_3 + x_5) \\ 0.002\sin(x_1)\cos(x_5) \\ 0.025\cos(x_4 + x_5)\sin(x_3 + x_2) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $f(x(t),u(t))$ is found

$$\Rightarrow \quad \|J_1\| \le 0.25$$

$$\Rightarrow \quad \|f(x_2(t),u(t)) - f(x_1(t),u(t))\| \le 0.25\|x_2(t) - x_1(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.25$.

$$(2) \quad g(x) = \begin{bmatrix} 0 \\ 0.016\sin(x_1 + x_5)\sin(x_2 + x_3) \\ 0.0175\cos(x_4 + x_5)\sin(x_3 + x_2) \\ 0.0275\sin(x_5)\cos(x_4) \\ 0.0025\cos(x_3 + x_1)\sin(x_2 + x_5) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $g(x(t))$ is found

$$\Rightarrow \quad \|J_2\| \le 0.062$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.062\|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.062$.

$$k + k_1 = 0.25 + 0.062 = 0.312, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0342}{0.0952} = 0.35$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.312 < 0.35$$

Hence, Lipschitz constant is satisfied.

***STEP (12)***

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -6z_1 + 5e_1$$

$$\dot{z}_2 = -7z_1 + 2z_3 + 3z_4 + 4z_5 + 0.9334x_1 + 2.5006x_2 + 4.174x_3 + 0.933x_4 + 8.2165e_2 + 0.4316e_3 + 1.1795e_4$$

$$\dot{z}_3 = 2z_2 - 5z_3 + 2z_4 + 5z_5 + 1.1668x_1 + 3.1258x_2 + 5.2175x_3 + 1.1668x_4 + 0.4516e_2 + 6.3696e_3 + 3.6417e_4$$

$$\dot{z}_4 = -3z_3 - 4z_4 + 3z_5 + 0.7001x_1 + 1.8755x_2 + 3.1305x_3 + 0.7001x_4 + 1.1795e_2 + 3.6417e_3 - 8.5433e_4$$

$$\begin{aligned}\dot{z}_5 = {} & -9z_5 - 1.9905x_1 - 5.0934x_2 - 9.1408x_3 - 1.8441x_4 + \\ & 0.4504 + 0.052325\sin(\hat{x}_1 + \hat{x}_2)\sin(\hat{x}_3 + \hat{x}_4) + 0.00392375 * \\ & \cos(\hat{x}_4 + \hat{x}_1)\cos(\hat{x}_3 + \hat{x}_5) + 0.0000432\sin(\hat{x}_1)\cos(\hat{x}_5) + \\ & 0.0013075\cos(\hat{x}_4 + \hat{x}_5)\sin(\hat{x}_3 + \hat{x}_2) + 0.00621555 * \\ & \sin(\hat{x}_1 + \hat{x}_5)\sin(\hat{x}_2 + \hat{x}_3) + 0.00549325\cos(\hat{x}_4 + \hat{x}_5) * \\ & \sin(\hat{x}_3 + \hat{x}_2) + 0.000594\sin(\hat{x}_5)\cos(\hat{x}_4) + 0.00013075 * \\ & \cos(\hat{x}_3 + \hat{x}_1)\sin(\hat{x}_2 + \hat{x}_5) - 6.0623e_2 - 7.3481e_3 - \\ & 8.5433e_4\end{aligned}$$

**STEP (13)**

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (21) of illustrations (2.6) case(1) to find the solutions of* $x_1$, $x_2$, $x_3$, $x_4$ *and* $x_5$ see ( *appendix ( C )* ).

**STEP (14)**

Find the solution of the error by the equation (2.20) .

By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. *Problem (22) of illustrations (2.6) case(1) to find the error solutions of* $e_1$, $e_2$, $e_3$, $e_4$ *and* $e_5$ see ( *appendix (C )* ). The numerical solution of error can be found in Figure (2.11).

**Case (II) :** *( When control* $u = \cos(t)$ *)*

**STEP (2), STEP (3), and STEP(4)**

Are the same as steps of *case ( I )*.

**STEPS (6) , (7) , (8), (9) and (10)**

Are the same as steps of *case ( I )* .

**STEP (11)**

Find the Lipschitz constant of $f(x(t),u(t))$ and $g(x(t))$, by the conditions : $k + k_1 < \dfrac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)}$

$$(1) \qquad f(x,u) = \begin{bmatrix} 0.125\cos(t)\sin(x_1 + x_2)\sin(x_3 + x_4) \\ 0 \\ 0.015\cos(t)\cos(x_4 + x_1)\cos(x_3 + x_5) \\ 0.002\cos(t)\sin(x_1)\cos(x_5) \\ 0.025\cos(t)\cos(x_4 + x_5)\sin(x_3 + x_2) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $f(x(t),u(t))$ is found

$$\Rightarrow \quad \|J_1\| \le 0.25$$

$$\Rightarrow \quad \|f(x_2(t),u(t)) - f(x_1(t),u(t))\| \le 0.25\|x_2(t) - x_1(t)\|$$

Thus, the non-linearity $f(x(t),u(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k = 0.25$.

$$(2) \qquad g(x) = \begin{bmatrix} 0 \\ 0.016\sin(x_1 + x_5)\sin(x_2 + x_3) \\ 0.0175\cos(x_4 + x_5)\sin(x_3 + x_2) \\ 0.0275\sin(x_5)\cos(x_4) \\ 0.0025\cos(x_3 + x_1)\sin(x_2 + x_5) \end{bmatrix}$$

From simple calculations the norm of Jacobian matrix for $g(x(t))$ is found

$$\Rightarrow \quad \|J_2\| \le 0.062$$

$$\Rightarrow \quad \|g(x(t)) - g(\hat{x}(t))\| \le 0.062\|x(t) - \hat{x}(t)\|$$

Thus, the non-linearity function $g(x(t))$ is satisfied a Lipschitz condition with Lipschitz constant $k_1 = 0.062$.

$$k + k_1 = 0.25 + 0.0622 = 0.312, \quad \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} = \frac{0.0342}{0.0952} = 0.35$$

$$\Rightarrow \quad k + k_1 < \frac{\delta_{\min}(\mathrm{P})}{\delta_{\max}(\mathrm{P}R)} \quad \Rightarrow \quad 0.312 < 0.35$$

Hence, Lipschitz constant is satisfied.

**STEP (12)**

Find the observer of this system, from eq. (2.40), and simplified the system, we get:

$$\dot{z}_1 = -6z_1 + 5e_1$$

$$\dot{z}_2 = -7z_1 + 2z_3 + 3z_4 + 4z_5 + 0.9334x_1 + 2.5006x_2 + 4.174x_3 + 0.933x_4 + 8.2165e_2 + 0.4316e_3 + 1.1795e_4$$

$$\dot{z}_3 = 2z_2 - 5z_3 + 2z_4 + 5z_5 + 1.1668x_1 + 3.1258x_2 + 5.2175x_3 + 1.1668x_4 + 0.4516e_2 + 6.3696e_3 + 3.6417e_4$$

$$\dot{z}_4 = -3z_3 - 4z_4 + 3z_5 + 0.7001x_1 + 1.8755x_2 + 3.1305x_3 + 0.7001x_4 + 1.1795e_2 + 3.6417e_3 - 8.5433e_4$$

$$\dot{z}_5 = -9z_5 - 1.9905x_1 - 5.0934x_2 - 9.1408x_3 - 1.8441x_4 + 0.4504\cos(t) + 0.052325\cos(t)\sin(\hat{x}_1 + \hat{x}_2) * \sin(\hat{x}_3 + \hat{x}_4) + 0.00392375\cos(t)\cos(\hat{x}_4 + \hat{x}_1) * \cos(\hat{x}_3 + \hat{x}_5) + 0.0000432\cos(t)\sin(\hat{x}_1)\cos(\hat{x}_5) + 0.0013075\cos(t)\cos(\hat{x}_4 + \hat{x}_5)\sin(\hat{x}_3 + \hat{x}_2) + 0.00621555 * \sin(\hat{x}_1 + \hat{x}_5)\sin(\hat{x}_2 + \hat{x}_3) + 0.00549325\cos(\hat{x}_4 + \hat{x}_5) * \sin(\hat{x}_3 + \hat{x}_2) + 0.000594\sin(\hat{x}_5)\cos(\hat{x}_4) + 0.00013075 * \cos(\hat{x}_3 + \hat{x}_1)\sin(\hat{x}_2 + \hat{x}_5) - 6.0623e_2 - 7.3481e_3 - 8.5433e_4$$

**STEP (13)**

Find the solutions of the dynamical system, By MATLAB, we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{x}$. *Problem (23) of illustrations (2.6) case (2) to find the solutions of $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$* see ( *appendix ( C )* ).

**STEP (14)**

Find the solution of the error by the equation (2.20) .

By MATLAB , we used the fourth-order ( Runge-Kutta ) method to find the solutions of $\dot{e}$. The numerical solution of error can be found in Figure (2.12). *Problem (24) of illustrations (2.6) case(2) to find the error solutions of* $e_1$, $e_2$, $e_3$, $e_4$ *and* $e_5$ see ( *appendix (C )* ). The results of error solution ( error solutions ) are ploted in the fllowing graphs.

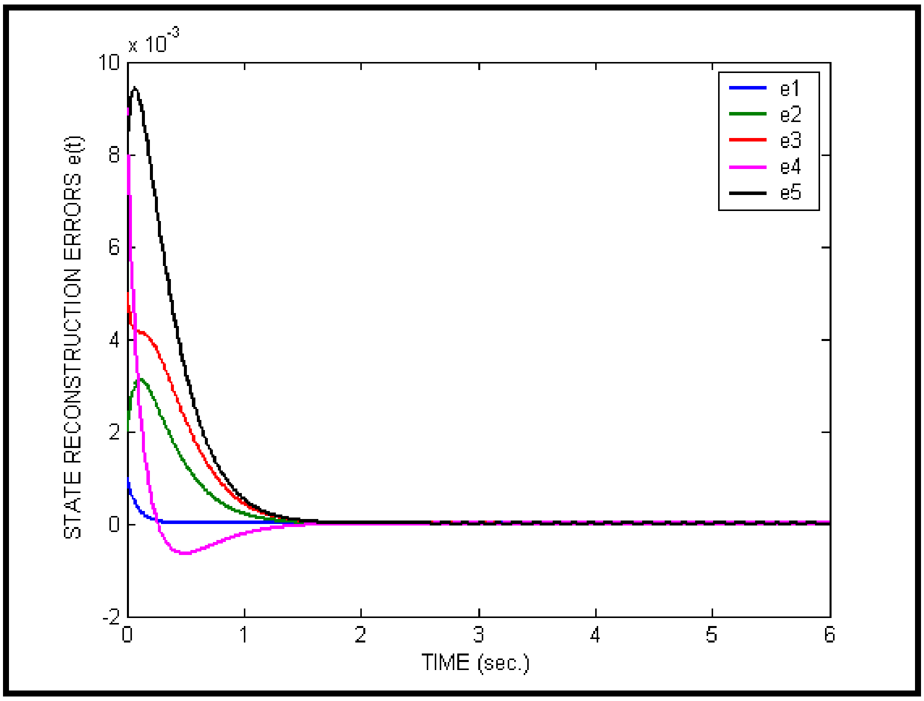


***Figure (2.11) The non-linear system of illustrations (2.6) If u =1 represented the states [$e_1$, $e_2$, $e_3$, $e_4$ and $e_5$] with initial conditions [$e_1(0)$, $e_2(0)$, $e_3(0)$, $e_4(0)$ and $e_5(0)$]=[0.001 0.002 0.005 0.009 0.008]***

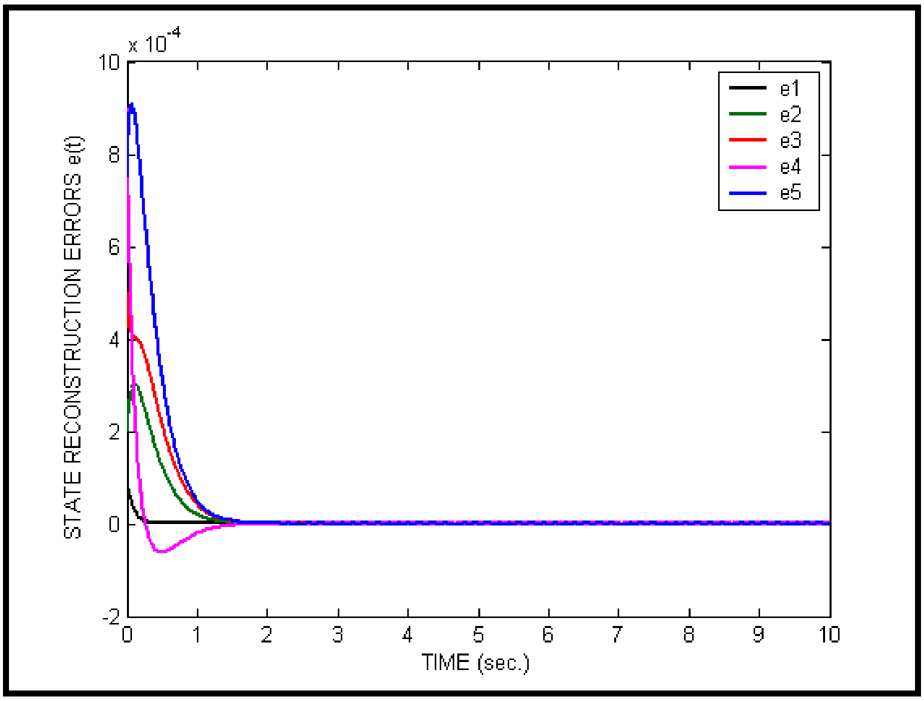


***Figure (2.12) The non-linear system of illustration (2.6) If u =cos(t) represented the states [$e_1$, $e_2$, $e_3$, $e_4$ and $e_5$] with initial conditions [$e_1(0)$, $e_2(0)$, $e_3(0)$, $e_4(0)$ and $e_5(0)$]=[0.0001 0.0002 0.0005 0.0009 0.00075]***

# *CONCLUSIONS*

A simple method has been presented for designing state-observers of non-linear singular systems. Assuming that the non-linear function is globally Lipschitz, it is permitted to set the dynamics of the linear part while ensuring the global stability.

Necessary and sufficient conditions for existence and stability of the proposed observer have been established. The observer system and its solution are found to be a suitable applicable procedure based on the theoretical result and have a very interested real life application. The stability behaviors of the dynamical systems are also been considered in this work.

# *Contents*

# *Future Work*

In the future work, one can precede and developed his work based on the present study to include a large class of nonlinear control systems;

**1.** The studying the non-linear dynamic control system observer in stochastic nature not in deterministic case.

**2.** Studying the non-linear dynamic fuzzy observer for some class of non-linear dynamic control system.

**3.** Studying the case when the non-singular matrix of singular dynamic control system are of type fat, skinny, etc. .

**4.** Generalized the present work to include a large class of system uncertainty.

# *INTRODUCTION*

Observers use the plant input and output signals to generate an estimate of the plant's state, which is then employed to close the control loop. Observers are utilized to augment or replace sensors in a control system. The observer was first proposed and developed by Luenberger in the early sixties of the last century ( Luenberger, 1966; 1971; 1979) [34], [35] and [36].

Since the early developments, observers for plants with both known and unknown inputs have been developed resulting in the so called unknown input observer architectures, such as, for example, those in ( Bhattacharyya, 1978 [1]; Chen *et al*., 1996 [8]; Corless and Tu, 1998 [10]; Darouach *et al*., 1994 [12]; Hostetter and Meditch, 1973 [17]; Hou and Müller, 1992 [18]; Hou *et al*., 1999 [19] ; Hui and ÿ Zak, 1993 [21]; 2005 [22]; Kudva *et al*., 1980 [30]; Kurek, 1983 [30]; Krzemi´ nski and Kaczorek, 2004 [31]; Sundareswaran *et al*., 1977 [46]; Wang *et al*., 1975 [54]; Yang and Wilde, 1988 [57] ).

More recently , observer architectures utilizing the concept of sliding modes were proposed for uncertain systems , see, for example, (Edwards and Spureon, 1998 [13]; Ha *et al*., 2003 [16]; Hui and ÿ Zak, 1990 [20]; Koshkouei and Zinober, 2004 [28]; Utkin *et al*., 1999 [48]; Walcott and ÿ Zak, 1987 [50]; 1988 [51]; Walcott and ÿ Zak ,1990 [52]; Hui and ÿ Zak, 1993 [21]).

Other methods of observer design for linear systems developed up to 1983 are reported by O'Reilly in (1983) [42]. Observers for systems with unknown inputs play an essential role in robust model-based fault detection ( Edwards *et al*., 2000 [14]; Edwards and Spurgeon, 1998 [13]; Jiang *et al*., 2004 [24]; Saif and Xiong, 2003 [43]).

The basic idea behind the use of observers for fault detection is to form residuals from the difference between the actual system outputs and the estimated outputs using an observer .

Once a fault occurs, the residuals are expected to react by becoming greater than a prespecified threshold. When the system under consideration is subject to unknown disturbances or unknown inputs, their effect has to be decoupled from the residuals to avoid false alarms.

State observation of non-linear dynamical systems is becoming a growing topic of investigation in the specialised literature ( Tsinias, 1989 [47]), (Walcott, 1987 [50]). The reconstruction of state variables remains a major problem both in control theory and process diagnosis (Magni, 1991 [37]). Researcher attention is being particularly focused on the design of adaptive observers for on-line process state estimation . There is increasing a wareness that to ensure robustness in performance requires simpler and stable adaptive observer schemes. Linear systems have received considerable attention leading to the several stable adaptive observer systems .

Linear observers involving unknown inputs have also been developed and analysed (Chang, 1995 [5]), (Gaddouna, 1996 [15]).

Nevertheless, the design of asymptotically stable observers for non-linear dynamic control system remains a hard task, even when the non-linearities are fully known. The nonsingular problem because a new task for design an stabilized observer.

Based on the results of [2], [5], [10], [15], [47], [38], [44], [50], and [53], our aim in this work is to design an stabilized full-order observer for some class of non-linear ( singular control system ).

In this work, some suggested scheme and suggested procedure for nonlinear dynamical control system are proposed and developed.

This thesis, consists of two chapters. The first chapter is devoted some basic mathematical concepts, dynamical control equation, the formulation of control problems, basic concepts and definitions, singular value decomposition, Lyapunove stability, controllability and observability.

The second chapter is concerned with the problem of observability and stabilizability of uncertain non-linear dynamical control system and some simulation and graphs conclude remarks, comments, useful mathematical facts, conclusions, future work, references and three appendices have also been presented.

# *REFERENCES*


[1] Bhattacharyya S. P., " Observer Design for Linear Systems with Unknown Inputs ", IEEE Trans. Automat. Contr., Vol. AC-23, No. 3, pp. 483–484, 1978.

[2] Boutayeb M., M. Darouach, HJ. Rafaralahy, G. Krzakala , " Asymptotic Observer for a Class of Non –Linear Singular System , Proceeding of the ACC, Baltimore, pp.1440-1551, 1994.

[3] Brian D. Hahn , " Essential MATLAB for Scientists and Engineering ", South Africa , Inc., 1997.

[4] Brogan W. L., " Modern Control Theory ", Prentice-Hall, Inc., 1985.

[5] Chang S.K., P.L. Hsu ," A Novel Design for The Unknown Input Fault Detection Observer ", Control Theory and Advanced Technology, pp.1029-1051, 1995**.**

[6] Chang S.-K., You W.-T., and Hsu P.-L., " Design of General Structured Observers for Linear Systems with Unknown Inputs ", J. Franklin Instit., Vol. 334B, No. 2, pp. 213–232, 1997.

[7] Chen C. T., "Linear System Theory and Design", CBS College Publishing New York, 1984.

[8] Chen J., Patton R. J., and Zhang H.-Y. , " Design of Un-known Input Observers and Robust Fault Detection Filters " , Int. J. Contr., Vol. 63, No. 1, pp. 85–105, 1996.

[9] Childs L. , " Note on the Singular Value Decomposition ", 5 November, 2003

[10] Corless M. and Tu J. ," State and Input Estimation for a Class of Uncertain Systems ", Automatica , Vol. 34 , No. 6 , pp. 757–764 , 1998.

[11] Cronin Jane, " Differential Equations: Introduction and Qualitative Theory ", by Marcel Dekker , New York , Inc., 1980.

[12] Darouach M., Zasadzinski M., and Xu S. J. , " Full- Order Observers for Linear Systems with Unknown Inputs ", IEEE Trans.Automat. Contr., Vol. 39 , No. 3 , pp. 606–609 , 1994.

[13] Edwards C. and Spurgeon S. K. , " Sliding Mode Control : Theory and Applications " , London : Taylor and Francis , Inc., 1998 .

[14] Edwards C., Spurgeon S. K., and Patton R. J. , " Sliding Mode Observers for Fault Detection and Isolation ", Automatica , Vol. 36 , No. 4 , pp. 541–553, 2000 .

[15] Gaddouna B. , S. Giuliani , J. Ragot , " Design of Observers for Descriptor Systems Affected by Unknown Inputs " , International Journal of Systems Science , pp465-471 , 1996 .

[16] Ha Q. P., Trinh H., Nguyen H. T., and Tuan H. , " Dynamic Output Feedback Sliding-Mode Control Using Pole Placement and Linear Functional Observers ", IEEE Trans.Industr. Electron. , Vol. 50 , No. 5 , pp. 1030–1037 , 2003.

[17] Hostetter G. and Meditch J. S. , " Observing Systems with Unmeasurable Inputs " , IEEE Trans. Automat. Contr., Vol. AC-18, No. 3, pp. 307–308 , 1973 .

[18] Hou M. and Müller P. C. , " Design of Observers for Linear with Systems Unknown Inputs ", IEEE Trans. Automat. Contr., Vol. 37, No. 6 , pp. 871–875 , 1992 .

[19] Hou M., Pugh A.C. and Müller P. C. , " Disturbance Decoupled-Functional Observers " , IEEE Trans. Automat. Contr., Vol. 44, No. 2, pp. 382–386 ,1999 .

[20] Hui S. and ÿ Zak S. H. , " Control and Observation of Uncertain systems: A variable Structure Systems Approach, In: Control and Dynamic Systems ; Advances in Theory and Applications ", Vol. 3: Advances in Control Mechanics, Part 1 of 2 (C. T. Leondes, Ed.), Inc. San Diego, Academic Press, pp. 175–204, 1990.

[21] Hui S. and ÿ Zak S. H., " Low-Order State Estimation and Compensators for Dynamical Systems with Unknown Inputs ", Syst. Contr. Lett., Vol. 2, No. 6, pp. 493–502, 1993.

[22] Hui S. and ÿ Zak S. H. , " Low-Order Unknown Input Observers ", Proc. Amer. Contr. Conf. , Portland, OR , pp. 4192–4197, 2005.

[23] Jayc H. and Andrew U.M., " Modern Control Principles and Applications ", New York, Mc Graw-Hill, Inc., 1968.

[24] Jiang B., Staroswiecki M., and Cocquempot V., " Fault Estimation in Nonlinear Uncertain Systems Using Robust/Sliding-Mode Observers ", IEE Proc. Contr. Theory Appl., Vol. 151, No. 1, pp. 29–37, 2004.

[25] Katsuhiko Ogata , " Modern Control Engineering ", Fourth Edition, Prentic-Hall, Inc., 2002.

[26] Kevin R Coombes, Brian R. Hunt, Ronald L. Lipsman, John E. Obsoon and Garrett J. Stuck, " Differential Equation with MATLAB ", New York, Inc., 2000.

[27] Kolman Bernard, " Introduction Linear Algebra with Applications ", Fourth Edition, New York, Inc., 1984.

[28] Koshkouei A. J. and Zinober A. S. I. , " Sliding Mode State Observation for Non-linear Systems ", Int. J. Contr., Vol. 77, No. 2, pp. 118–127, 2004.

[29] Kudva P., Viswanadham N., and Ramakrishna A. , " Observers for Linear Systems with Unknown Inputs ", IEEE Trans. Automat. Contr., Vol. AC-25, No. 1, pp. 113–115, 1980.

[30] Kurek J. E. , " The State Vector Reconstruction for Linear Systems with Unknown Inputs ", IEEE Trans. Automat. Contr., Vol. AC-28, No. 12, pp. 1120–1122, 1983.

[31] Krzemi´nski S. and Kaczorek T. , " Perfect Reduced-Order Unknown-Input Observer for Standard Systems " , Bull. Polish Acad. Sci., Techn. Sci. , Vol. 52, No. 2, pp. 103–107, 2004.

[32] Kwakernaak H. and Raphael S., " Linear Optimal Control Systems ", John Wiley & Sons, Inc., 1972 .

[33] Lapidus L. and Luus R., " Optimal Control of Engineering Processes", Blalsdell Publishing Company a Dlicision of Gin and Company, USA, Inc., 1967.

[34] Luenberger D. G., " Observers for Multivariable Systems ", IEEE Trans. Automat. Contr., Vol. AC-11, No. 2, pp. 190–197, 1966.

[35] Luenberger D. G. , " An Introduction to Observers ", IEEE Trans. Automat. Contr., Vol. AC-16, No. 6, pp. 596–602 , 1971.

[36] Luenberger D. G., " Introduction to Dynamic Systems: Theory, Models, and Applications ", New York: Wiley , Inc., 1979.

[37] Magni J.F. and P. Mouyon , " A Generalised Approach to Observers for Fault Dagnosis " , Proc. of the 30th IEEE Conference on Decision and Control, Brighton, pp. 2236-2241, 1991.

[38] Misawa E.A. and J.K. Hedrick, " Non-Linear Observer – A state of the Art Survey ", Transactions of the ASME, pp. 344-352, 1989.

[39] Ogata K. , " State Space Analysis of Control System ", Prentice Hall, Inc., Englewood Cliffs , USA , 1967 .

[40] Ogata Katsuhiko , " Modern Control Engineering ", Second Edition , Prentic-Hall of India, Inc., 1996.

[41] O’Reilly J., " Observers for Linear Systems ", London Academic Press , Inc., 1983.

[42] Saif M. and Xiong Y. , " Sliding Mode Observers and their Application in Fault Diagnosis " , In : Fault Diagnosis and Fault Tolerance for Mechatronic Systems ( F. Caccavale and L. Villani, Eds.), Berlin: Springer , Inc., pp. 1–57 , 2003.

[43] S. Hui and S.H . ÿZak Gantmacher F. R. , " The Theory of Matrices " , Vol. 1, second edition , New York , Chelsea Publishing Company ,Inc., 1990 .

[44] Söffker D. , T.J. Yu , P.C. Müller , " State Estimation of Dynamical Systems with Non-Linearities by using Proportional-Integral Observer ", International Journal of Systems Sciences, pp. 1571-1582, 1995.

[45] Suddhendu Biswas, " A Text Book of Matrix Algebra ", Second Edition, New Delhi, Inc., 1997.

[46] Sundareswaran K. K., McLane P. J. and Bayoumi M. M., " Observers for Linear Systems with Arbitrary Plant Disturbances " , IEEE Trans. Automat. Contr., Vol. AC-22, No. 5, pp. 870–871, 1977.

[47] Tsinias J., " Observer Design for Non-Linear Systems ", Systems and Control Letters, pp. 135-142, 1989.

[48] Utkin V., Guldner J., and Shi J. , " Sliding Mode Control in Electromechanical Systems ", London: Taylor and Francis ,Inc.,1999.

[49] Walcott B.L. and S.H. Zak , " Observation of Dynamical Systems in the Presence of Bounded Non-Linearities / Uncertainties ", Proc. of 25th IEEE Conference on Decision and Control , Athens, pp. 961-966, 1986.

[50] Walcott B.L. and ÿ Zak S. H., " State Observation of Nonlinear Uncertain Dynamical Systems ", IEEE Trans. Automat. Contr., Vol. AC-32, No. 2, pp. 166–170, 1987.

[51] Walcott B. L. and ÿ Zak S. H. , " Combined Observer Controller Synthesis for Uncertain Dynamical Systems with Applications ", IEEE Trans. Sys., Man Cybern., Vol. 18, No. 1, pp. 88–104, 1988.

[52] Walcott B. L. and ÿ Zak S. H., " State Observation of Nonlinear Control Systems via the Method of Lyapunov ", In: Deterministic Control of Uncertain Systems (A. S. I. Zinober, Ed.), London: Peter, Peregrinus Ltd, Ch. 16, pp. 333–350, 1990.

[53] Walcott B. L., and Hui S., ÿ Zak S. H., " Variable Structure Control and Observation of Nonlinear/Uncertain Systems ", In: Variable Structure Control for Robotics and Aerospace Applications ( K.-K. D. Young, Ed. ), Amsterdam: Elsevier, Inc., Ch. 4, pp. 59–88, 1993.

[54] Wang S.-H., Davison E. J., and Dorato P., " Observing the States of Systems with Unmeasurable Disturbances ", IEEE Trans. Automat.Contr. Vol. AC-20, No. 5, pp. 716–717, 1975.

[55] Xia X.H. and W.B. Gao, " On Exponential Observer for Non-Linear Systems ", Systems and Control Letters, pp. 319-325,1988.

[56] Xiang J., Su H., and Chu J., " On the Design of Walcott-Zak Sliding Mode Observer ", Proc.Amer. Contr. Conf. Portland, OR, pp. 2451–2456, 2005.

[57] Yang F. and Wilde R. W., " Observers for Linear Systems with Unknown Inputs ", IEEE Trans. Automat. Contr., Vol. 33, No. 7, pp. 677–681, 1988.

# *ABSTRACT*

This work deals with the problem of dynamic state estimator for a class of non- linear singular dynamic systems. Sufficient conditions for the existence of such observers are provided and the design of such observer is examined.

Since our nonlinear dynamic system is singular and assuming that the singular matrix is full row rank involves many calculation simplifications; therefore special emphasis in the computational aspect of the observer matrices is discussed supported by some useful comments.

In this work, scheme and procedure work for nonlinear dynamical observer control system are proposed and developed to estimate the unmeasured state space points.

The proofs of the theorems of this scheme as well as their computational algorithm have been developed and presented. The concluding and necessary remarks and lemmas have also been discussed.

Finally, several problems are given to demonstrate the validity and the effectiveness of present work before and after their behaviors are simulated and are shown in graphs.

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*Noor Kamaran Al-Chawishly*

*April,* 2007

# *DEDICATION*

*To My Parents and My Brothers with Love and Respects*

*NOOR*

# *Examining Committee's Certification*

We certify that we read this thesis entitled " ***NONLINEAR DYNAMIC OBSERVER FOR SOME CLASS OF NONLINEAR SINGULAR DYNAMIC CONTROL SYSTEM*** " and as examining committee examined the student, ***Noor Kamaran Ihssan*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

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# *Suppervisors Certification*

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science***, ***Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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In view of the available recommendations; I forward this thesis for debate by examination committee.

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*Ministry of Higher Education*
*and Scientific Research*
*Al-Nahrain University*
*College of Science*

# *Nonlinear Dynamic Observer for Some Class of Nonlinear Singular Dynamic Control System*

*A Thesis*

*Submitted to the Department of Mathematics, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics*

*By*

*Noor Kamaran Ihssan*

*(B.Sc. 2004)*

*Under Supervision of*

*Assist. Prof. Dr. Radhi Ali Zaboon*

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# المستخلص


يتعلق هذا العمل بمسألة مخمن دالة الحالة الديناميكية (state estimation) لبعض الأنظمة الديناميكية غير الخطية الشاذة(Nonlinear singular dynamic system) . حصلنا على الشروط الكافية لوجود النظام المخمن (observer) مع إختيار التصميم .

أعطي إهتمام خاص لحساب المصفوفات الشاذة (singular matrix) ذات الرتبة الكاملة (full rank) لإرتباطها بالتصميم .

في هذا العمل ، لقد تم وضع و عرض أسلوبية عمل لتصميم نظام دينامي لبعض أنظمة السيطرة مدعوماً بالنظريات الأساسية و براهينها بالإضافة إلى خوارزميات عددية لغرض تصميم المخمن .

لقد نوقشت الاستنتاجات الضرورية و الملاحظات الرياضية المتعلقة بها .

أخيراً ، لدراسة الأهمية و كفاءة الأسلوبية ، لقد تم عرض و محاكاة بعض المسائل الرياضية الافتراضية و دراسة سلوكها الكمي الرياضي و تم عرض النتائج على شكل رسومات (graphs) .