List of Common Symbols using in MATLAB Programs:

$\mathrm{ctrb}(\mathrm{A},\mathrm{B})$	Computes the controllability matrix, where A and B are system matrices $\dot{x} = \mathrm{A}x + \mathrm{B}u$
$\mathrm{obsv}(\mathrm{A},C)$	Computes the observability matrix, where A and C are system input-output matrices.
$\mathrm{place}(\mathrm{A},\mathrm{B},\mu)$	Compute an appropriate gain matrix K, where A, B are matrices of the system, μ is the vector of the desired closed-loop poles.
A'	The transpose of the matrix A.
$\mathrm{place}(\mathrm{A}',C',\eta)'$	Compute an appropriate observer gain matrix L, where A,C are system input-output matrices, η is the vector of the observer poles.
eye (n)	Generates an $n \times n$ identity matrix.
zeros (n)	Generates an $n \times n$ matrix of zeros.
ode45	(fourth, fifth order) to implement Runge-Kutta method.
inv (A)	Returns the inverse of the square matrix A.
plot (x,y)	Graphs y as a function of x.
norm (x)	Calculates the norm.
eig (A)	Finds eigenvalues of the square matrix A.
$\mathrm{care}(G,H,Q)$	Computes the unique solution P of the algebraic Riccati equation $G'\mathrm{P} + \mathrm{P}G - \mathrm{P}HH'\mathrm{P} + Q = 0$
rank (A)	Returns the rank of the square matrix A.
exp(.)	Exponential.
label	Appears beneath its respective axis in a two-dimensional plot.
A(: , j)	Returns the j-th column of the matrix.
inline	Construct A MATLAB inline function from a stringe expression.

MATLAB program (A1)

$\mathrm{A} = [0 \quad 1 \quad 0;0 \quad 0 \quad 1;6 \quad -11 \quad -6];$

$\mathrm{B} = [0;0;10];$

$C = [1 \quad 0 \quad 0];$

$\mathrm{M} = \mathrm{ctrb}(\mathrm{A},\mathrm{B})$

rank (M)

$\mathrm{K} = [2.2 \quad 0.9 \quad -0.1]$

$\mathrm{N} = \mathrm{obsv}(\mathrm{A}, C)$

rank (N)

$L = [14;255;-474]$

Q =eye(3) ;

H = [0 ; 0 ; 0] ;

$G = (\mathrm{A} - LC)$;

$\mathrm{P} = \mathrm{care}(G, H, Q);$

$n = \mathrm{eig}\ (\mathrm{P})$;

$\sigma_1 = 20$; $\sigma_2 = 1$; $\sigma_3 = 0.2$;

$\mathrm{T} = [\sigma_1 \quad 0 \quad 0;0 \quad \sigma_2 \quad 0;0 \quad 0 \quad \sigma_3]$;

$\mathrm{T}^{-1} = \mathrm{inv}(\mathrm{T});$

$\overline{G} = \mathrm{T} * G * \mathrm{T}^{-1};$

$\overline{\mathrm{P}} = \mathrm{care}(\overline{G}, H, Q);$

$\overline{n} = \mathrm{eig}(\overline{\mathrm{P}})$

$[\mathrm{T}*(\mathrm{A}-\mathrm{B}*\mathrm{K})*\mathrm{T}^{-1} \quad \mathrm{T}*\mathrm{B}*\mathrm{K}*\mathrm{T}^{-1}; zeros(3,3) \quad \mathrm{T}*(\mathrm{A}-L*C)*\mathrm{T}^{-1}]$

$\Psi = inline($'$[20*z(2)+0.2*\cos((z(1)/20)+z(2));5*z(3)+0.06*\sin(z(2))*\cos(z(2))+0.1*z(1)*\sin^2(t);-0.16*z(1)-4*z(2)-5*z(3)+0.22*z(4)+1.8*z(5)-z(6)+0.002*\sin^2(z(3)/0.2)+0.02*z(2)*\cos(2*t);-14*z(4)+20*z(5)+0.2*\cos((z(1)/20)+z(2))-0.2*\cos(((z(1)-z(4))/20)+(z(2)-z(5)));-11.25*z(4)+5*z(6)+0.06*\sin(z(2))*\cos(z(2))-0.06*\sin(z(2)-z(5))*\cos(z(2)-z(5))+$

$0.01*z(1)*\sin^2(t)-0.01*(z(1)-z(4))*\sin^2(t);4.8*z(4)-2.2*z(5)-6*z(6)+$
$*z(4)-2.2*z(5)-6*z(6)+0.002*\sin^2(z(3)/0.2)-0.002*\sin^2((z(3)-z(6$
$))/0.2)+0.02*z(2)*\cos(2*t)-0.02*(z(2)-z(5))*\cos(2*t)]','t','z');$

$[t,za]=ode45(\Psi,[0:0.01:10],[10 \quad 1 \quad 0.4 \quad 10 \quad 0 \quad 0.06]);$

$z_1=za(:,1);$

$z_2=za(:,2);$
$z_3=za(:,3);$
$E_1=za(:,4);$
$E_2=za(:,5);$
$E_3=za(:,6);$

$x_1=z_1/\sigma_1$
$x_2=z_2/\sigma_2$
$x_3=z_3/\sigma_3$

$e_1=E_1/\sigma_1$
$e_2=E_2/\sigma_2$
$e_3=E_3/\sigma_3$

$\hat{x}_1=x_1-e_1$
$\hat{x}_2=x_2-e_2$
$\hat{x}_3=x_3-e_3$

$cc_1=[\hat{x}_1 \quad x_1]$
$cc_2=[\hat{x}_2 \quad x_2]$
$cc_3=[\hat{x}_3 \quad x_3]$

figure (1), plot (t,cc_1)

$x$label (' Time (sec)')

y label (' the state x_1 and its observer ')

figure (2), plot (t,e_1)

$x$label (' Time (sec)')

y label (' the state variable $e_1(t)$ ')

figure (3), plot (t,cc_2)

$x$label (' Time (sec)')

y label (' the state x_2 and its observer ')

figure (4), plot (t, e_2)

$x$label (' Time (sec)')

y label (' the state variable $e_2(t)$ ')

figure (5), plot (t, cc_3)

$x$label (' Time (sec)')

y label (' the state x_3 and its observer ')

figure (6), plot (t, e_3)

$x$label (' Time (sec)')

y label (' the state variable $e_3(t)$ ')

MATLAB program (A2)

$\mathrm{A} = [0 \quad 1 \quad 0 \quad 0; -48.6 \quad -12.4 \quad 48.6 \quad 0; 0 \quad 0 \quad 0 \quad 1; 19.4 \quad 0 \quad -19.4 \quad 0];$

$\mathrm{B} = [0; 21.6; 0; 0];$

$C = [1 \quad 0 \quad 0 \quad 0; 0 \quad 1 \quad 0 \quad 0];$

$\mathrm{M} = \mathrm{ctrb}(\mathrm{A}, \mathrm{B})$

rank (M)

$\mathrm{K} = [2.2 \quad 0.9 \quad -0.1];$

$\mathrm{N} = \mathrm{obsv}(\mathrm{A}, C)$

rank (N)

$\eta = [14.4 - i * 19.2 \quad -14.4 + i * 19.2 \quad -16 \quad -20];$

$L = \mathrm{place}(\mathrm{A}', c', \eta)'$

Q = eye(4) ;

H = [0 ; 0 ; 0 ; 0] ;

$G = (\mathrm{A} - LC)$;

P = care(G, H, Q);

n = eig (P) ;

$\sigma_1 = 10$; $\sigma_2 = 10$; $\sigma_3 = 20$; $\sigma_4 = 0.6$;

$\mathrm{T} = [\sigma_1 \quad 0 \quad 0 \quad 0;0 \quad \sigma_2 \quad 0 \quad 0;0 \quad 0 \quad \sigma_3 \quad 0;0 \quad 0 \quad 0 \quad \sigma_4]$;

$\mathrm{T}^{-1} = \mathrm{inv}(\mathrm{T})$;

$\overline{G} = \mathrm{T} * G * \mathrm{T}^{-1}$;

$\overline{\mathrm{P}} = \mathrm{care}(\overline{G}, H, Q)$;

$\overline{n} = \mathrm{eig}(\overline{\mathrm{P}})$

$[\mathrm{T}*(\mathrm{A}-\mathrm{B}*\mathrm{K})*\mathrm{T}^{-1} \quad \mathrm{T}*\mathrm{B}*\mathrm{K}*\mathrm{T}^{-1}; zeros(4,4) \quad \mathrm{T}*(\mathrm{A}-L*C)*\mathrm{T}^{-1}]$

$\Psi = inline('[z(2);-101.4*z(1)-16.2*z(2)+32.1433*z(3)-132.0619*z(4)+$
$52.8*z(5)+3.8*z(6)-7.8433*z(7)+132.0619*z(8);33.3333*z(4);1.1640*$
$z(1)-0.5820*z(3)+(0.6*-33.2*\sin(z(3)/20));-17.9596*z(5)+0.7544*z($
$6);1.0943*z(5)-46.8404*z(6)+24.3*z(7);-0.5963*z(5)-44.1560*z(6)+$
$33.333*z(8);1.7117*z(5)-11.6216*z(6)-0.5820*z(7)+0.6*-33.2*(\sin($
$z(3)/20)-\sin((z(3)-z(7))/20))]', `t', `z');$
$[t, za] = ode45(\Psi,[0:0.01:10],[1 \quad 0 \quad -2 \quad 0 \quad 0.5 \quad -1 \quad -0.5 \quad 0]);$

$z_1 = za(:,1);$
$z_2 = za(:,2);$
$z_3 = za(:,3);$
$z_4 = za(:,4);$
$E_1 = za(:,5);$
$E_2 = za(:,6);$
$E_3 = za(:,7);$
$E_4 = za(:,8);$

$x_1 = z_1/\sigma_1$
$x_2 = z_2/\sigma_2$
$x_3 = z_3/\sigma_3$
$x_4 = z_4/\sigma_4$

$e_1 = E_1/\sigma_1$
$e_2 = E_2/\sigma_2$

$e_3 = E_3/\sigma_3$

$e_4 = E_4 / \sigma_4$

$\hat{x}_1 = x_1 - e_1$
$\hat{x}_2 = x_2 - e_2$
$\hat{x}_3 = x_3 - e_3$

$\hat{x}_4 = x_4 - e_4$

$cc_1 = [\hat{x}_1 \quad x_1]$
$cc_2 = [\hat{x}_2 \quad x_2]$
$cc_3 = [\hat{x}_3 \quad x_3]$
$cc_4 = [\hat{x}_4 \quad x_4]$

figure (1), plot (t, cc_1)

*x*label (' Time (sec)')

y label (' the state x_1 and its observer ')

figure (2), plot (t, e_1)

*x*label (' Time (sec)')

y label (' the state variable $e_1(t)$ ')

figure (3), plot (t, cc_2)

*x*label (' Time (sec)')

y label (' the state x_2 and its observer ')

figure (4), plot (t, e_2)

*x*label (' Time (sec)')

y label (' the state variable $e_2(t)$ ')

figure (5), plot (t, cc_3)

*x*label (' Time (sec)')

y label (' the state x_3 and its observer ')

figure (6), plot (t, e_3)

*x*label (' Time (sec)')

y label (' the state variable $e_3(t)$ ')

figure (7), plot (t, cc_4)

$x$label (' Time (sec)')

y label (' the state x_4 and its observer ')

figure (6), plot (t, e_4)

$x$label (' Time (sec)')

y label (' the state variable $e_4(t)$ ')

MATLAB program (A3)

$\mathrm{A} = [0 \quad 1, -9 \quad -3.6];$

$\mathrm{B} = [0;1];$

$C = [1 \quad 0];$

$\mathrm{N} = \mathrm{obsv}(\mathrm{A}, C)$

rank (N)

$\eta = [-7.2 + 9.6 * i \quad -7.2 - 9.6 * i];$

$L = \mathrm{place}(\mathrm{A}', C', \eta)'$

Q = [10 0; 0 10];

H = [0 ; 0] ;

$G = (\mathrm{A} - LC)$;

P = care(G, H, Q);

n = eig (P) ;

no = norm (L)

$\Psi = inline(`[x(2) + 0.001 * \sin(x(1)) + 0.001 * \cos(x(2)) + 0.01 * x(2) * \cos(t/2); -$
$9 * x(1) - 3.6 * x(2) + \exp(-t) + 0.002 * \cos(x(2)) + 0.003 * x(1) * \sin^2(t)]', `t', `x');$
$[t, xa] = ode45(\Psi, [0 : 0.01 : 10], [-1 \quad 2]);$

$\Phi = inline(`[-10.8 * \hat{x}(1) + \hat{x}(2) - 0.0054 * \sin(\hat{x}(1)) * \cos(\hat{x}(2)) + 0.001 * \sin(\hat{x}(1))$
$+ 0.001 * \cos(\hat{x}(2)) + 0.01 * \hat{x}(2) * \cos(t/2) + 0.0065; -105.12 * \hat{x}(1) - 3.6 * \hat{x}(2) +$
$\exp(-t) - 0.0481 * \sin(\hat{x}(t)) * \cos(\hat{x}(2)) + 0.002 * \cos(\hat{x}(2)) + 0.003 * \hat{x}(1) * \sin^2($
$t) + 0.0577]', `t', `\hat{x}');$
$[t, \hat{x}a] = ode45(\Phi, [0 : 0.01 : 10], [-1 \quad 2]);$

$x_1 = xa(:,1);$
$x_2 = xa(:,2);$
$\hat{x}_3 = \hat{x}a(:,1);$

$\hat{x}_4 = x_4 - e_4$

$e_1 = x_1 - \hat{x}_1$

$e_2 = x_2 - \hat{x}_2$

$cc_1 = [\hat{x}_1 \quad x_1]$
$cc_2 = [\hat{x}_2 \quad x_2]$

figure (1), plot (t, cc_1)

$x$label (' Time (sec)')

y label (' the state x_1 and its observer ')

figure (2), plot (t, e_1)

$x$label (' Time (sec)')

y label (' the state variable $e_1(t)$ ')

figure (3), plot (t, cc_2)

$x$label (' Time (sec)')

y label (' the state x_2 and its observer ')

figure (4), plot (t, e_2)

$x$label (' Time (sec)')

y label (' the state variable $e_2(t)$ ')

MATLAB program (A4)

$\mathrm{A} = [2 \quad 0 \quad 0;0 \quad 2 \quad 0;0 \quad 3 \quad 1];$

$\mathrm{B} = [0 \quad 1;1 \quad 0;0 \quad 1];$

$C = [1 \quad 1 \quad 1;1 \quad 2 \quad 3];$

$\mathrm{M} = \mathrm{ctrb}(\mathrm{A}, \mathrm{B})$

rank (M)

$\mu = [-1+i \quad -1-i \quad -1];$

$\mathrm{K} = \mathrm{place}(\mathrm{A}, \mathrm{B}, \mu);$

$\mathrm{N} = \mathrm{obsv}(\mathrm{A}, C)$

rank (N)

$\eta = [-4 + i * 4 \quad -4 - i * 4 \quad -4];$

$L = \mathrm{place}(\mathrm{A}', C', \eta)'$

Q = 3*eye(3)

H = [0 ; 0 ; 0] ;

$G = (\mathrm{A} - LC)$;

P = care(G, H, Q);

n = eig (P) ;

no = norm (L)

$[\mathrm{A} - \mathrm{B} * \mathrm{K} \quad \mathrm{B} * \mathrm{K}; zeros(3,3) \quad \mathrm{A} - L * C]$

$\Psi = inline($`[0.101* x(1) − 1.1554* x(2) − 0.7188* x(3) + 1.8990* x(4) + 1.155 * x(5) + 0.7188* x(6) + 0.1* cos(x(1)) + 0.01* sin(x(3)) + 0.03* sin(x(2))* cos (x(2));2.7088* x(1) − 3.3822* x(2) − 1.8972* x(3) − 2.7088* x(4) + 5.3822* x (5) + 1.8972* x(6) + 0.02* cos(x(1)) + 0.06* sin(x(2))* cos(x(2)) + 0.03* x(3) * sin(t);–1.899* x(1) + 1.8446* x(2) + 0.2812* x(3) + 1.899* x(4) + 1.1554* x (5) + 0.7188* x(6) + 0.06* sin(x(3)) + 0.01* sin(x(2))* cos(x(2)) + 0.06* x(2) * sin(2* t);–4.4518* x(4) + 0.4207* x(5) + 7.2933* x(6) + 0.1* (cos(x(1)) − cos(x(1) − x(4))) + 0.01* (sin(x(3)) − sin(x(3) − x(6))) + 0.03* (sin(x(2))* cos (x(2)) − sin(x(2) − x(5))* cos(x(2) − x(5))) + 0.0266488* (cos(x(3))* sin(x(3)) − cos(x(3) − x(6))* sin(x(3) − x(6))) − 0.0343625* (sin(x(1)) − sin(x(1) − x(4)));0.9686* x(4) − 4.9715* x(5) − 14.9117* x(6) + 0.02* (cos(x(1)) − cos(x(1) − x(4))) + 0.06* (sin(x(2))* cos(x(2)) − sin(x(2) − x(5))* cos(x(2) − x(5))) + 0.03* (x(3) − (x(3) − x(6)))* sin(t) − 0.0178176* (cos(x(3))* sin(x(3)) − cos(x(3) − x(6))* sin(x(3) − x(6))) + 0.039701* (sin(x(1)) − sin(x(1) − x(4)));–0.45 * x(4) + 0.9854* x(5) − 2.5767* x(6) + 0.06* (sin(x(3) − sin(x(3) − x(6))) + 0.01 + 0.06* (sin(x(3)) − sin(x(3) − x(6))) + 0.01* (sin(x(2))* cos(x(2)) − sin(x(2) − x (5))* cos(x(2) − x(5))) + 0.06* (x(2) − (x(2) − x(5)))* sin(2* t) − 0.0022192* (cos(x(3))* sin(x(3)) − cos(x(3) − x(6))* sin(x(3) − x(6))) + 0.0078105* sin(x

$(1)) - \sin(x(1) - x(4)))]', `t', `x');$

$[t, xa] = ode45(\Psi, [0:0.01:10], [1 \quad 0 \quad -1 \quad 0 \quad 0.5 \quad -0.5]);$

$x_1 = xa(:,1);$
$x_2 = xa(:,2);$

$x_3 = xa(:,3);$
$e_1 = xa(:,4);$
$e_2 = xa(:,5);$
$e_3 = xa(:,6);$

$\hat{x}_1 = x_1 - e_1$
$\hat{x}_2 = x_2 - e_2$
$\hat{x}_3 = x_3 - e_3$

$cc_1 = [\hat{x}_1 \quad x_1]$
$cc_2 = [\hat{x}_2 \quad x_2]$
$cc_3 = [\hat{x}_3 \quad x_3]$

figure (1), plot (t, cc_1)

$x$label (' Time (sec)')

y label (' the state x_1 and its observer ')

figure (2), plot (t, e_1)

$x$label (' Time (sec)')

y label (' the state variable $e_1(t)$ ')

figure (3), plot (t, cc_2)

$x$label (' Time (sec)')

y label (' the state x_2 and its observer ')

figure (4), plot (t, e_2)

$x$label (' Time (sec)')

y label (' the state variable $e_2(t)$ ')

figure (5), plot (t, cc_3)

$x$label (' Time (sec)')

y label (' the state x_3 and its observer ')

figure (6), plot (t, e_3)

$x$label (' Time (sec)')

y label (' the state variable $e_3(t)$ ')

MATLAB program (A5)

$\mathrm{A} = [0 \quad 1; 0 \quad 0];$

$\mathrm{B} = [0; 1];$

$C = [1 \quad 0];$

$\mathrm{M} = \mathrm{ctrb}(\mathrm{A}, \mathrm{B})$

rank (M)

$\mu = [-0.7071 + i * 0.7071 \quad -0.7071 - i * 0.7071];$

$\mathrm{K} = \mathrm{place}(\mathrm{A}, \mathrm{B}, \mu)$

$\mathrm{N} = \mathrm{obsv}(\mathrm{A}, C)$

rank (N)

$\eta = [-2.8284 + i * 2.8284 \quad -2.8284 - i * 2.8284];$

$L = \mathrm{place}(\mathrm{A}', C', \eta)'$

Q = eye(2)

H = [0 ; 0] ;

$G = (\mathrm{A} - LC)$;

P = care(G, H, Q);

n = eig (P) ;

no = norm (L)

$O = [1 \quad 1];$

$\mathrm{N}_1 = \mathrm{obsv}(\mathrm{A}, O)$

rank (N_1)

$L_1 = \mathrm{place}(\mathrm{A}', O', \eta)'$

$G_1 = (\mathrm{A} - L_1 O)$;

$\mathrm{P}_1 = \mathrm{care}(G_1, H, Q)$;

$n_1 = \mathrm{eig}\ (\mathrm{P}_1)$;

$\sigma_1 = 30$; $\sigma_2 = 20$;

$\mathrm{T} = [\sigma_1 \quad 0; 0 \quad \sigma_2]$;

$\mathrm{T}^{-1} = \mathrm{inv}(\mathrm{T})$;

$\overline{G}_1 = \mathrm{T} * G_1 * \mathrm{T}^{-1}$;

$\overline{\mathrm{P}}_1 = \mathrm{care}(\overline{G}_1, H, Q)$;

$\overline{n}_1 = \mathrm{eig}(\overline{\mathrm{P}}_1)$

$[\mathrm{T} * (\mathrm{A} - \mathrm{B} * \mathrm{K}) * \mathrm{T}^{-1} \quad \mathrm{T} * \mathrm{B} * \mathrm{K} * \mathrm{T}^{-1}; zeros(2,2) \quad \mathrm{T} * (\mathrm{A} - L_1 * O) * \mathrm{T}^{-1}]$

$\Psi = inline(`[1.5 * z(2) + 30 * \sin(z(1)/30) + 0.3 * z(1) * \cos^3(t); -0.6667 * z(1) - 1.4142 * z(2) + 0.6667 * z(3) + 1.4142 * z(4) + 20 * \cos(z(1)/30) + z(2) * \sin^2(t); 10.3429 * z(3) + 17.0143 * z(4) + 30 * (\sin(z(1)/30) - \sin((z(1) - z(3))/30)) + 0.3 * (z(1) - (z(1) - z(3))) * \cos^3(t); -10.6665 * z(3) - 15.9997 * z(4) + 20 * (\cos(z(1)/30) - \cos((z(1) - z(3))/30)) + (z(2) - (z(2) - z(4))) * \sin^2(t)]', `t', `z');$

$[t, za] = ode45(\Psi, [0:0.01:100], [120 \quad 66 \quad 20 \quad -33]);$

$z_1 = za(:,1);$
$z_2 = za(:,2);$
$E_1 = za(:,3);$
$E_2 = za(:,4);$

$x_1 = z_1 / \sigma_1$
$x_2 = z_2 / \sigma_2$

$e_1 = E_1 / \sigma_1$
$e_2 = E_2 / \sigma_2$

$\hat{x}_1 = x_1 - e_1$
$\hat{x}_2 = x_2 - e_2$

$cc_1 = [\hat{x}_1 \quad x_1]$

$cc_2 = [\hat{x}_2 \quad x_2]$

figure (1), plot (t, cc_1)

$x$label (' Time (sec)')

y label (' the state x_1 and its observer ')

figure (2), plot (t, e_1)

$x$label (' Time (sec)')

y label (' the state variable $e_1(t)$ ')

figure (3), plot (t, cc_2)

$x$label (' Time (sec)')

y label (' the state x_2 and its observer ')

figure (4), plot (t, e_2)

$x$label (' Time (sec)')

y label (' the state variable $e_2(t)$ ')

This chapter presents basic concepts of modern control theory that are needed later on. The chapter is divided into five sections, the first section contains the basic definitions of control theory and some mathematical preliminaries, the second section is about the mathematical control equations, the third section concerns with Lyapunov direct method and stability of the non-linear dynamical system, the fourth section discusses the controllability and observability of dynamical control system, the last section is about the design of control system by using linear state feedback control and full order state observer.

1.1 PRELIMINARY CONCEPTS

Before discussing the modern control theory, some basic terminology's must be defined:

Definition (1.1) (System) [20]:

A combination of components that acts together and performs a certain objective is called a system.

Definition (1.2) (Disturbance) [20]:

A disturbance is a signal that tends to adversely affect the value of the output of a system. If a disturbance is generated within the system, it is called internal while an external disturbance is generated outside the system and is an input.

Definition (1.3) (State Vector) [4]:

The state of a system can be represented by a finite-dimensional column vector X called the state vector. The components of X are called the state variables.

Definition (1.4) (State Space) [20]:

The n-dimensional space whose coordinate axes consist of the x_1-axis, x_2-axis, ..., x_n- axis is called state space. Any state can be represented by a point in the state space.

Definition (1.5) (Feedback) [20]:

Feedback control refers to an operation, that in the presence of disturbances, tends to reduce the difference between the output of system and some reference input and does so on the basis of this difference.

Feedback controls do not depend explicitly on time t, but instead depend only on the state.

Definition (1.6) (Closed- Loop Control System) [20]:

Feedback control systems are often referred to as closed loop control systems. In closed-loop control system the actuating error signal, which is the difference between the input signal and the feedback signal is fed to the controller so as to reduce the error and bring the output of the system to a desired value. The term closed-loop control always implies the use of feedback control action in order to reduce system error.

Definition (1.7) (Open- Loop System) [20]:

Those systems in which the output has no effect on the control action are called open-loop control systems. In other words, in an open-loop control system the output is neither measured nor fed back for comparison with the input.

Definition (1.8) (Time–Invariant Control System) [20]:

A time-invariant control system (constant coefficient control system) is that one whose parameters do not vary with time. The response of such a system is independent of the time at which an input is applied.

Definition (1.9) (Time- Varying Control System) [20]:

A time–varying control system is a system in which one or more of its parameters vary with time.

Definition (1.10) (Observation) [21]:

Estimation of unmeasurable state variables is commonly called observation.

Definition (1.11) (Positive Definite Matrix) [6]:

A real symmetric matrix A is called positive definite if $x^{\mathrm{T}} \mathrm{A}\, x > 0$ for every nonzero vector $x \in R^n$.

Lemma (1.1) [10]:

A symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive.

Definition (1.12) (Negative Definite Matrix) [6]:

An $n \times n$ real symmetric matrix A is negative definite if the determinant of A is positive if n is even and negative if n is odd, and the successive principal minors of even order be positive and the successive principal minors of odd order be negative, i.e.,

$$a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \quad \cdots$$

$\det(\mathrm{A}) > 0$ if n is even, $(a_{ij} = a_{ji})$

$\det(A) < 0$ if n is odd.

Definition (1.13) (Positive Definite Function) [20]:

A scalar function V(x) is said to be positive definite in a region Ω (which includes the origin of the state space) if V(x) > 0 for all nonzero states x in the region Ω and V(0) = 0.

Definition (1.14) (Negative Definite Function) [20]:

A scalar function V(x) is said to be negative definite if –V(x) is positive definite.

1.2 MATHEMATICAL CONTROL EQUATIONS [4]:

The set of n-first order differential equations that describes the unique relations between the input (control), output and state is called dynamical control equation.

In this work, some non-linear dynamical control systems have been discussed:

$$\dot{x}(t) = \mathrm{F}(x(t), u(t), t) \qquad \text{(state equation)} \qquad (1.1a)$$

$$y(t) = \mathrm{G}(x(t), u(t), t) \qquad \text{(output equation)} \qquad (1.1b)$$

or, more explicitly,

$$\left.\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t)\\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t)\\
&\vdots\\
\dot{x}_n(t) &= f_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t)
\end{aligned}\right\} \qquad (1.2\text{ a})$$

$$\left.\begin{array}{l} y_1(t) = g_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t) \\ y_2(t) = g_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t) \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad . \\ y_m(t) = g_m(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_p(t), t) \end{array}\right\} \quad (1.2b)$$

where $x = [x_1 \;\; x_2 \ldots x_n]^{\mathrm{T}}$ is an n-dimensional state vector, $x_1, x_2, \ldots, x_n$ are the state variables, $y = [y_1 \quad y_2 \ldots y_m]^{\mathrm{T}}$ is an m-dimensional output vector, $y_1, y_2, \ldots, y_m$ are output variables, $u = [u_1 \quad u_2 \ldots u_p]^{\mathrm{T}}$ is an p-dimensional input (control) vector and $u_1, u_2, \ldots, u_p$ are input (control) variables. The dynamical control system is specified by n-dimensional vector-valued function F and m-dimensional vector-valued function G.

The control u, the output y and the state *x* are real-valued functions of t defined over the real line R.

Equation (1.1) can be represented in the block diagram shown in figure (1.1). In this figure, the flow of a vector quantity is represented by double-line arrow.

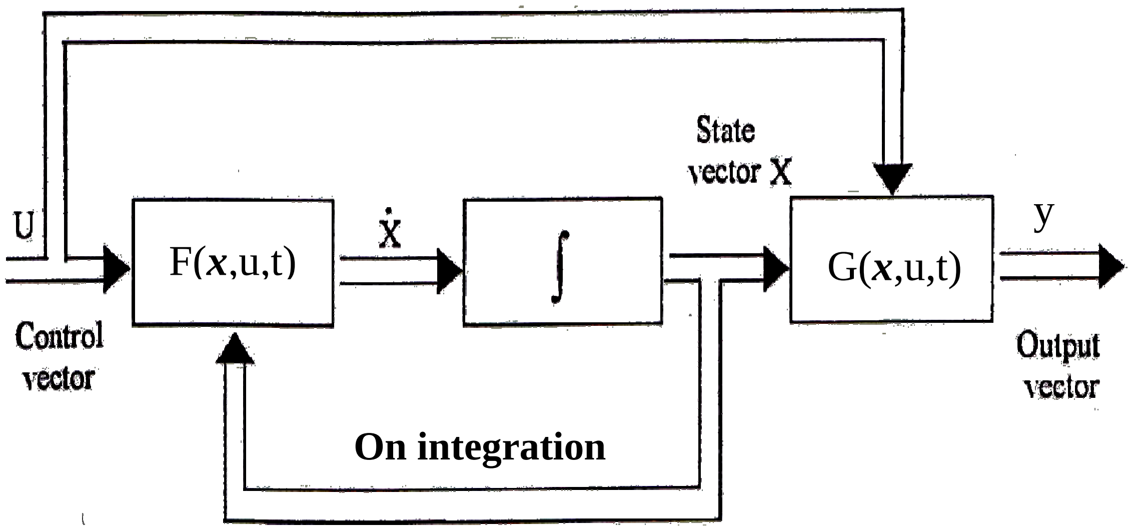


Figure (1.1) The block diagram of a system represented by equation (1.1a) and (1.1 b) in vector representation.

If a vector-valued function F and/or G in (1.1a) and (1.1b) involve time explicitly, then the system is called a time-varying control system, otherwise, the system is called a time-invarying control system.

1.2.1 Linear Dynamical Control System [16]:

Consider the following linear dynamical system

$$\mathrm{F}(x(t),u(t),t) \equiv \mathrm{A}(t)x(t)+\mathrm{B}(t)u(t)$$

and

$$\mathrm{G}(x(t),u(t),t) \equiv C(t)x(t)+D(t)u(t)$$

where A, B, C and D are , n×n, n×p, m×n and m×p matrices respectively. Hence, an n-dimensional linear dynamical control equation is of the form:

$$\dot{x}(t)=\mathrm{A}(t)x(t)+\mathrm{B}(t)u(t) \qquad \text{(state equation)} \qquad (1.3a)$$

$$y(t)=C(t)x(t)+D(t)u(t) \qquad \text{(output equation)} \qquad (1.3b)$$

since the values of $\mathrm{A}(\cdot)$, $\mathrm{B}(\cdot)$,$\mathrm{C}(\cdot)$ and $\mathrm{D}(\cdot)$ changewith time, the dynamical control equation in (1.3) is more suggestively called a linear time varying dynamical control equation.

For linear time – invariant dynamical control equation, the matrices $\mathrm{A}(\cdot)$, $\mathrm{B}(\cdot)$, $\mathrm{C}(\cdot)$, and $\mathrm{D}(\cdot)$ are independent of timeand the equation reduces to:

$$\dot{x}(t)=\mathrm{A}\,x(t)+\mathrm{B}u(t) \qquad (1.4a)$$

$$y(t)=C\,x(t)+Du(t) \qquad (1.4b)$$

1.3 Lyapunov Stability [6]:

We present here the second method of Lyapunov stability analysis, which is applicable to both linear and nonlinear systems and provides stability information on linear and nonlinear differential equations without solving the system explicitly, hence the second method is called the direct method of Lyapunov, the direct method is most useful for investigating stability of non-linear systems. It gives sufficient conditions for asymptotic stability of equilibrium states of nonlinear systems and linear time-invariant systems.

Definition (1.15) (Equilibrium States) [19]:

Consider the system $\dot{x} = f(x,t)$, a state x_e where $f(x_e,t) = 0, \forall t$ is called an equilibrium state of the system.

Definition (1.16)(Lyapunov Stability) [6]:

An equilibrium state x_e of the dynamical system $\dot{x} = f(x,t)$ is stable (or stable in the sense of Lyapunov) if for every $\varepsilon > 0$, there exists $\delta > 0$ $(\delta(\varepsilon, t_0))$ such that

$\|x_0 - x_e\| < \delta$ implies $\|x(t, x_0) - x_e\| \le \varepsilon$, for all $t \ge t_0$

where $\|.\|$ denotes the Euclidean norm of a vector.

Definition (1.17) (Asymptotic Stability) [6]:

An equilibrium state x_e of the system $\dot{x} = f(x,t)$ is asymptotically stable if

1- It is stable in the sense of Lyapunov.

2- For all t_0 there exists a $\rho(t_0) > 0$ (possibly depending on t_0) such that

$$\|x_0 - x_e\| < \rho \text{ implies that } \|x(t, x_0) - x_e\| \to 0 \text{ as } t \to \infty.$$

Remarks (1.1)

1- An equilibrium state x_e is said to be unstable if it is not stable, ***[19]***.

2- An equilibrium state x_e of a free dynamical system is unstable if there exists an ε such that no δ can be found to satisfy the conditions of definition of stability, ***[5]***.

3-The system described by equation

$$\dot{x}(t) = \mathrm{A}x(t) \tag{1.5}$$

where A is a constant matrix, is asymptotically stable if and only if all eigenvalues of A have negative real parts, ***[27]***.

Theorem (1.1) [14]:

The time-invariant linear system (1.5):

$$\dot{x}(t) = \mathrm{A}\,x(t)$$

is stable in the sense of Lyapunov if and only if:

1. All of the characteristic values of A has non-positive real parts, and,

2. To any characteristic value on the imaginary axis with multiplicity m there correspond exactly m characteristic vectors of the matrix A.

Definition (1.18) (Asymptotically Stable Matrix) [14]:

The $n \times n$ constant matrix A is asymptotically stable if all its characteristic values have strictly negative real parts. The characteristic values of A are the roots of the characteristic polynomial $\det(\lambda \mathrm{I} - \mathrm{A})$.

1.3.1 The Direct Method of Lyapunov [20]:

The second method of Lyapunov attempts to give information on the stability of equilibrium state of linear and nonlinear systems without any prior knowledge of their solutions.

The second method of Lyapunov is based on generalization of the idea that if the system has an asymptotically stable equilibrium state, then the stored energy of the system displaced within the domain of attraction decays with increasing time until it finally assumes its minimum value at the equilibrium state. The method consists of determination of a fictitious (energy) function called the Lyapunov function which is more general than that of energy and is more widely applicable.

Definition (1.19) (Attraction Domain) [20]:

The largest region of asymptotic stability is called domain of attraction. It is a part of the state space in which asymptotically stable trajectories originate.

Definition (1.20) (Quadratic Form) [20]:

A class of scalar function that plays an important role in the stability analysis, based on the second method of Lyapunov is the quadratic form. An example is

$$V(x) = x^{\mathrm{T}}\mathrm{P}x = [x_1 \ \ x_2 \cdots x_n] \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \dots & \mathrm{P}_{1n} \\ \mathrm{P}_{12} & \mathrm{P}_{22} & \dots & \mathrm{P}_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \mathrm{P}_{1n} & \mathrm{P}_{2n} & \dots & \mathrm{P}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Note that *x* is a real vector and P is a real symmetric matrix.

Definition (1.21) (Hermitian Form) [20]:

If x is a complex *n*-vector and P is a Hermitian matrix, then the complex quadratic form is called the Hermitian form. An example is

$$V(x) = x^{*}\mathrm{P}x = [\bar{x}_1 \ \ \bar{x}_2 \cdots \bar{x}_n] \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \dots & \mathrm{P}_{1n} \\ \bar{\mathrm{P}}_{12} & \mathrm{P}_{22} & \dots & \mathrm{P}_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \bar{\mathrm{P}}_{1n} & \bar{\mathrm{P}}_{2n} & \dots & \mathrm{P}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Note that $\bar{\mathrm{P}}_{ij}$ is the complex conjugate of P_{ij}. For the quadratic form, $\bar{\mathrm{P}}_{ij} = \mathrm{P}_{ij}$.

Theorem (1.2) [20]:

Suppose that a system is described by

$$\dot{x} = \mathrm{f}(x, \mathrm{t}) \tag{1.6}$$

where f(0,t) = 0, for all t

If there exists a scalar function V(x,t) having continuous first partial derivatives and satisfying the following conditions,

1- V(x,t) is positive definite

2- $\dot{\mathrm{V}}(x,\mathrm{t})$ is negative definite

Then the equilibrium state at the origin is uniformly asymptotically stable.

If, in addition, $\mathrm{V}(x,t) \to \infty$ as $\|x\| \to \infty$, then the equilibrium state at the origin is uniformly asymptotically stable in the large.

Remark (1.2) [20]:

If an equilibrium state x = 0 of a system (1.6) is unstable, then there exists a scalar function W(x,t) which determines the instability of the equilibrium state. We shall present a theorem on instability in the following.

Theorem (1.3) [20]:

Suppose a system is described by

$$\dot{x} = \mathrm{f}(x,t)$$

where f(0,t) = 0, for all t ≥ t_o

If there exists a scalar function W(x,t) having continuous first partial derivatives and satisfying the following conditions,

1- $\mathrm{W}(x,t)$ is positive definite in some region about the origin.

2- $\dot{\mathrm{W}}(x,t)$ is positive definite in the same region.

Then the equilibrium state at the origin is unstable.

1.3.2 Lyapunov Stability Analysis of Linear Time–Invariant Systems [19]

Consider the following linear time – invariant system:

$$\dot{x} = \mathrm{A}\,x$$

where x is a state vector (n-vector) and A is n×n constant matrix. We assume that A is non singular. Then the only equilibrium state is the origin $x = 0$. The stability of the equilibrium state of the linear time – invariant system can be investigated easily by use of the second method of Lyapunov. For the system defined by equation (1.5), let us choose a possible Lyapunov function as

$$V(x) = x^* Px$$

where P is a positive – definite Hermitian matrix (if x is a real vector and A is a real matrix, then P can be chosen to be a positive–definite real symmetric matrix). The time derivative of V(x) along any trajectory is

$$\begin{aligned}\dot{V}(x) &= \dot{x}^* Px + x^* P\dot{x} \\ &= (Ax)^* Px + x^* PAx \\ &= x^* A^* Px + x^* PAx \\ &= x^*(A^*P + PA)x\end{aligned}$$

Since V(x) was chosen to be positive definite, we require, for asymptotic stability, that $\dot{V}(x)$ be negative definite. Therefore, we require that:

$$\dot{V}(x) = -x^* Qx$$

where

$$Q = -(A^*P + PA) \text{ positive definite}$$

Hence, for the asymptotic stability of the system of equation (1.5) it is sufficient that Q be positive definite. For a test of positive definiteness of an $n \times n$ matrix, we apply Sylvester's criterion, which states that a necessary and sufficient condition that the matrix be positive definite is that the determinants of all the successive principal minors of the matrix be positive.

Instead of first specifying a positive – definite matrix P and examining whether Q is positive definite, it is convenient to specify a positive – definite matrix Q first and then examine whether P determined from

$$A^{*}P + PA = -Q$$

is positive definite. Note that P being positive definite is a necessary and sufficient condition.

We shall summarize what we have just stated in the form of a theorem.

Theorem (1.4) [19]:

Consider the system described by

$$\dot{x} = A\,x$$

where x is a state vector (n - vector) and A is an $n \times n$ constant nonsingular matrix. A necessary and sufficient condition that the equilibrium state $x = 0$ be asymptotically stable in the large is that, given any positive–definite Hermitian (or real symmetric) matrix Q, there exists a positive–definite Hermitian (or real symmetric) matrix P such that

$$A^{*}P + PA = -Q$$

The scalar function $x^{*}Px$ is a Lyapunov function for this system. (Note that in the linear system considered, if the equilibrium state (the origin) is asymptotically stable, then it is asymptotically stable in the large).

In applying this theorem, several important remarks are in order.

Remarks (1.3) [19]:

1- If the system involves only real state vector x and real state matrix A, then the Lyapunov function $x^{*}Px$ becomes $x^{T}Px$ and the Lyapunov equation becomes

$$A^{T}P + PA = -Q$$

2- To determine the elements of the P matrix, we equate the matrices $A^{*}P + PA$ and –Q element by element. This results in $n(n+1)/2$ linear equations for the determination of the elements of $p_{ij} = \overline{p}_{ji}$ of P. If we denote the eigenvalues of A by $\lambda_1, \lambda_2, \ldots, \lambda_n$ each repeated as often as its multiplicity as a root of the characteristic equation, and if for every sum of two roots:

$$\lambda_j + \lambda_k \neq 0$$

Then the elements of P are uniquely determined. Note that if the matrix A represents a stable system, then the sums $\lambda_j + \lambda_k$ are always non zero.

3- In determining whether there exists a positive – definite Hermitian or a real symmetric matrix P, it is convenient to choose Q = I, where I is the identity matrix. Then the elements of P are determined from

$$A^{*}P + PA = -I$$

and the matrix P is tested for positive definiteness.

1.4 CONTROLLABILITY AND OBSERVABILITY [28]:

There are two basic problems we need to consider. The first one is the coupling between the input and the state or to control the state by using the information about the input. This is a controllability problem. Another problem is the relationship between the state and the output, i.e., the information about the state can be observed from the output. This is an observability problem. The concept of observability is dual to that of controllability. Roughly speaking, controllability studies the possibility of steering the state from the input; observability studies the possibility of estimating the state from the output. If a dynamical equation is controllable, all the modes of the equation can be excited from the input; if a dynamical equation is observable, all the modes of the equation can be observed at the output. These two concepts are defined under the assumption that we have the complete knowledge of a dynamical equation.

Definition (1.22) (Controllable System) [28]:

A system is said to be controllable at time t_0 if it is possible to find an unconstrained control vector to transfer any initial state to the origin in a finite time interval.

Stated mathematically, the system is controllable at t_0 if for any $x(t_0)$, there exists $u_{[t_0,t_1]}$ that gives $x(t_1) = 0\,(t_1 > t_0)\cdot$

If this true for all initial time t_0 and all initial states $x(t_0)$, the system is completely controllable.

Theorem (1.5) [4]:

The n-dimensional linear time-invariant state equation (1.4a)

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

is completely state controllable if and only if the n×(np) matrix

M = [B: AB: A^2B:…:A^{n-1}B] has rank n.

Definition (1.23) (Observable System) [28]:

A system is said to be observable at time t_0 if it is possible to determine the state $x(t_0)$ from the output function over a finite time interval.

Mathematically, the system is observable at time t_0 if any $x(t_0)$ can be estimated by the observation of $y_{[t_0,t_1]}(t_1 > t_0)$.

If this true for all time t_0 and all states $x(t_0)$, the system is completely observable.

Theorem (1.6) [20]:

The n-dimensional linear time-invariant dynamical control equation (1.4)

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

$$y(t) = C\,x(t) + Du(t)$$

is completely state observer if and only if the n×(nm) observability matrix

N = [C*: A*C*:…: A* $^{n-1}$ C*]

has rank n (* denoted to conjugate transpose matrix).

Theorem (1.7) [20]:

Consider the dynamical control equation

$$\dot{x} = \mathrm{A}x + \mathrm{B}u\,, \quad y = Cx + Du \tag{1.7}$$

and the dynamical control equation defined by:

$$\dot{z} = \mathrm{A}^{*}z + C^{*}v\,, \quad n = \mathrm{B}^{*}z + D^{*}v \tag{1.8}$$

where A^*, B^*, C^* and D^* are the complex conjugate transposes of A, B, C and D respectively. Then equation (1.7) is completely state controllable (observable) if and only if the equation (1.8) is completely state observable (controllable).

1.5 POLE PLACEMENT AND FULL ORDER STATE OBSERVER

The pole placement approach requires the feedback of all state variables. Therefore, it becomes necessary that all state variables are assumed to be available for measurement as outputs. However, some state variables may be unmeasurable and may not be available for feedback. Then, we need to estimate such unmeasurable state variables by using state observers.

1.5.1Design of Linear Dynamical Control System Via Pole Placement[20]:

Consider a linear state dynamical control equation:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \tag{1.9}$$

where A and B are, respectively, constant matrices with an appropriate dimensions. The control $u(t)$ shall be assumed

$$u(t) = -\mathrm{K}x(t) \tag{1.10}$$

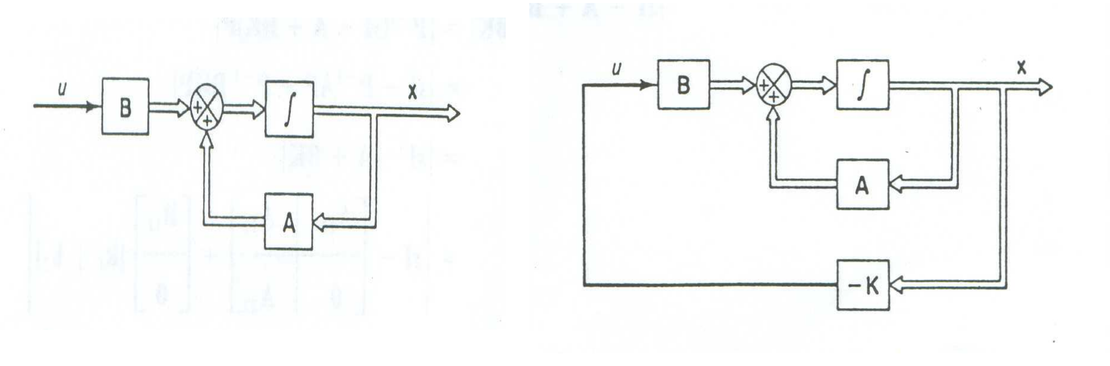
where K is constant matrix with an appropriate dimension. Substituting equation (1.10) into equation (1.9) gives the closed loop

$$\dot{x}(t) = (\mathrm{A} - \mathrm{BK})x(t) \tag{1.11}$$

the solution of this equation (1.11) is given by:

$$x(t) = e^{(\mathrm{A}-\mathrm{BK})t}x(0) \tag{1.12}$$

where $x(0)$ is the initial state (may be caused by external disturbances). The stability and transient response characteristic are determined by the eigenvalues of matrix $(\mathrm{A} - \mathrm{BK})$. If the matrix K is chosen properly, then the matrix $(\mathrm{A} - \mathrm{BK})$ can be made as asymptotically stable matrix, and for all $x(0) \neq 0$ it is possible to make $x(t)$ approach 0 as t approaches infinity. The eigenvalues of matrix $(\mathrm{A} - \mathrm{BK})$ are sometimes called the regulator poles. The problem of placing the closed- loop poles at the desired location is called a ***Pole Placement*** problem. Figure (1.2) (a) shows the system defined by equation (1.9). It is an open- loop control system, because the state x is not fed back to the control u. Figure (1.2) (b) shows the system with linear state feedback control. This is closed-loop control system, because the state x is fed back to the control u.



(a) **(b)**

Figure (1.2)

(a) Open-loop control system

(b) Closed-loop control system with feedback control $u = -\mathrm{K}x$.

Theorem (1.8) [13]:

Consider the linear state time invariant dynamical control state equation (1.9):

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

with linear state feedback control (1.10)

$$u(t) = -\mathrm{K}x(t)$$

Then, the closed-loop characteristic values (regulators poles), that is ,the characteristic values of $(\mathrm{A} - \mathrm{BK})$, can be arbitrarily located in the complex plane (with the restriction that complex characteristic values occur in complex conjugate pairs) by choosing K suitably if and only if the system (1.9) is completely state controllable.

Algorithm (1.1):'' pole placement design, single variable case'' [20]:

Consider the single variable time invariant equation:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

where $\mathrm{A} \in R^{n\times n}$ and $\mathrm{B} \in R^{n\times 1}$ and linear state feedback $u(t) = -\mathrm{K}x(t)$ where $\mathrm{K} \in R^{1\times n}$

Step (1): Check the controllability condition for the system. If the system is completely state controllable, i.e.,(A,B) is controllable, then use the following steps.

Step (2): From the characteristic polynomial for matrix A,

$$|\lambda \mathrm{I} - \mathrm{A}| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$$

and then determine the values of $a_1, a_2, \ldots, a_n$

Step (3): Determine the transformation matrix T that transforms the system state equation into the controllable canonical form see the following remarks (1.4) (If the given system equation is already in the controllable canonical form, then $\mathrm{T} = \mathrm{I}$). It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the transformation matrix T which is given by

$$\mathrm{T} = \mathrm{MW} \tag{1.13}$$

where M is the controllability matrix

$$\mathrm{M} = [\mathrm{B} \vdots \mathrm{AB} \vdots \mathrm{A}^2\mathrm{B} \vdots \ \ldots \ \vdots \mathrm{A}^{n-1}\mathrm{B}] \tag{1.14}$$

and W is defined by

$$\mathrm{W} = \begin{pmatrix} a_{n-1} & a_{n-2} & \ldots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{1.15}$$

where the a_i's are coefficients of the characteristic polynomial of step (2).

Step (4): Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial as:

$$(\lambda - \mu_1)(\lambda - \mu_2)\ldots(\lambda - \mu_n) \equiv \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n$$

where the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be determined.

Step (5): The required state feedback gain matrix K can be determined from the following equation

$$\mathrm{K} = [\alpha_n - a_n \vdots \alpha_{n-1} - a_{n-1} \vdots \ \ldots \ \vdots \alpha_2 - a_2 \vdots \alpha_1 - a_1]\mathrm{T}^{-1} \tag{1.16}$$

Remarks (1.4) [20]:

1- The matrix A is said to be in a controllable canonical form if it is can be written as:

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{pmatrix}$$

where the coefficients a_i, $i = 1,2,\ldots,n$ are computed by

$$|\lambda \mathrm{I} - \mathrm{A}| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$$

2- The discussed method of subsection (1.5.1) is called Pole Placement method. There are also different methods like Ackermann's formula, see for information in ***[20]***.

3- Note that if the system is of low order $n \le 3$, then direct substitution of a matrix K into the desired characteristic polynomial may be simpler. For example, if $n = 3$, then write the state feedback gain matrix K as

$$\mathrm{K} = [k_1 \quad k_2 \quad k_3]$$

Substitute K into the desired characteristic polynomial $|\lambda \mathrm{I} - \mathrm{A} + \mathrm{BK}|$ and equate it to $(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)$, or

$$|\lambda \mathrm{I} - \mathrm{A} + \mathrm{BK}| = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)$$

Since both sides of this characteristic equation are polynomials in λ, then by equating the coefficients of the same powers of λ on both sides, it is possible to determine the values of k_1, k_2, and k_3. This approach is convenient if $n = 2$ or 3. (For $n = 4,5,6,\ldots$, this approach may become very tedious).

Algorithm (1.2) "pole placement design multivariable case" [4]:

Consider the n-dimensional linear time invariant multivariable state equation

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

where $\mathrm{A} \in R^{n \times n}$ and $\mathrm{B} \in R^{n \times p}$, and linear state feedback $u(t) = -\mathrm{K}x(t)$, where $\mathrm{K} \in R^{p \times n}$.

Step (1): Check the controllability condition if the system is completely state controllable, then we use the following steps.

Step (2): Choose an arbitrary $n \times n$ matrix F, which has no eigenvalue common with those of A .

Step (3): Choose an arbitrary $p \times n$ matrix k such that (F, k) is completely state observer.

Step (4): Solve the unique T in the matrix equation:

$$\mathrm{A\,T} - \mathrm{T\,F} = \mathrm{B\,k}. \tag{1.17}$$

Step (5): If T is nonsingular, then we have K = $\mathrm{k\,T}^{-1}$ and A-BK has the same eigenvalues as those of F.

If T is singular, then choose a different F or a different k and repeat the steps.

Remark (1.5) [4]:

If T is nonsingular, the equation matrix AT- TF = Bk. Implies that

$$\text{A-BK} = \mathrm{TF\,T}^{-1} \tag{1.18}$$

Hence A-BK and F are similar and have the same set of eigenvalues.

1.5.2 States Observers [20]:

In the pole placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables. A device (or a computer program) that estimates or observes the state variables is called a state observer, or simply an observer. If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a full-order state observer.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a reduced-order state observer or, simply, a reduced-order observer. If the order of the reduced-order state observer is the minimum possible, the observer is called a minimum-order state observer or minimum-order observer.

In this work, the full-dimensional state observer has been discussed.

1.5.3 Full-Dimension Linear State Observer [4]:

Consider the n – dimensional linear time invariant dynamical equation

$$\dot{x}(t) = \mathrm{A}\,x(t) + \mathrm{B}\,u(t) \tag{1.19 a}$$

$$y(t) = C\,x(t) \tag{1.19 b}$$

where A, B and C are respectively n×n, n×p and m×n real constant matrices. We assume now that the state variables are not accessible. Note that although the state variables are not accessible, the matrices A, B and C are assumed to be completely known. Hence the problem is that of estimating or generating $x(t)$ from the available input u and the output y with the knowledge of the matrices A , B and C.

Consider the state observer shown in figure (1.3)

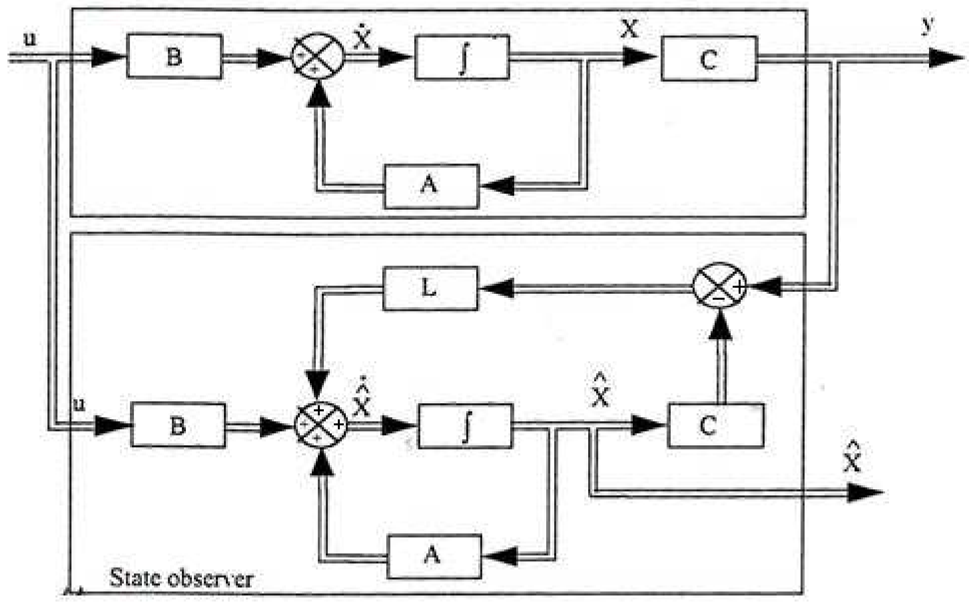


Figure (1.3) a full – order observer state

The observer is driven by the input as well as the output of the original system. The output of (1.19 b), $y = C\,x$, is compared with $\hat{y} = C\,\hat{x}$ and their difference is used to serve as a correcting term. The difference $y - C\hat{x}$, is multiplied by n×m real constant matrix L and fed into the input of the integrators of the observer. This observer is called linear full–order state observer.

The linear dynamical control equation of the full – order observer shown in figure (1.3) is given by:

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + L(y(t) - C\,\hat{x}) \tag{1.20}$$

where $\hat{x}$ is the state observer.

$$\dot{\hat{x}}(t) = (A - L\,C\,)\,\hat{x}(t) + B\,u + L\,y(t) \tag{1.21}$$

Define

$$e(t) = x(t) - \hat{x}(t) \tag{1.22}$$

Clearly $e(t)$ is the dynamical error between the actual state and the state observer.

Subtracting (1.20) from (1.19 a), we obtain:

$$\dot{e}(t) = (A - L\,C\,)\,e(t) \tag{1.23}$$

If the eigenvalues of $(A - LC)$ can be chosen arbitrarily, then the behavior of the error $e(t)$ can be controlled. For example, if all the eigenvalues of $(A - LC)$ have negative real parts smaller than -σ, then all the elements of $e(t)$ will approach zero at rates faster than $e^{-\sigma t}$. Consequently, even if there is a large error between $\hat{x}(t_0)$ and $x(t_0)$ at initial time t_0, the vector $\hat{x}$ will approach *x* rapidly.

<u>Theorem (1.9) [15]:</u>

Consider the linear time – invariant full order state observer (1.20):

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + L[y(t) - C\,\hat{x}(t)]$$

For the linear time invariant dynamical control equations

$$\left.\begin{aligned} \dot{x}(t) &= \mathrm{A}\,x(t) + \mathrm{B}\,u(t) \\ y(t) &= C\,x(t) \end{aligned}\right\} \tag{1.24}$$

Then , the observer poles , that is , the characteristic values of $(\mathrm{A} - LC)$, can be arbitrarily located in the complex plane (with the restriction that the complex characteristic values occur in complex conjugate Pairs), by choosing the constant matrix L suitably, if and only if the system (1.24) is completely state observer .

1.5.4 Design Steps for Full Order Observer [21]:

Consider the linear time invariant dynamical control system defined by (1.24). In designing the full – order state observer, we may solve the dual problem, that is, solve the pole placement problem for the dual system

$$\left.\begin{aligned} \dot{z} &= \mathrm{A}^{*} z + C^{*} v \\ n &= \mathrm{B}^{*} z \end{aligned}\right\} \tag{1.25}$$

where A^*, B^* and C^* are the transpose conjugate of A, B and C respectively. Assume the control v to be:

$$v = -\mathrm{K}\,\mathrm{z} \tag{1.26}$$

If the dual system is completely state controllable, then the state feedback gain matrix K can be determined such that matrix $\mathrm{A}^* - C^*\mathrm{K}$ will yield a set of the desired eigenvalues. Noting that the eigenvalues of $\mathrm{A}^* - C^*\mathrm{K}$ and those of $\mathrm{A} - \mathrm{K}^* C$ are the same, we have:

$$\left| s\,\mathrm{I} - (\mathrm{A}^* - C^*\mathrm{K}) \right| = \left| s\,\mathrm{I} - (\mathrm{A} - \mathrm{K}^* C) \right| \tag{1.27}$$

Comparing the characteristic polynomial $\left| s\mathrm{I} - (\mathrm{A} - \mathrm{K}^* C) \right|$ and the characteristic Polynomial $\left| s\mathrm{I} - (\mathrm{A} - LC) \right|$ for the observer system, we find that L and K^* are related by

$$L = \mathrm{K}^* \tag{1.28}$$

Thus, using the matrix K determined by the pole placement approach in the dual system, the observer gain matrix L for the original system can be determined by using the relationship $L = \mathrm{K}^*$.

In this chapter, the problem of designing dynamical state observer for inherently non-linear system is considered. A sufficient conditions to design a state observer of non-linear system are developed.

A computational algorithm based on theorem for design a deterministic state dynamic observer has been developed and presented. An approximate procedure is proposed to design approximate observer based controllers.

Finally, several problems are given to demonstrate the validity of our results.

3.1 SUFFICIENT CONDITIONS FOR DESIGN A STATE OBSERVER OF NON-LINEAR SYSTEM

The sufficient conditions to select observer gain matrix L that the state observer $\hat{x}(t)$ for inherently non-linear dynamical control system will converge to the actual state of non-linear dynamical control system are obtained in the following theorem:

Theorem (3.1)

Consider the non-linear dynamical system

$$\begin{aligned} \frac{dx(t)}{dt} &= \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t),t) \\ y(t) &= C\,x(t) + h(x(t)) \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

and assume that the state variables are not available for measurement. Consider the observer of non-linear dynamical control system (3.1) is

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &= \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - (C\,\hat{x}(t) + h(\hat{x}(t)))) \\ \hat{y}(t) &= C\,\hat{x}(t) + h(\hat{x}(t)) \\ \hat{x}(0) &= \hat{x}_0 \end{aligned} \tag{3.2}$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$, $\mathrm{A} \in R^{n \times n}$, $\mathrm{B} \in R^{n \times p}$, $D \in R^{n \times n}$, $C \in R^{m \times n}$, $f : R^n \to R^n$, $g : R^n \times R \to R^n$, $h : R^n \to R^m$, the following conditions are assumed to be satisfied

1. The pair (A, C) of a non-linear dynamical control system (3.1), is completely state observer.
2. The non-linearity function $Df(x(t)) : R^n \to R^n$ is assumed to be globally Lipschitz condition with Lipschitz constant γ, i.e.,

   $$\|Df(x(t)) - Df(\hat{x}(t))\| \le \gamma \|x(t) - \hat{x}(t)\| \tag{3.3}$$

3. The non-linearity function $g(x(t), t) : R^n \times R \to R^n$ is assumed to be globally Lipschitz condition with Lipschitz constant β, i.e.,

   $$\|g(x(t), t) - g(\hat{x}(t), t)\| \le \beta \|x(t) - \hat{x}(t)\| \text{ for } t \in R \tag{3.4}$$

4. The non-linearity function $h(x(t)) : R^n \to R^m$ is assumed to be globally Lipschitz condition with Lipschitz constant δ, i.e.,

   $$\|h(x(t)) - h(\hat{x}(t))\| \le \delta \|x(t) - \hat{x}(t)\| \tag{3.5}$$

5. The observer gain L can be selected such that $(\mathrm{A} - LC)$ is asymptotically stable matrix.
6. The Riccati equation $(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$ (3.6)

   has a unique positive definite solution P for arbitrary positive definite selection matrix Q.
7. On using the Lyapunov function stability $V(e(t)) = e^{\mathrm{T}}(t) P e(t)$, where

   $$e(t) = x(t) - \hat{x}(t) \tag{3.7}$$

   and P satisfy equation (3.6)

If

$$\gamma + \beta + \delta \|L\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \tag{3.8}$$

Then the dynamical error

$$\frac{de(t)}{dt} = (A - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t) - L(h(x(t)) - h(\hat{x}(t))) \tag{3.9}$$

is asymptotically stable via a single observer gain parameter L.

Proof

From the non-linear dynamical system (3.1) and state observer (3.2) as well as $e(t) = x(t) - \hat{x}(t)$, one can get the following

$$\frac{de(t)}{dt} = (A - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t) - L(h(x(t)) - h(\hat{x}(t)))$$

with $e(0) = x(0) - \hat{x}(0)$

On using the Lyapunov function and its derivative

$$V(t) \equiv V(e(t)) = e^{T}(t)P\,e(t)$$

$$\frac{dV(t)}{dt} = \dot{e}^{T}(t)P\,e(t) + e^{T}(t)P\,\dot{e}(t)$$

From (3.9), we have that

$$\frac{dV(t)}{dt} = e^{T}(t)[(A - LC)^{T}P + P(A - LC)]e(t) + (Df(x(t)) - Df(\hat{x}(t)))^{T}Pe(t) + e^{T}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{T}Pe(t) + e^{T}(t)P(g(x(t),t) - g(\hat{x}(t),t)) + (-L(h(x(t)) - h(\hat{x}(t))))^{T}Pe(t) + e^{T}(t)P(-L(h(x(t)) - h(\hat{x}(t)))) \tag{3.10}$$

From (3.6) we have

$$\frac{dV(t)}{dt} = e^{T}(t)(-Q)e(t) + (Df(x(t)) - Df(\hat{x}(t)))^{T}P\,e(t) + e^{T}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{T}P\,e(t) + e^{T}(t)P(g(x(t),t) - g(\hat{x}(t),t)) + (-L(h(x(t)) - h(\hat{x}(t))))^{T}Pe(t) + e^{T}(t)P(-L(h(x(t)) - h(\hat{x}(t)))) \tag{3.11}$$

From (3.3) and (3.7) one deduces.

$$(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}} Pe(t) \le \gamma \|e(t)\|^2 \lambda_{\max}(P) \tag{3.12}$$

$$e^{\mathrm{T}}(t) P(Df(x(t)) - Df(\hat{x}(t))) \le \gamma \|e(t)\|^2 \lambda_{\max}(P) \tag{3.13}$$

From (3.4) and (3.7) one deduces.

$$(g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}} Pe(t) \le \beta \|e(t)\|^2 \lambda_{\max}(P) \tag{3.14}$$

$$e^{\mathrm{T}}(t) P(g(x(t),t) - g(\hat{x}(t),t)) \le \beta \|e(t)\|^2 \lambda_{\max}(P) \tag{3.15}$$

From (3.5) and (3.7) one deduces.

$$(-L(h(x(t)) - h(\hat{x}(t))))^{\mathrm{T}} Pe(t) \le \delta \|L\| \|e(t)\|^2 \lambda_{\max}(P) \tag{3.16}$$

$$e^{\mathrm{T}}(t) P(-L(h(x(t)) - h(\hat{x}(t)))) \le \delta \|L\| \|e(t)\|^2 \lambda_{\max}(P) \tag{3.17}$$

where $\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

Substituting (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17) into (3.11) gives:

$$\frac{dV(t)}{dt} \le e^{\mathrm{T}}(t)(-Q)e(t) + 2(\gamma + \beta + \delta \|L\|) \|e(t)\|^2 \lambda_{\max}(P) \tag{3.18}$$

with $e^{\mathrm{T}}(t)\, Q\, e(t) \ge \lambda_{\min}(Q) \|e(t)\|^2$ (3.19)

One deduces from (3.19):

$$\frac{dV(t)}{dt} \le \left(-\lambda_{\min}(Q) + 2(\gamma + \beta + \delta \|L\|)\lambda_{\max}(P)\right) \|e(t)\|^2 \tag{3.20}$$

From (3.20) and (3.8) we have

$$\dot{V}(e(t)) < 0 \tag{3.21}$$

since P is unique positive definite solution and it is clear that $V(e(t)) = e^{\mathrm{T}}(t) P e(t) > 0$, $V(0) = 0$ and by (3.21) we have conclude that the error dynamic system (3.9) is asymptotically stable via a single observer gain parameter L. Thus $x(t) \cong \hat{x}(t)$ as $t \to \infty$. And this complete the proof.

Remarks (3.1)

1. The problem tackled in this theorem can be stated as follows:
   Find the condition on the observer gain matrix L that the dynamic behavior of the dynamical error (3.9) is asymptotically stable or the unmeasurable actual state $x(t)$ of non-linear dynamical control system (3.1) will converge to the state observer $\hat{x}(t)$ as t tend to infinite.
2. The assumption that *Df*, *g* and *h* are Lipschitz globally condition may be relaxed to only locally Lipschitz condition.
3. An observer gain matrix L is selected such that (3.8) is satisfied. That is, by the single observer gain L, we can always guarantee that the state observer $\hat{x}(t)$ converge to the actual state $x(t)$ of non-linear dynamical control system (3.1).

Algorithm (3.1)

The following algorithm is presented in order to design a deterministic observer that estimates the original non-linear dynamical states given in (3.1). Based on the result of the theorem (3.1).

Step (0): Consider the non-linear dynamical system

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t), t)$$
$$y(t) = C\,x(t) + h(x(t))$$
$$x(0) = x_0$$

where $x(t) \in R^n$ is unmeasurable state vector, $u(t)$ is the control input and $y(t) \in R^m$ is the output vector. Suppose that the matrices A, B, C and D have a constant entries and appropriate dimensions. The non-linearity functions $Df(x(t)): R^n \to R^n$, $g(x(t), t): R^n \times R \to R^n$ and $h(x(t)): R^n \to R^m$ are assumed to be globally Lipschitz with a Lipschitz constants γ, β and δ

respectively.

Step (1): If (A, C) is observable, go to ***Step (2)***, otherwise, the system should be modified to satisfy the observable condition.

Step (2): Check the following Lipschitz conditions

$$\|Df(x(t)) - Df(\hat{x}(t))\| \le \gamma \|x(t) - \hat{x}(t)\|$$

$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le \beta \|x(t) - \hat{x}(t)\| \quad \text{for } t \in R$$

$$\|h(x(t)) - h(\hat{x}(t))\| \le \delta \|x(t) - \hat{x}(t)\|$$

and design the observer dynamic by

$$\begin{aligned}
&\frac{d\hat{x}(t)}{dt} = \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - (C\,\hat{x}(t) + h(\hat{x}(t))))\\
&\hat{y}(t) = C\,\hat{x}(t) + h(\hat{x}(t))\\
&\hat{x}(0) = \hat{x}_0
\end{aligned}$$

Step (3): Select L that makes $(\mathrm{A} - LC)$ asymptotically stable by using dual of the pole placement. (see subsection 1.5.4). And compute $\|L\|$.

Step (4): Let the dynamic error $e(t) = x(t) - \hat{x}(t)$ and $e(0) = x(0) - \hat{x}(0)$

$$\begin{aligned}
\frac{de(t)}{dt} &= (\mathrm{A} - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t)\\
&\quad - L(h(x(t)) - h(\hat{x}(t)))\\
e(0) &= x(0) - \hat{x}(0)
\end{aligned}$$

Step (5): Set $V(t) \equiv V(e(t)) = e^{\mathrm{T}}(t)P\,e(t)$

where P is the unique positive definite solution of $(\mathrm{A} - LC)^{\mathrm{T}}P + P(\mathrm{A} - LC) = -Q$, for arbitrary positive definite matrix Q.

Step (6): Check $\gamma + \beta + \delta\|L\| < \dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$

where γ is the Lipschitz constant of $Df(x(t))$, β is the Lipschitz constant of $g(x(t),t)$ and δ is the Lipschitz constant of $h(x(t))$. $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q and $\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

If ***Step (6)*** is not satisfied go to ***Step (3)*** and select another L such that ***Step (6)*** satisfy.

Problem (3.1)

Consider a non-linear dynamical system described by the following dynamical equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -3.6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.001 & 0.001 \\ 0 & 0.002 \end{bmatrix} \begin{bmatrix} \sin(x_1) \\ \cos(x_2) \end{bmatrix} + \begin{bmatrix} 0.01 x_2 \cos\left(\dfrac{t}{2}\right) \\ 0.003 x_1 \sin^2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0.0005 \sin(x_1)\cos(x_2)$$

$$x(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathrm{T}},\ \hat{x}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathrm{T}} \tag{3.22}$$

Step (1): Check the observability condition for the system

$$\mathrm{N} = (C^* \vdots \mathrm{A}^* C^*)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence rank (N) = 2. Therefore (A, C) is completely state observable.

Step (2): To verify the non-linearity $Df(x(t))$, $g(x(t),t)$ and $h(x(t))$ satisfy Lipschitz condition:

$$(1)\ Df(x(t)) = \begin{bmatrix} 0.001\sin(x_1) + 0.001\cos(x_2) \\ 0.002\cos(x_2) \end{bmatrix}$$

The Jacobian matrix for $Df(x(t))$ is

$$J_1 = \begin{bmatrix} 0.001\cos(x_1) & -0.001\sin(x_2) \\ 0 & -0.002\sin(x_2) \end{bmatrix}$$

where

$$\|J_1\| \le 0.002449489.$$

which implies that:

$$\left\| Df(x(t)) - Df(\hat{x}(t)) \right\| \le 0.002449489 \left\| x(t) - \hat{x}(t) \right\| \tag{3.23}$$

Thus, the non-linearity $Df(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\gamma = 0.002449489$.

(2) $g(x(t),t) = \begin{bmatrix} 0.01x_2 \cos\left(\dfrac{t}{2}\right) \\ 0.003x_1 \sin^2(t) \end{bmatrix}$

The Jacobian matrix for $g(x(t),t)$ is

$$J_2 = \begin{bmatrix} 0 & 0.01\cos\left(\dfrac{t}{2}\right) \\ 0.003\sin^2(t) & 0 \end{bmatrix}$$

where

$\left\| J_2 \right\| \le 0.010440306$.

hence:

$$\left\| g(x(t),t) - g(\hat{x}(t),t) \right\| \le 0.010440306 \left\| x(t) - \hat{x}(t) \right\| \tag{3.24}$$

Thus, the non-linearity function $g(x(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\beta = 0.010440306$.

(3) $h(x(t)) = 0.0005\sin(x_1)\cos(x_2)$

The Jacobian matrix for the function $h(x(t))$ is

$$J_3 = \left[0.0005\cos(x_1)\cos(x_2), \quad -0.0005\sin(x_1)\sin(x_2)\right]$$

where

$\left\| J_3 \right\| \le 0.000707106$.

hence:

$$\left\| h(x(t)) - h(\hat{x}(t)) \right\| \le 0.000707106 \left\| x(t) - \hat{x}(t) \right\| \tag{3.25}$$

Thus, the non-linearity function $h(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\delta = 0.000707106$.

Step (3): Suppose that we use the dual of the pole placement approach to compute observer gain matrix L and the desired poles for this system are selected as:

$$\eta_1 = -7.2 + i9.6,\ \eta_2 = -7.2 - i9.6.$$

The state observer gain matrix L can be obtained (by using MATLAB) as shown in program (A3) in Appendix A.

$$L = \begin{bmatrix} 10.8 \\ 96.12 \end{bmatrix} \tag{3.26}$$

Hence

$$\|L\| = 96.7248. \tag{3.27}$$

Step (4): To find P which is the solution of this Riccati equation:

$$(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$$

On solving it to get a unique positive definite solution P, on selection of:

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

hence

$$\begin{bmatrix} -10.8 & -105.12 \\ 1 & -3.6 \end{bmatrix}\begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} \\ \mathrm{P}_{12} & \mathrm{P}_{22} \end{bmatrix} + \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} \\ \mathrm{P}_{12} & \mathrm{P}_{22} \end{bmatrix}\begin{bmatrix} 10.8 & 1 \\ -105.12 & -3.6 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 27.0235 & -2.7288 \\ -2.7288 & 0.6309 \end{bmatrix} \tag{3.28}$$

$$\lambda_1(P) = 0.3517,\ \lambda_2(P) = 27.3027.$$

Then P is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 10$, $\lambda_{\max}(P) = 27.3027$.

Step (5): From ***step (2)*** and ***step (3)*** it is clear that

$$\gamma + \beta + \delta\|L\| = 0.081284556. \tag{3.29}$$

Now, check $\gamma + \beta + \delta\|L\| < \dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} = 0.1831.$ (3.30)

Finally, we shall obtain the response of the system to the given initial condition: $x(0) = [-1 \quad 2]^{\mathrm{T}}$, $\hat{x}(0) = [-1 \quad 2]^{\mathrm{T}}$.

Assuming that the control is an open- loop controller, i.e., $u(t)$ is a function of t only and assuming that no any disturbance to the system occurred. We take $u(t) = \exp(-t)$ then Equation (3.22), becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -3.6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0.001 & 0.001 \\ 0 & 0.002 \end{bmatrix} \begin{bmatrix} \sin(x_1) \\ \cos(x_1) \end{bmatrix} + \begin{bmatrix} 0.01 x_2 \cos\left(\dfrac{t}{2}\right) \\ 0.003 x_1 \sin^2(t) \end{bmatrix} \tag{3.31}$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0.0005 \sin(x_1)\cos(x_2)$$

Now, the observer can be estimated by:

$$\frac{d\hat{x}(t)}{dt} = (\mathrm{A} - LC)\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t), t) + Ly(t)$$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -10.8 & 1 \\ -105.12 & -3.6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0.001\sin(\hat{x}_1) + 0.001\cos(\hat{x}_2) \\ 0.002\cos(\hat{x}_2) \end{bmatrix} + \begin{bmatrix} 0.01\hat{x}_2 \cos\left(\dfrac{t}{2}\right) \\ 0.003\hat{x}_1 \sin^2(t) \end{bmatrix} + \begin{bmatrix} 1.2080 \\ 10.7516 \end{bmatrix} \tag{3.32}$$

A MATLAB program using the fourth-order Runge-kutta method is used to obtain the states x_1 , x_2 , $\hat{x}_1$ and $\hat{x}_2$, and simulate the dynamics errors are shown in MATLAB program (A3), in Appendix A.

The numerical results and estimators based on the algorithm for problem (3.1) have been shown in the following plotted graphs.

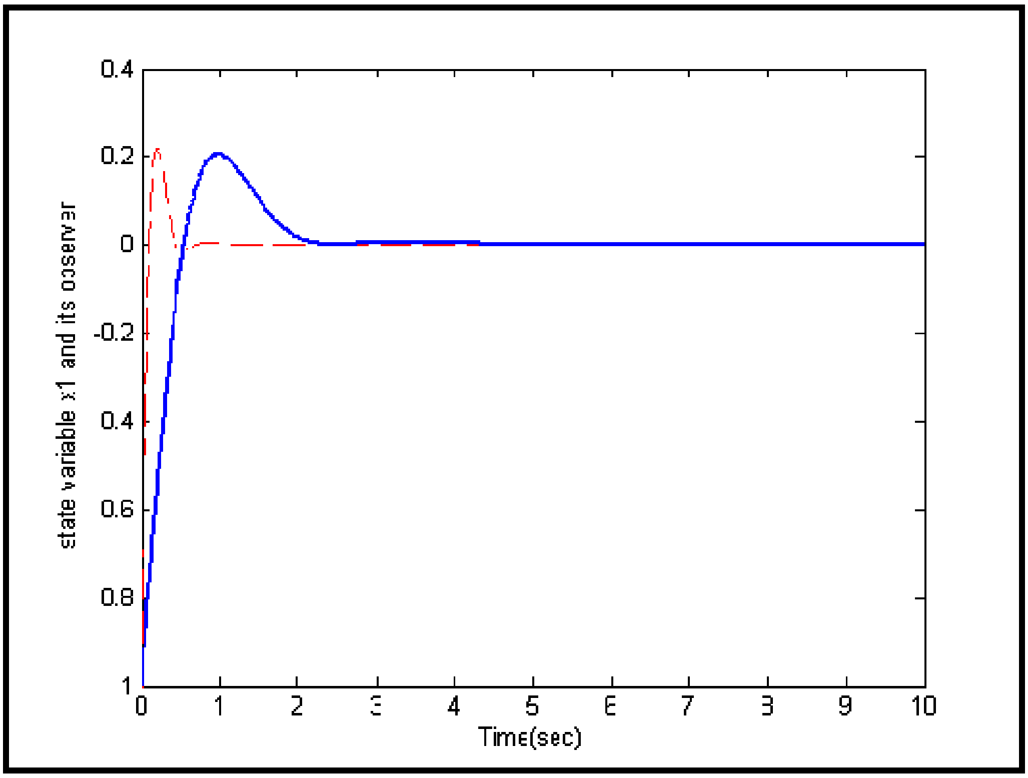


Fig (3.1) Observer performance: state variable x_1 (solid curve). and its observer $\hat{x}_1$ (broken curve) for $x_1(0) = -1$, $\hat{x}_1(0) = -1$ of problem (3.1).

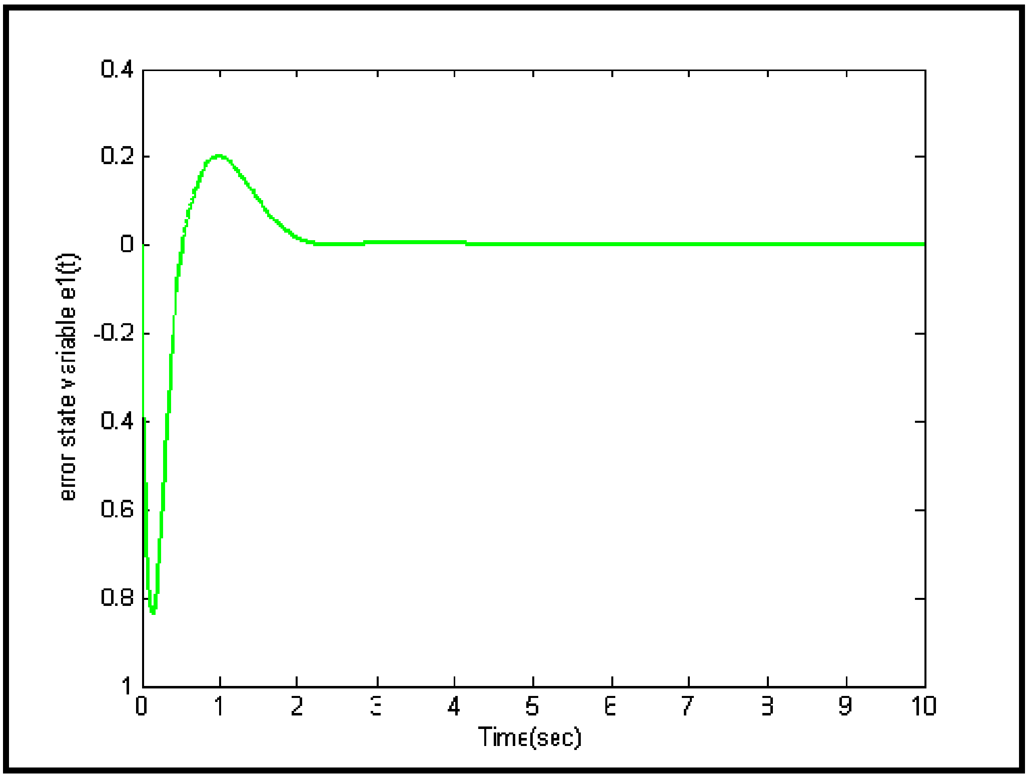


Fig (3.2) error between $x_1(t)$ and its observer $\hat{x}_1(t)$ of problem (3.1).

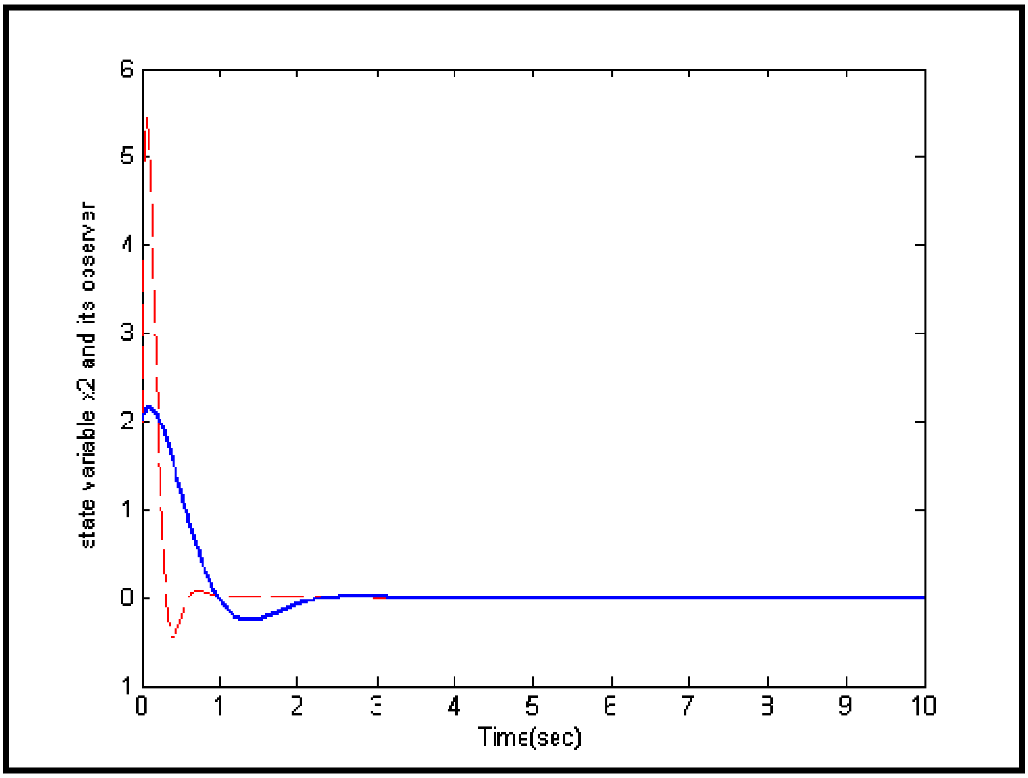


Fig (3.3) Observer performance: state variable x_2 ***(solid curve). and its observer*** $\hat{x}_2$ ***(broken curve) for*** $x_2(0)=2$**,** $\hat{x}_2(0)=2$ ***of problem (3.1).***

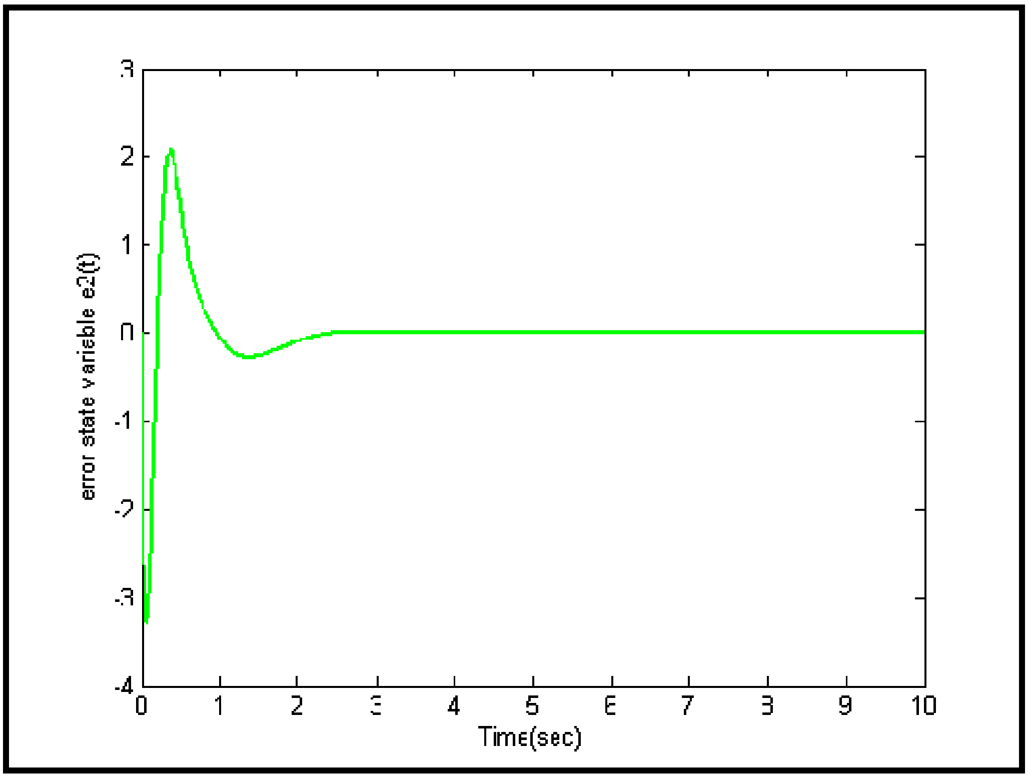


Fig (3.4) error between $x_2(t)$ ***and its observer*** $\hat{x}_2(t)$ ***of problem (3.1).***

Problem (3.2)

Consider a non-linear dynamical control system described by the following dynamical equations:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}2&0&0\\0&2&0\\0&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}u+\begin{bmatrix}0.1&0.01&0.03\\0.02&0&0.06\\0&0.06&0.01\end{bmatrix}\begin{bmatrix}\cos(x_1)\\ \sin(x_3)\\ \sin(x_2)\cos(x_2)\end{bmatrix}$$

$$+\begin{bmatrix}0\\ 0.003x_3\sin(t)\\ 0.006x_2\sin(2t)\end{bmatrix}$$

$$y(t)=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}+\begin{bmatrix}0.002\cos(x_3)\sin(x_3)\\ 0.005\sin(x_1)\end{bmatrix}$$

$$x(0)=[1\quad 0\quad -1]^{\mathrm{T}},\ \hat{x}(0)=[1\quad -0.5\quad -0.5]^{\mathrm{T}} \tag{3.33}$$

(1) the first stage: the feedback gain K is obtained by algorithm (1.2) as follows:

Step (1): Check the controllability condition for the system

$$\mathrm{M}=(\mathrm{B}\vdots\mathrm{AB}\vdots\mathrm{A}^2\mathrm{B})$$

$$=\begin{pmatrix}0&1&0&2&0&4\\1&0&2&0&4&0\\0&1&3&1&9&1\end{pmatrix}$$

Hence rank (M) = 3. Therefore (A, B) is completely state controllable.

Step (2): Suppose that we use the pole placement approach to compute feedback gain matrix K and the control poles for this system are selected as:

$$\mu_1=-1+i,\ \mu_2=-1-i,\ \mu_3=-1.$$

The feedback gain matrix K can be obtained (by using MATLAB as shown in program (A4) in Appendix A.

$$\mathrm{K}=\begin{pmatrix}-2.7088&5.3822&1.8972\\1.8990&1.1554&0.7188\end{pmatrix}. \tag{3.34}$$

(2) the second stage: the observer gain L by algorithm (3.1) is obtained as follows:

Step (1): Check the observability condition for the system

$$N=\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & 1 \\ 2 & 13 & 3 \\ 4 & 13 & 1 \\ 4 & 35 & 3 \end{pmatrix}$$

Hence rank (N) = 3. Therefore (A, C) is completely state observable.

Step (2): To verify the non-linearity $Df(x(t))$, $g(x(t),t)$ and $h(x(t))$ satisfy Lipschitz condition:

$$(1)\ Df(x(t))=\begin{pmatrix} 0.1\cos(x_1)+0.01\sin(x_3)+0.03\sin(x_2)\cos(x_2) \\ 0.02\cos(x_1)+0.06\sin(x_2)\cos(x_2) \\ 0.06\sin(x_3)+0.01\sin(x_2)\cos(x_2) \end{pmatrix}$$

The Jacobian matrix for $Df(x(t))$ is

$$J_1=\begin{bmatrix} -0.1\sin(x_1) & 0.03\cos(2x_2) & 0.01\cos(x_3) \\ -0.02\sin(x_1) & 0.06\cos(2x_2) & 0 \\ 0 & 0.01\cos(2x_2) & 0.06\cos(x_3) \end{bmatrix}$$

where

$$\|J_1\| \le 0.1367$$

which implies that:

$$\|Df(x(t))-Df(\hat{x}(t))\| \le 0.1367\|x(t)-\hat{x}(t)\| \tag{3.35}$$

Thus, the non-linearity $Df(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\gamma=0.1367$.

$$(2)\ \ g(x(t),t)=\begin{bmatrix}0\\ 0.03x_3\sin(t)\\ 0.06x_2\sin(2t)\end{bmatrix}$$

The Jacobian matrix for the function $g(x(t),t)$ is

$$J_2=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0.03\sin(t)\\ 0 & 0.06\sin(2t) & 0\end{bmatrix}$$

where

$$\|J_2\|\le 0.06708$$

hence:

$$\|g(x(t),t)-g(\hat{x}(t),t)\|\le 0.06708\|x(t)-\hat{x}(t)\| \tag{3.36}$$

Thus, the non-linearity function $g(x(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\beta=0.06708$.

$$(3)\ \ h(x(t))=\begin{bmatrix}0.002\cos(x_3)\sin(x_3)\\ 0.005\sin(x_1)\end{bmatrix}$$

The Jacobian matrix for the function $h(x(t))$ is

$$J_3=\begin{bmatrix}0 & 0 & 0.002\cos(2x_3)\\ 0.005\cos(x_1) & 0 & 0\end{bmatrix}$$

where

$$\|J_3\|\le 0.00538.$$

hence:

$$\|h(x(t))-h(\hat{x}(t))\|\le 0.00538\|x(t)-\hat{x}(t)\| \tag{3.37}$$

Thus, the non-linearity function $h(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\delta=0.00538$.

Step (3): Suppose that we use the dual of the pole placement approach to compute observer gain matrix L and the desired poles for this system are selected as:

$\eta_1 = -4 - i4$, $\eta_2 = -4 + i4$, $\eta_3 = -4$.

The state observer gain matrix L can be obtained (by using MATLAB) as shown in program (A4) in Appendix A.

$$L = \begin{pmatrix} 13.3244 & -6.8725 \\ -8.9088 & 7.9402 \\ -1.1096 & 1.5621 \end{pmatrix} \tag{3.38}$$

Hence

$$\|L\| = \left(\sum_{i=1}^{3} \sum_{j=1}^{2} |a_{ij}|^2 \right)^{1/2}$$

$$= 19.1014 \tag{3.39}$$

Step (4): To find P which is the solution of this Riccati equation:

$$(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$$

On solving it to get a unique positive definite solution P, on selection of:

$$Q = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tag{3.40}$$

hence

$$\begin{bmatrix} -4.4518 & 0.9686 & -0.4525 \\ 0.4207 & -4.9715 & 0.9854 \\ 7.2933 & -14.9117 & -2.5767 \end{bmatrix} \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \mathrm{P}_{13} \\ \mathrm{P}_{12} & \mathrm{P}_{22} & \mathrm{P}_{23} \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} \end{bmatrix} + \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \mathrm{P}_{13} \\ \mathrm{P}_{12} & \mathrm{P}_{22} & \mathrm{P}_{23} \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} \end{bmatrix}$$

$$\begin{bmatrix} -4.4518 & 0.4207 & 7.2933 \\ 0.9686 & -4.9715 & -14.9117 \\ -0.4525 & 0.9854 & -2.5767 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 0.3426 & 0.0601 & 0.0725 \\ 0.0601 & 0.2661 & -0.2055 \\ 0.0725 & -0.2055 & 1.9765 \end{bmatrix} \tag{3.41}$$

$\lambda_1(P) = 0.2069$, $\lambda_2(P) = 0.3750$, $\lambda_3(P) = 2.0034$.

Then P is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q)=3, \lambda_{\max}(P)=2.0034$.

Step (5): From ***step (2)*** and ***step (3)*** of stage 2 it is clear that

$$\gamma+\beta+\delta\|L\|=0.3065. \tag{3.42}$$

Now, check $\gamma+\beta+\delta\|L\|<\dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ = 0.7486. (3.43)

Finally, we shall obtain the response of the system to the following initial condition:

$$x(0)=\begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad e(0)=\begin{bmatrix}0\\0.5\\-0.5\end{bmatrix} \tag{3.44}$$

In applying theorem (3.1), the Eq. (2.27) becomes

$$\dot{e}(t)=(\mathrm{A}-LC)e(t)+\zeta(x(t),\hat{x}(t),t) \tag{3.45}$$

where

$$\zeta(x(t),\hat{x}(t),t)=Df(x(t))-Df(\hat{x}(t))+g(x(t),t)-g(\hat{x}(t),t)$$
$$-L(h(x(t))-h(\hat{x}(t)))$$

where the observer gain L is determined such that the inequality (3.8), is satisfied.

Hence Eq. (2.28) becomes

$$\begin{bmatrix}\dot{x}(t)\\\dot{e}(t)\end{bmatrix}=\begin{bmatrix}\mathrm{A}-\mathrm{BK} & \mathrm{BK}\\0 & \mathrm{A}-LC\end{bmatrix}\begin{bmatrix}x(t)\\e(t)\end{bmatrix}+\begin{bmatrix}Df(x(t))+g(x(t),t)\\\zeta(x(t),\hat{x}(t),t)\end{bmatrix} \tag{3.46}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}=\begin{bmatrix}0.1 & -1.1554 & -0.7188 & 1.899 & 1.1554 & 0.7188\\ 2.7088 & -3.3822 & -1.8972 & -2.7088 & 5.3822 & 1.8972\\ -1.899 & 1.8446 & 0.2812 & 1.899 & 1.1554 & 0.7188\\ 0 & 0 & 0 & -4.4518 & 0.4207 & 7.2933\\ 0 & 0 & 0 & 0.9686 & -4.9715 & -14.9117\\ 0 & 0 & 0 & -0.4525 & 0.9854 & -2.5767\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ e_1\\ e_2\\ e_3\end{bmatrix}+$$

$$\begin{bmatrix}0.1\cos(x_1)+0.01\sin(x_3)+0.03\sin(x_2)\cos(x_2)\\ 0.02\cos(x_1)+0.06\sin(x_2)\cos(x_2)+0.03x_3\sin(t)\\ 0.06\sin(x_3)+0.01\sin(x_2)\cos(x_2)+0.06x_2\sin(2t)\\ 0.1(\cos(x_1)-\cos(x_1-e_1))+0.01(\sin(x_3)-\sin(x_3-e_3))+0.03(\sin(x_2)\cos(x_2)-\\ -\sin(x_2-e_2)\cos(x_2-e_2))-0.0266488(\cos(x_3)\sin(x_3)-\cos(x_3-e_3)\sin(x_3-e_3))\\ +0.0343625(\sin(x_1)-\sin(x_1-e_1))\\ 0.02(\cos(x_1)-\cos(x_1-e_1))+0.06(\sin(x_2)\cos(x_2)-\sin(x_2-e_2)\cos(x_2-e_2))+0.03\\ (x_3-(x_3-e_3))\sin(t)+0.01781(\cos(x_3)\sin(x_3)-\cos(x_3-e_3)\sin(x_3-e_3))\\ -0.0397(\sin(x_1)-\sin(x_1-e_1))\\ 0.06(\sin(x_3)-\sin(x_3-e_3))+0.01(\sin(x_2)\cos(x_2)-\sin(x_2-e_2)\cos(x_2-e_2))+0.06(x_2\\ -(x_2-e_2))\sin(2t)+0.002219(\cos(x_3)\sin(x_3)-\cos(x_3-e_3)\sin(x_3-e_3))-0.007810\\ (\sin(x_1)-\sin(x_1-e_1))\end{bmatrix}$$

(3.47)

A MATLAB program using the fourth-order Runge-kutta method is used to obtain the response is shown in MATLAB program (A4), in Appendix A.

The numerical results and estimators based on the algorithm for problem (3.2) have been shown in the following plotted graphs.

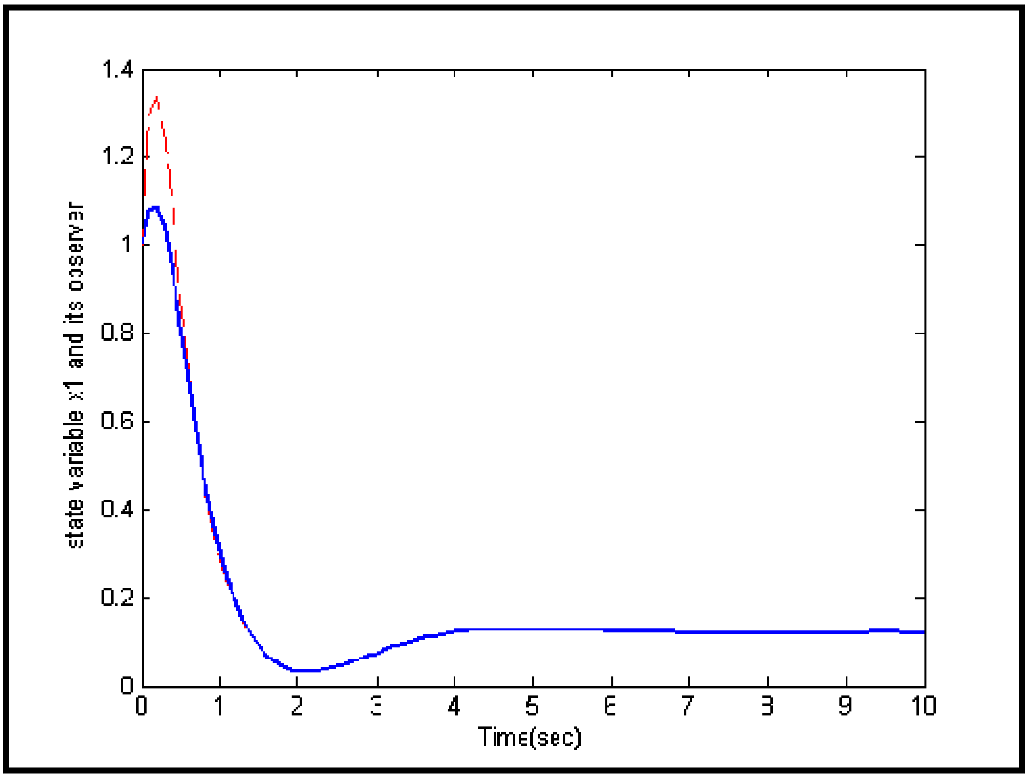


Fig (3.5) Observer performance: state variable x_1 (solid curve). and its observer $\hat{x}_1$ (broken curve) for $x_1(0)=1$, $\hat{x}_1(0)=1$ of problem (3.2).

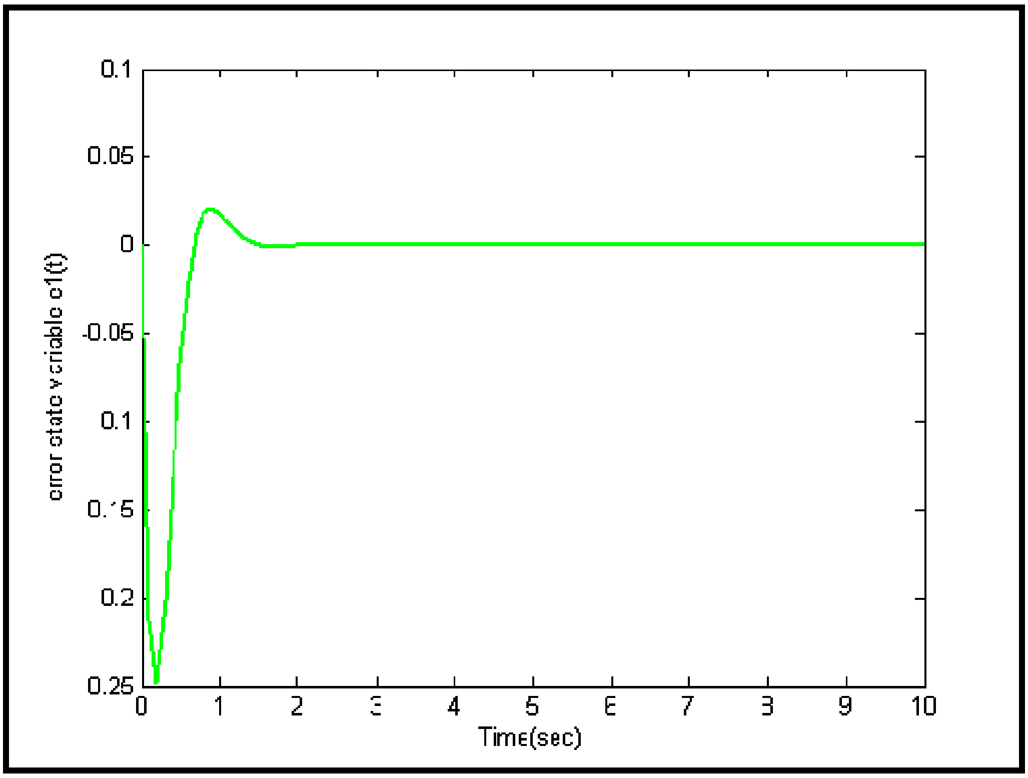


Fig (3.6) error between $x_1(t)$ and its observer $\hat{x}_1(t)$ of problem (3.2).

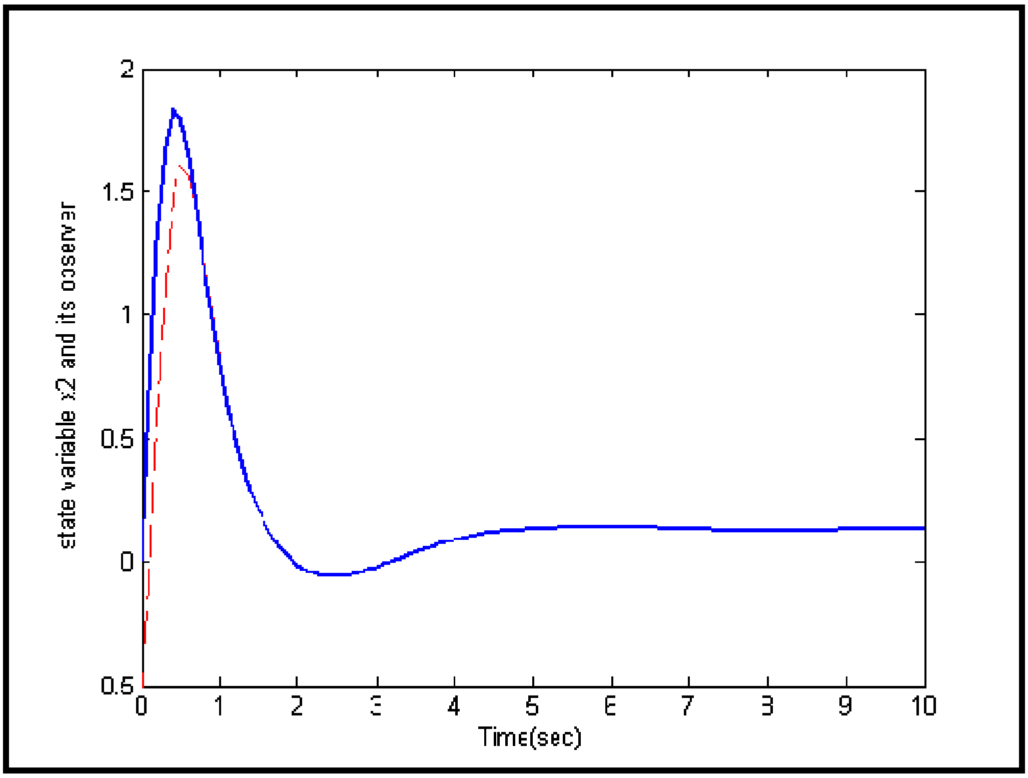


Fig (3.7) Observer performance: state variable x_2 ***(solid curve). and its observer*** $\hat{x}_2$ ***(broken curve) for*** $x_2(0)=0$**,** $\hat{x}_2(0)=-0.5$ ***of problem (3.2).***

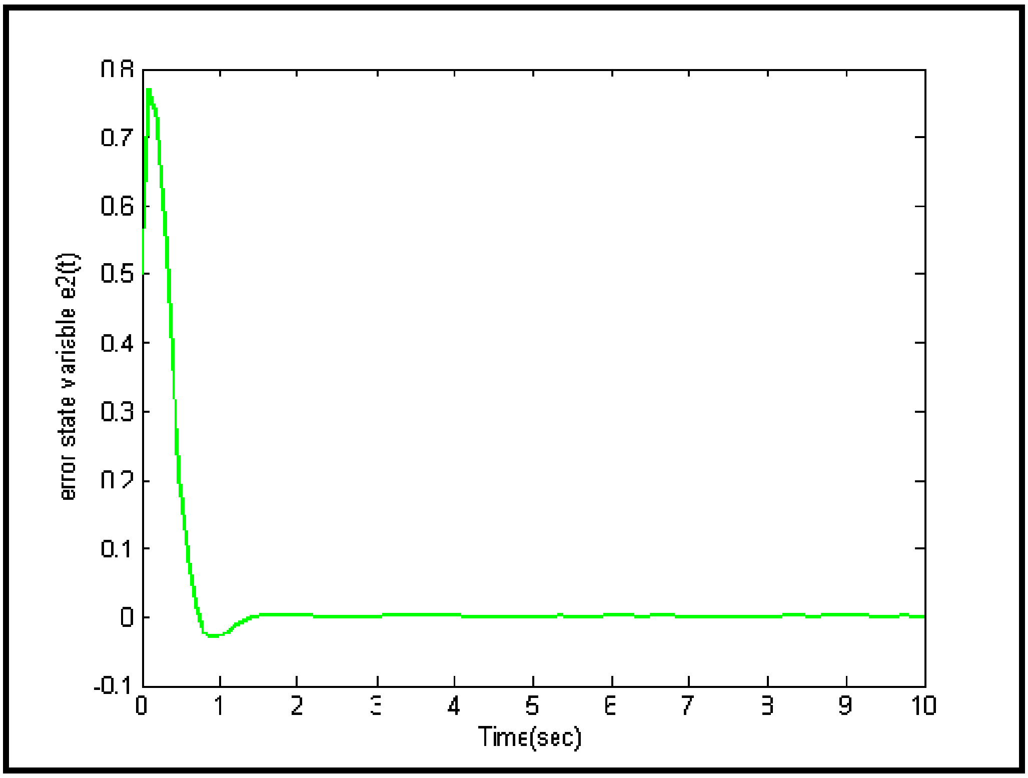


Fig (3.8) error between $x_2(t)$ ***and its observer*** $\hat{x}_2(t)$ ***of problem (3.2).***

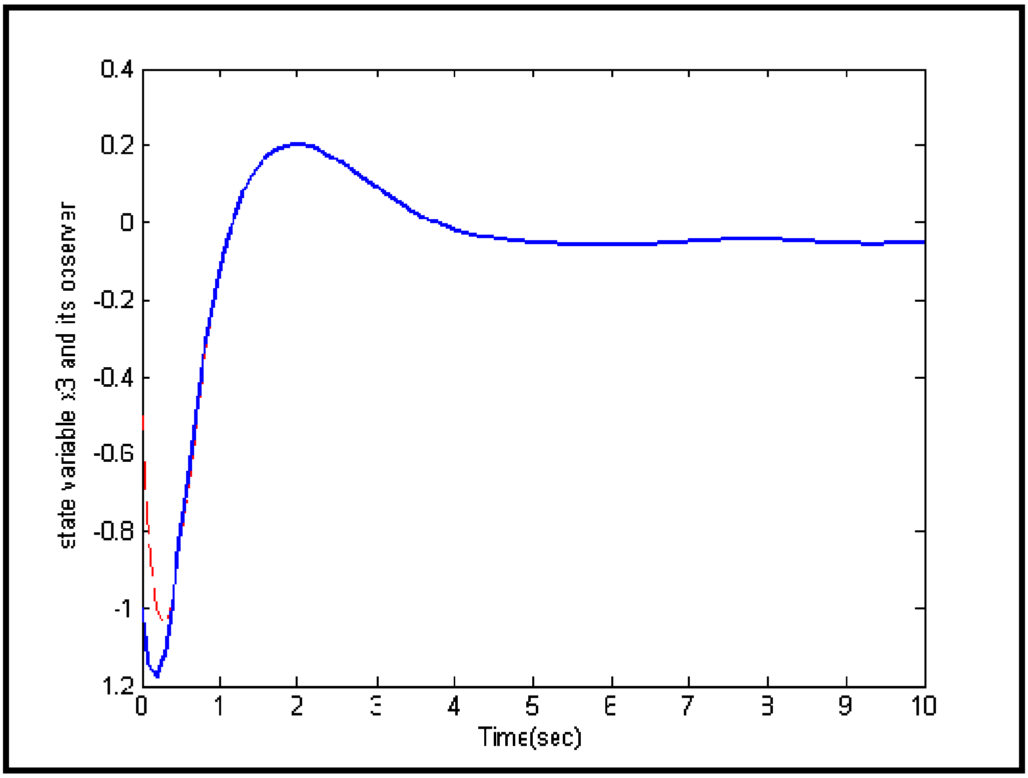


Fig (3.9) Observer performance: state variable x_3 ***(solid curve). and its observer*** $\hat{x}_3$ ***(broken curve) for*** $x_3(0) = -1$***,*** $\hat{x}_3(0) = -0.5$ ***of problem (3.2).***

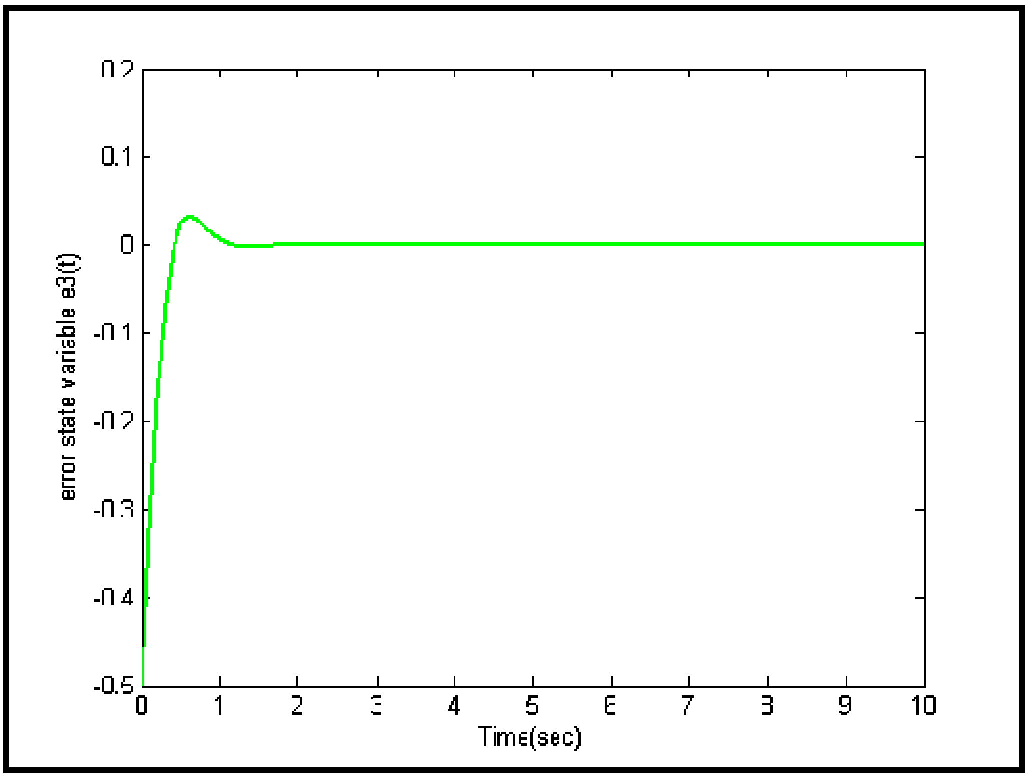


Fig (3.10) error between $x_3(t)$ ***and its observer*** $\hat{x}_3(t)$ ***of problem (3.2).***

3.2 AN APPROXIMATE STATE SPACE OBSERVER

If we fail to find a single gain L that can lead to stable error dynamics when we use closed loop controller $u = -\mathrm{K}\hat{x}$ such that $-\eta_i$ be approximately equal to $-4\mu_i$ where $-\eta_i$ are the observer roots, and $-\mu_i$ are the control roots. Then we can find approximate observer as follows:

If $h(\cdot)$ is continuously differentiable function, with $h(0) = 0$. Let us denote

$$C_1 = \left(\frac{\partial h}{\partial x}\right)_{x=0}$$

Then the given system (3.1) can be expanded as

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u + Df(x(t)) + g(x(t),t) \\ y(t) &= C\,x(t) + C_1\,x(t) + h_1(x(t)) \end{aligned} \qquad (3.48)$$

where $h_1(x(t))$ is obtained from expanding $h(x(t))$ in a Taylor series about $x = 0$, as $h(x(t)) = h(0) + C_1\,x(t) + h_1(x(t))$

For the observer design we will neglect the function $h_1(x(t))$. The observer so designed will be approximate, since we neglect the higher-order terms of h.

One can now find the highest value of γ and β for which an observer design is possible for the following system.

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u + Df(x(t)) + g(x(t),t) \\ y(t) &= O\,x(t) \end{aligned} \qquad (3.49)$$

where $O = C + C_1$

Then the observer, for the non-linear system (3.49) is implemented as follows.

$$\begin{aligned} \dot{\hat{x}}(t) &= (\mathrm{A} - LC)\hat{x}(t) + \mathrm{B}u + Df(\hat{x}(t)) + g(\hat{x}(t),t) + Ly(t) \\ \hat{y}(t) &= O\,\hat{x}(t) \end{aligned} \qquad (3.50)$$

Note that Eq. (3.49) is the same as Eq. (2.1) so we can make a suitable transformation and satisfy the inequality (2.9) as we discussed in Lemma (2.2).

Problem (3.3)

Consider a non-linear dynamical control system described by the following dynamical equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(x_1) \\ \cos(x_1) \end{bmatrix} + \begin{bmatrix} 0.3x_1 \cos^3(t) \\ x_2 \sin^2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sin(x_2)$$

$$x(0) = \begin{bmatrix} 4 & 3.3 \end{bmatrix}^{\mathrm{T}}, \ \hat{x}(0) = \begin{bmatrix} 3.33 & 4.95 \end{bmatrix}^{\mathrm{T}} \tag{3.51}$$

(1) the first stage: the feedback gain K is obtained by algorithm (1.1) as follows:

Step (1): Check the controllability condition for the system

$$\mathrm{M} = (\mathrm{B} \vdots \mathrm{AB})$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence rank (M) = 2. Therefore (A, B) is completely state controllable.

Step (2): Suppose that we use the pole placement approach to compute feedback gain matrix K and the control poles for this system are selected as:

$$\mu_1 = -0.7071 + i0.7071, \ \mu_2 = -0.7071 - i0.7071.$$

The feedback gain matrix K can be obtained (by using MATLAB) as shown in program (A5) in Appendix A.

$$\mathrm{K} = \begin{pmatrix} 1 & 1.4142 \end{pmatrix} \tag{3.52}$$

(2) the second stage: the observer gain L by algorithm (3.1) is obtained as follows:

Step (1): Check the observability condition for the system

$$\mathrm{N} = \left(C^* \vdots \mathrm{A}^* C^*\right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence rank (N) = 2. Therefore (A, C) is completely state observable.

Step (2): To verify the non-linearity $Df(x(t))$, $g(x(t),t)$ and $h(x(t))$ satisfy Lipschitz condition:

$$(1)\ Df(x(t)) = \begin{pmatrix} \sin(x_1) \\ \cos(x_1) \end{pmatrix}$$

The Jacobian matrix for $Df(x(t))$ is

$$J_1 = \begin{bmatrix} \cos(x_1) & 0 \\ -\sin(x_1) & 0 \end{bmatrix}$$

where

$$\|J_1\| \le 1.414213562$$

which implies that:

$$\|Df(x(t)) - Df(\hat{x}(t))\| \le 1.414213562 \|x(t) - \hat{x}(t)\| \qquad (3.53)$$

Thus, the non-linearity $Df(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\gamma = 1.414213562$.

$$(2)\ g(x(t),t) = \begin{bmatrix} 0.3x_1 \cos^3(t) \\ x_2 \sin^2(t) \end{bmatrix}$$

The Jacobian matrix for $g(x(t),t)$ is

$$J_2 = \begin{bmatrix} 0.3\cos^3(t) & 0 \\ 0 & \sin^2(t) \end{bmatrix}$$

where

$$\|J_2\| \le 1.044030651$$

hence:

$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le 1.044030651 \|x(t) - \hat{x}(t)\| \qquad (3.54\ )$$

Thus, the non-linearity function $g(x(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\beta = 1.044030651$.

$$(3)\ h(x(t)) = \sin(x_2)$$

The Jacobian matrix for the function $h(x(t))$ is

$$J_3 = [0, \quad \cos(x_2)]$$

where

$$\|J_3\| \le 1$$

hence:

$$\|h(x(t)) - h(\hat{x}(t))\| \le \|x(t) - \hat{x}(t)\| \tag{3.55}$$

Thus, the non-linearity function $h(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\delta = 1$.

Step (3): Suppose that we use the dual of the pole placement approach to compute observer gain matrix L and the desired poles for this system are selected as:

$$\eta_1 = -2.8284 + i2.8284,\ \eta_2 = -2.8284 - i2.8284.$$

The state observer gain matrix L can be obtained (by using MATLAB) as shown in program (A5) in Appendix A.

$$L = \begin{bmatrix} 5.6568 \\ 15.9997 \end{bmatrix} \tag{3.56}$$

Hence

$$\|L\| = 16.9703. \tag{3.57}$$

Step (4): To find P which is the solution of this Riccati equation:

$$(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$$

On solving it to get a unique positive definite solution P, on selection of:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

hence

$$\begin{bmatrix} -5.6568 & -15.9997 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} \\ \mathrm{P}_{12} & \mathrm{P}_{22} \end{bmatrix} + \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} \\ \mathrm{P}_{12} & \mathrm{P}_{22} \end{bmatrix}\begin{bmatrix} -5.6568 & 1 \\ -15.9997 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 1.5026 & -0.5 \\ -0.5 & 0.2707 \end{bmatrix} \tag{3.58}$$

$\lambda_1(P) = 0.0933$, $\lambda_2(P) = 1.68$.

Then P is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 1$, $\lambda_{\max}(P) = 1.68$.

Step (5): From ***step (2)*** and ***step (3)*** of stage 2, it is clear that

$$\gamma + \beta + \delta\|L\| = 19.428544213 \;. \tag{3.59}$$

$$\text{Now, check } \gamma + \beta + \delta\|L\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} = 0.2976. \tag{3.60}$$

Since the observer gain was unsuccessful and $h(x) = \sin(x_2)$ is continuously differentiable function, with $h(0) = \sin(0) = 0$. Then the output of the (3.51) can be approximate in order to overcome this difficulties and as follows:

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.61}$$

so another observer gain which leads to stable error dynamics by using approximate output (3.61), has been adapted.

Step (6): Check the observability condition for the system

$$\mathrm{N}_1 = \left(O^* \vdots \mathrm{A}^* O^*\right)$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Hence rank (N_1) = 2.Therefore (A, O) is completely state observable.

Step (7): The state observer gain matrix L_1 can be obtained (by using MATLAB) as shown in program (A5) in Appendix A.

$$L_1 = \begin{bmatrix} -10.3429 \\ 15.9997 \end{bmatrix} \tag{3.62}$$

Step (8): Find P_1 which is the solution of this Riccati equation:

$$(\mathrm{A}-L_1O)^{\mathrm{T}}P_1+P_1(\mathrm{A}-L_1O)=-Q$$

On solving it to get a unique positive definite solution P_1, on selection of:

$$Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

therefore

$$\begin{bmatrix}10.342 & -15.999\\ 11.342 & -15.999\end{bmatrix}\begin{bmatrix}\mathrm{P}_{11} & \mathrm{P}_{12}\\ \mathrm{P}_{12} & \mathrm{P}_{22}\end{bmatrix}+\begin{bmatrix}\mathrm{P}_{11} & \mathrm{P}_{12}\\ \mathrm{P}_{12} & \mathrm{P}_{22}\end{bmatrix}\begin{bmatrix}10.342 & 11.342\\ -15.999 & -15.999\end{bmatrix}=$$
$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$\rightarrow$

$$P_1=\begin{bmatrix}2.9168 & 1.9168\\ 1.9168 & 1.3901\end{bmatrix} \tag{3.63}$$

$$\lambda_1(P_1)=0.0903,\ \lambda_2(P_1)=4.2167\,.$$

Then P_1 is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q)=1, \lambda_{\max}(P_1)=4.2167$.

Step (9): Now, check $\gamma+\beta<\dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P_1)}$ = 0.1186. (3.64)

Since the observer gain in original coordinates was unsuccessful. Using transformation of coordinates, $z=\mathrm{T}x$ where

$$\mathrm{T}=\mathrm{diag}\begin{pmatrix}20 & 30\end{pmatrix} \tag{3.65}$$

then

$$\mathrm{T}^{-1}=\mathrm{diag}\begin{pmatrix}0.0333 & 0.05\end{pmatrix} \tag{3.66}$$

Step (10): Compute

$$\overline{\mathrm{K}}=\mathrm{KT}^{-1}=\begin{pmatrix}1 & 1.4142\end{pmatrix}\begin{pmatrix}0.0333 & 0\\ 0 & 0.05\end{pmatrix}$$

$$= \begin{pmatrix} 0.0333 & 0.0707 \end{pmatrix} \tag{3.67}$$

Step (11): To verify the non-linearity $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ and $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy Lipschitz condition:

$$\mathrm{T}Df(\mathrm{T}^{-1}z(t)) = \begin{bmatrix} 30 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \sin\left(\frac{z_1}{30}\right) \\ \cos\left(\frac{z_1}{30}\right) \end{bmatrix} = \begin{bmatrix} 30\sin\left(\frac{z_1}{30}\right) \\ 20\cos\left(\frac{z_1}{30}\right) \end{bmatrix}$$

The Jacobian matrix for $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ is

$$\overline{J}_1 = \begin{bmatrix} \cos\left(\frac{z_1}{30}\right) & 0 \\ \frac{-20}{30}\sin\left(\frac{z_1}{30}\right) & 0 \end{bmatrix}$$

where

$$\left\|\overline{J}_1\right\| \le 1.201850425$$

hence:

$$\left\|\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))\right\| \le 1.201850425\left\|\mathrm{T}^{-1}z - \mathrm{T}^{-1}\hat{z}\right\| \tag{3.68}$$

Thus, the non-linearity $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\overline{\gamma} = 1.201850425$, and,

$$\mathrm{T}g(\mathrm{T}^{-1}z(t),t) = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} 0.6\frac{z_1}{30}\cos^3(t) \\ \frac{z_2}{20}\sin^2(t) \end{pmatrix} = \begin{pmatrix} 0.6z_1\cos^3(t) \\ z_2\sin^2(t) \end{pmatrix}$$

The Jacobian matrix for the function $g(x(t),t)$ is

$$J_2 = \begin{bmatrix} 0.6\cos^3(t) & 0 \\ 0 & \sin^2(t) \end{bmatrix}$$

where

$$\left\|\overline{J}_2\right\| \le 1.166190379.$$

hence:

$$\left\| \mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t) \right\| \le 0.02236 \left\| \mathrm{T}^{-1}z - \mathrm{T}^{-1}\hat{z} \right\| \qquad (3.69)$$

Thus, the non-linearity function $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\overline{\beta} = 0.02236$.

Step (12): Compute

$$\overline{L}_1 = \mathrm{T}L_1 = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} -10.3429 \\ 15.9997 \end{pmatrix} = \begin{pmatrix} 169.7040 \\ 319.9939 \end{pmatrix} \qquad (3.70)$$

Step (13): To find $\overline{P}_1$ which is the solution of this Riccati equation:

$$(\overline{\mathrm{A}} - \overline{L}_1 \overline{O})^{\mathrm{T}} \overline{P}_1 + \overline{P}_1 (\overline{\mathrm{A}} - \overline{L}_1 \overline{O}) = -Q$$

On solving it to get a unique positive definite solution $\overline{P}$, on selection of:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore

$$\begin{bmatrix} 10.342 & -10.666 \\ 17.014 & -15.999 \end{bmatrix} \begin{bmatrix} \overline{\mathrm{P}}_{11} & \overline{\mathrm{P}}_{12} \\ \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{22} \end{bmatrix} + \begin{bmatrix} \overline{\mathrm{P}}_{11} & \overline{\mathrm{P}}_{12} \\ \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{22} \end{bmatrix} \begin{bmatrix} 10.342 & 17.014 \\ -10.666 & -15.999 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\rightarrow$

$$\overline{P}_1 = \begin{bmatrix} 2.1311 & 2.1134 \\ 2.1134 & 2.2786 \end{bmatrix} \qquad (3.71)$$

$$\lambda_1(\overline{P}_1) = 0.0902, \;\; \lambda_2(\overline{P}_1) = 4.3195.$$

Then $\overline{P}_1$ is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 1$, $\lambda_{\max}(\overline{P}_1) = 4.3195$, $\lambda_{\max}(\mathrm{T}^{-1}) = 0.05$.

Step (14): From ***step(11)*** it is clear that $\overline{\gamma} + \overline{\beta} = 1.224210425$.

Now, check $\bar{\gamma}+\bar{\beta}<\dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(\bar{P}_1)\lambda_{\max}(\mathrm{T}^{-1})}$ = 2.3151. (3.72)

Finally, we shall obtain the response of the system to the following initial condition:

$$z(0)=\begin{bmatrix}120\\66\end{bmatrix},\quad E(0)=\begin{bmatrix}20\\-33\end{bmatrix} \tag{3.73}$$

Referring to Equation (2.65),

$$\begin{bmatrix}\dot{z}(t)\\ \dot{E}(t)\end{bmatrix}=\begin{bmatrix}\mathrm{T(A-BK)T^{-1}} & \mathrm{TBKT^{-1}}\\ 0 & \mathrm{T(A}-LC)\mathrm{T}^{-1}\end{bmatrix}\begin{bmatrix}z(t)\\E(t)\end{bmatrix}+\begin{bmatrix}\mathrm{T}Df(\mathrm{T}^{-1}z(t))+\mathrm{T}g(\mathrm{T}^{-1}z(t),t)\\ \xi(\mathrm{T}^{-1}z(t),\mathrm{T}^{-1}\hat{z}(t),t)\end{bmatrix}$$

where

$$\xi(\mathrm{T}^{-1}z(t),\mathrm{T}^{-1}\hat{z}(t),t)=\mathrm{T}Df(\mathrm{T}^{-1}z(t))-\mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))+\mathrm{T}g(\mathrm{T}^{-1}z(t),t)-\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)$$

the response to the initial condition can be determined from

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{E}_1\\ \dot{E}_2\end{bmatrix}=\begin{bmatrix}0 & 1.5 & 0 & 0\\ -0.6667 & -1.4142 & 0.6667 & 1.4142\\ 0 & 0 & 10.3429 & 17.0143\\ 0 & 0 & -10.6665 & -15.9997\end{bmatrix}\begin{bmatrix}z_1\\z_2\\E_1\\E_2\end{bmatrix}+\begin{bmatrix}30\sin\left(\frac{z_1}{30}\right)+0.6z_1\cos^3(t)\\ 20\cos\left(\frac{z_1}{30}\right)+z_2\sin^2(t)\\ 30\left(\sin\left(\frac{z_1}{30}\right)-\sin\left(\frac{z_1-E_1}{30}\right)\right)+0.6\left(z_1-(z_1-E_1)\right)\cos^3(t)\\ 20\left(\cos\left(\frac{z_1}{30}\right)-\cos\left(\frac{z_1-E_1}{30}\right)\right)+\left(z_2-(z_2-E_2)\right)\sin^2(t)\end{bmatrix} \tag{3.74}$$

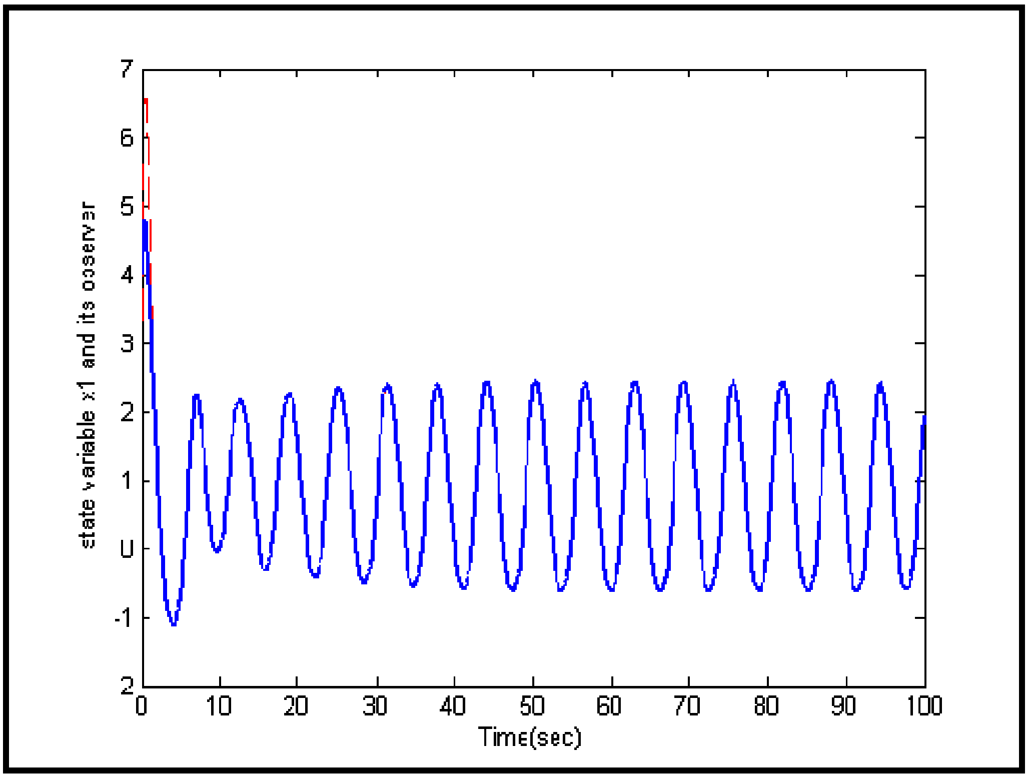


Fig (3.11) Observer performance: state variable x_1 ***(solid curve). and its observer*** $\hat{x}_1$ ***(broken curve) for*** $x_1(0)=4$, $\hat{x}_1(0)=3.33$ ***of problem (3.3).***

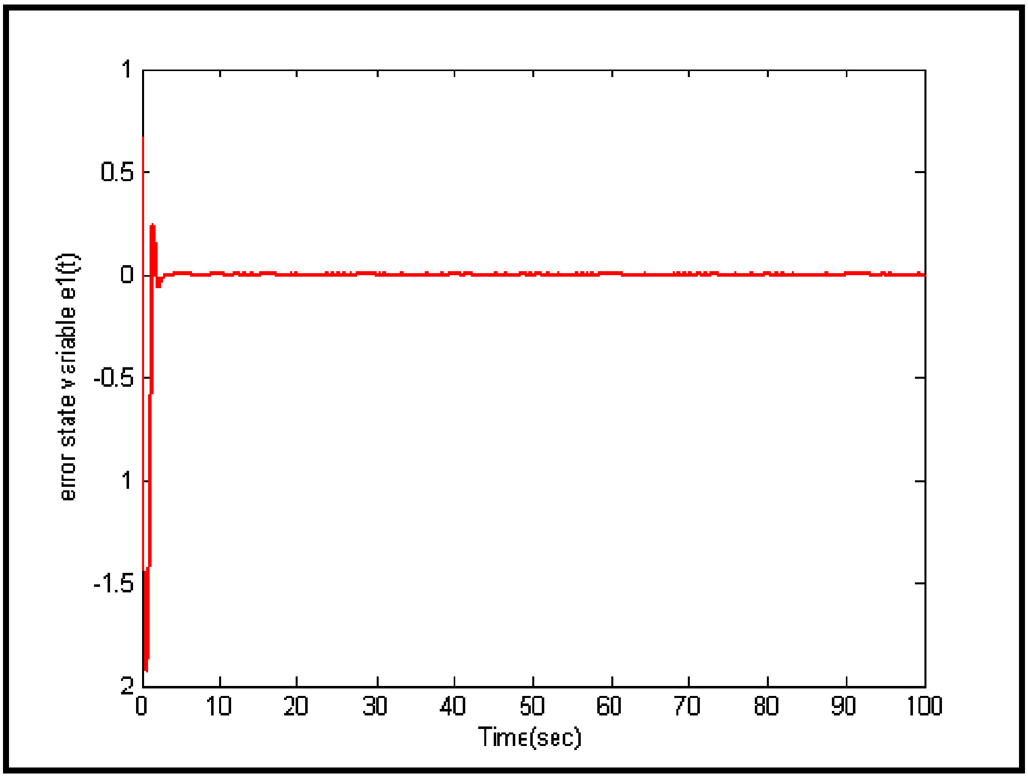


Fig (3.12) error between $x_1(t)$ ***and its observer*** $\hat{x}_1(t)$ ***of problem (3.3).***

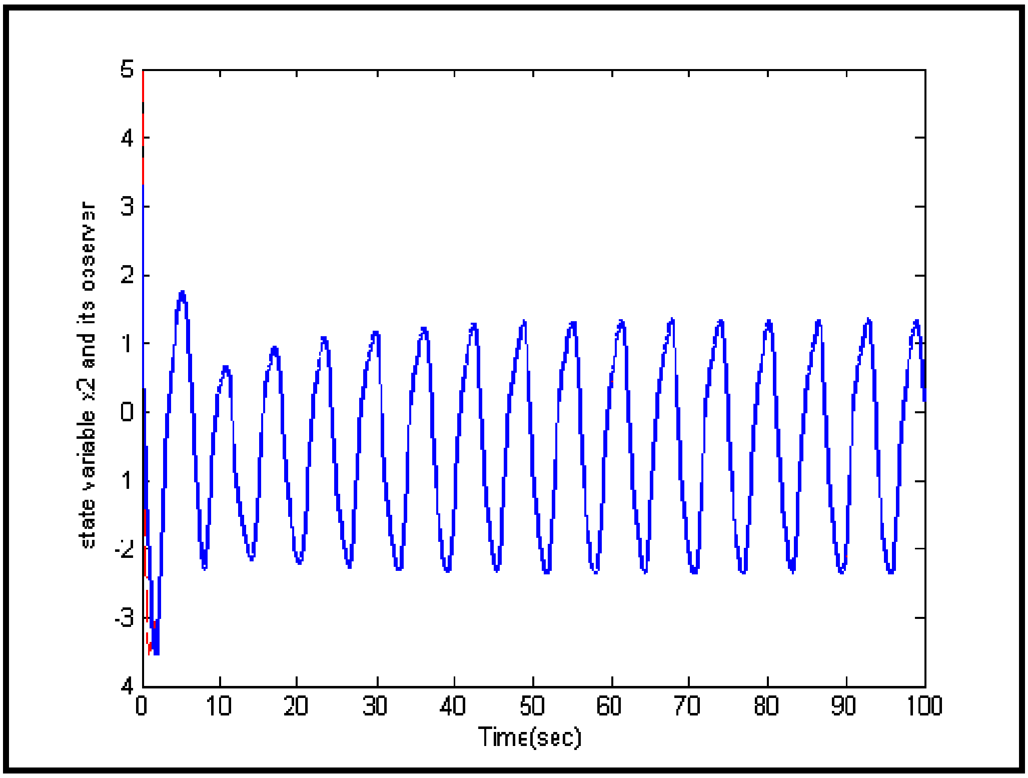


Fig (3.13) Observer performance: state variable x_2 (solid curve). and its observer $\hat{x}_2$ (broken curve) for $x_2(0)=3.3$, $\hat{x}_2(0)=4.95$ of problem (3.3).

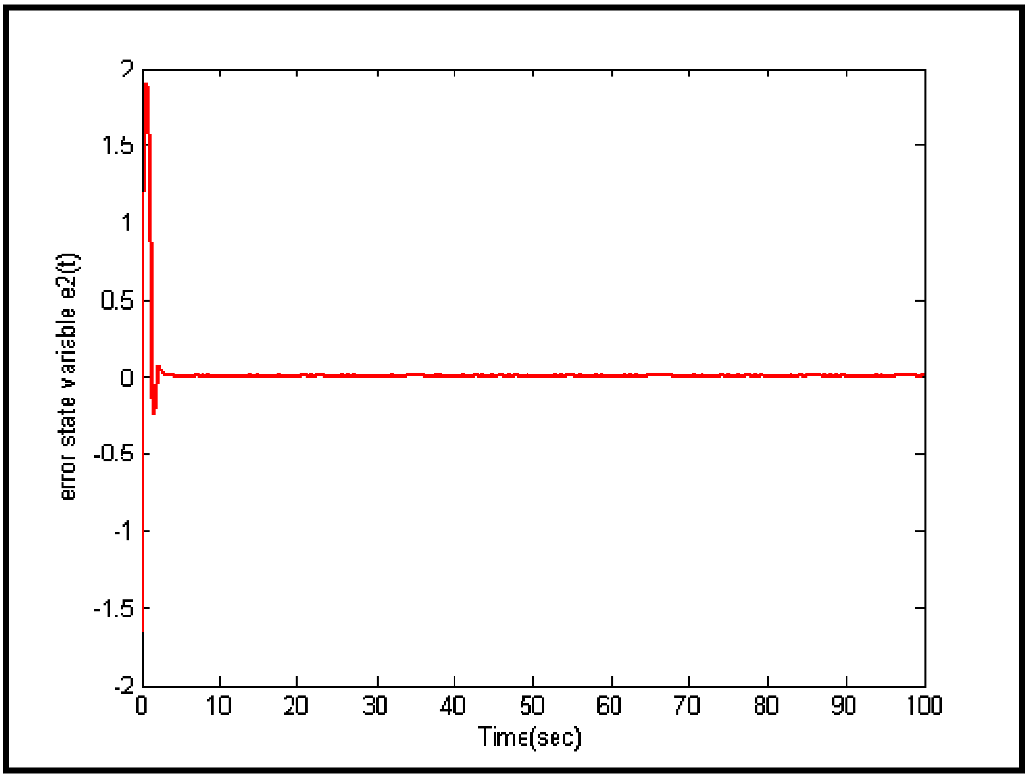


Fig (3.14) error between $x_2(t)$ and its observer $\hat{x}_2(t)$ of problem (3.3).

This chapter represents an effort towards developing a suitable method to design and implement observers for inherently nonlinear dynamical control systems. These systems are driven by nonlinear functions which are Lipschitz in nature. The underlying theory makes use of the methods developed for the quadratic stabilization of uncertain systems.

A computational algorithm based on theorem for design a deterministic state dynamic observer has been presented. Observer-based control law for non-linear system is studied, several problems are demonstrated the validity of our results. The proposed theory is used to design an observer for a single-link flexible joint robot.

2.1 MATHEMATICAL PRELIMINARIES

The following are some necessary mathematical principal that will be needed in our work.

Remarks (2.1) [4]:

1. The Euclidean norm of $n \times n$ matrix can be defined as:

$$\|\mathrm{A}\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} \right|^2 \right)^{1/2}$$

where $|a_{ij}|$ is the absolute value of the matrix element a_{ij}.

2. The norm of $n \times n$ matrix is also defined as:

$$\|\mathrm{A}\| = \left(\lambda_{\max} (\mathrm{A}^{\mathrm{T}} \mathrm{A}) \right)^{1/2}$$

where A^{T} is the transpose of A, $\lambda_{\max}$ is the maximum eigenvalue of $(\mathrm{A}^{\mathrm{T}} \mathrm{A})$, provided that $\mathrm{A}^{\mathrm{T}} \mathrm{A}$ is positive semi definite matrix.

Lemma (2.1) [4]:

Let A be a hermitian matrix and let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and maximum eigenvalues of A, respectively, then:

$$\lambda_{\min(\mathrm{A})}\|x\|^2 \le x^* \mathrm{A}\, x \le \lambda_{\max(\mathrm{A})}\|x\|^2$$

for any x in the n-dimensional complex vector space C^n, where

$\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$, x_i is the i-th component of x.

Definition (2.1) [24]:

A (vector-valued) function $f(x)$ is said to be globally Lipschitz if there exists a constant γ such that for all $x_1, x_2 \in R^n$, the following inequality holds:

$$\| f(x_1) - f(x_2)\| \le \gamma \|x_1 - x_2\|.$$

In this case γ is said to be the Lipschitz constant of f.

Remark (2.2) [24]:

If $f(x), x \in R^n$ is differentiable function with bounded partial derivatives, then γ is simply is the upper bound of the norm of the Jacobian matrix for the function $f(x)$, the upper bound taken over the entire R^n. However, in general, a Lipschitz function may not be differentiable.

2.2 SYSTEM DESCRIPTION AND MOTIVATION

The suggested non-linear dynamical control system described as follows:

$$\begin{aligned} &\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t), t) \\ &y(t) = C\,x(t) \qquad (2.1) \\ &x(0) = x_0 \end{aligned}$$

where $x(t) \in R^n$ is unmeasurable state vector, $u(t)$ is the control input and $y(t) \in R^m$ is the output vector. Suppose that the matrices A, B, C and D have a constant entries of appropriate dimensions. The non-linear functions $Df(x(t)) : R^n \to R^n$ and $g(x(t),t) : R^n \times R \to R^n$ are assumed to be globally Lipschitz in *x* with a Lipschitz constants γ and β, respectively.

The assumption that Df and g are Lipschitz globally may be relaxed to assume that Df and g are only locally Lipschitz.

Systems of type (2.1) are common mechanical systems frequently contain Lipschitz-type nonlinearities-trignometric nonlinearities which occur in robotic applications, a non-linear softening spring,…,etc. Non-linearities which are square or cubic in nature are not globally Lipschitz: however, they are locally so. Moreover, when such functions occur in physical system, they frequently have a saturation in their growth rate, making them globally Lipschitz functions. Frequently, we make the system measurements part of the system state, so the assumption that the output is linear in the state is justified. The assumption that the system dynamics are linear in the input is usually true for mechanical systems because the input is usually a torque or force which enters the dynamics linearly due to form of Newton's laws. In summary, while the class of systems we are addressing is not exhaustive, it is fairly large from an engineering point-of-view, as well as the mathematics.

Since the state variables of a non-linear dynamical control system (2.1) are not available for measurement as an output, then a dynamical state observer of non-linear dynamical control system (2.1) is constructed as follows:

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &= \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - C\,\hat{x}(t)) \\ \hat{y}(t) &= C\,\hat{x}(t) \end{aligned} \qquad (2.2)$$

where the observed state is denoted by $\hat{x}(t)$, and L is the observer gain matrix with an appropriate dimension.

Define:

$$e(t) = x(t) - \hat{x}(t) \tag{2.3}$$

Clearly, $e(t)$ is the dynamical error between the actual state $x(t)$ and state observer $\hat{x}(t)$. Then, the dynamical error in state observer (2.2) of the non-linear dynamical control system (2.1) and has the following dynamic equation:

$$\frac{de(t)}{dt} = (\mathrm{A} - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t) \tag{2.4}$$

$$e(0) = x(0) - \hat{x}(0) \tag{2.5}$$

If the dynamic behavior of dynamical error (2.4) is asymptotically stable, then the dynamical error (2.4) will tend to zero with an adequate speed as the time tend to infinity. That is, the unmeasurable actual state $x(t)$ given in (2.1), will converge to the state observer $\hat{x}(t)$, given in (2.2), regardless of the values of $x(0)$ and $\hat{x}(0)$ as t tends to infinity.

The following theorem is developed to design the nonlinear dynamic state observer (2.2) for the presented problem (2.1).

<u>Theorem (2.1)</u>

Consider the non-linear dynamical system (2.1)

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t),t)$$

$$y(t) = C\,x(t)$$

$$x(0) = x_0$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$, $\mathrm{A} \in R^{n\times n}$, $\mathrm{B} \in R^{n\times p}$, $D \in R^{n\times n}$, $C \in R^{m\times n}$, $f : R^n \to R^n$, $g : R^n \times R \to R^n$, and assume that the state variables are not available for measurement. Consider the observer of non-linear dynamical control system (2.1) is given in (2.2)

$$\frac{d\hat{x}(t)}{dt} = \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - C\,\hat{x}(t))$$
$$\hat{y}(t) = C\,\hat{x}(t)$$
$$\hat{x}(0) = \hat{x}_0$$

The following conditions are assumed to be satisfied

1. The pair (A, C) of a non-linear dynamical control system (2.1), is completely state observer.
2. The non-linearity function $Df(x(t)) : R^n \to R^n$ is assumed to be globally Lipschitz condition with Lipschitz constant γ, i.e.,
$$\|Df(x(t)) - Df(\hat{x}(t))\| \le \gamma \|x(t) - \hat{x}(t)\| \tag{2.6}$$
3. The non-linearity function $g(x(t),t) : R^n \times R \to R^n$ is assumed to be globally Lipschitz condition with Lipschitz constant β, i.e.,
$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le \beta \|x(t) - \hat{x}(t)\| \text{ for } t \in R \tag{2.7}$$
4. The observer gain L can be selected such that $(\mathrm{A} - LC)$ is asymptotically stable matrix.
5. The Riccati equation $(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$ (2.8) has a unique positive definite solution P for arbitrary positive definite selection matrix Q.
6. On using the Lyapunov function stability $V(e(t)) = e^{\mathrm{T}}(t)Pe(t)$, where $e(t) = x(t) - \hat{x}(t)$ and P satisfy equation (2.8)

If

$$\gamma + \beta < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \tag{2.9}$$

Then the dynamical error (2.4)

$$\frac{de(t)}{dt} = (\mathrm{A} - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t)$$

is asymptotically stable via a single observer gain parameter L.

Proof

The non-linear dynamical control system (2.1)

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t),t)$$
$$y(t) = C\,x(t)$$
$$x(0) = x_0$$

the state observer of non-linear dynamical control system (2.1) is given as follows (2.2)

$$\frac{d\hat{x}(t)}{dt} = \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - C\,\hat{x}(t))$$
$$\hat{y}(t) = C\,\hat{x}(t)$$
$$\hat{x}(0) = \hat{x}_0$$

The dynamical error in state observer (2.2) of non-linear dynamical control system is obtained to subtract (2.2) from (2.1) as follows

$$\frac{de(t)}{dt} = (\mathrm{A} - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t)$$

where

$$e(0) = x(0) - \hat{x}(0)$$

and thus

$$\dot{e}^{\mathrm{T}}(t) = e^{\mathrm{T}}(t)(\mathrm{A} - LC)^{\mathrm{T}} + (Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}} + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}} \tag{2.10}$$

To examine the stability of $e(t)$, we consider the following quadratic Lyapunov function,

$$V(t) \equiv V(e(t)) = e^{\mathrm{T}}(t)P\,e(t)$$

$$\frac{dV(t)}{dt} = \dot{e}^{\mathrm{T}}(t)P\,e(t) + e^{\mathrm{T}}(t)P\,\dot{e}(t)$$

On using (2.4) and (2.10) we have that

$$\frac{dV(t)}{dt} = [e^{\mathrm{T}}(t)(\mathrm{A}-LC)^{\mathrm{T}} + (Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}} + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}]$$
$$P\,e(t) + e^{\mathrm{T}}(t)P[(\mathrm{A}-LC)e(t) + (Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))]$$

$$\frac{dV(t)}{dt} = e^{\mathrm{T}}(t)[(\mathrm{A}-LC)^{\mathrm{T}}P + P(\mathrm{A}-LC)]e(t) + (Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}Pe(t)$$
$$+ e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t)) \tag{2.11}$$

From (2.8) we have

$$\frac{dV(t)}{dt} = e^{\mathrm{T}}(t)(-Q)e(t) + (Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}P\,e(t) + e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}P\,e(t) + e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t)) \tag{2.12}$$

Now

$$(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}P\,e(t) + e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t)) \le \Big\|(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}} Pe(t) + e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t))) + (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t))\Big\| \le \|(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}Pe(t)\| + \|e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t)))\| + \|(g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}}Pe(t)\| + \|e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t))\| \tag{2.13}$$

The first part of (2.13) can be simplified as follows:

$$\|(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}Pe(t)\| \le \|Df(x(t)) - Df(\hat{x}(t))\|\,\|P\|\,\|e(t)\|$$

Since $Df(x(t))$ satisfy Lipschitz condition with Lipschitz constant γ i.e,.

$$\|Df(x(t)) - Df(\hat{x}(t))\| \le \gamma\|x(t) - \hat{x}(t)\| = \gamma\|e(t)\|$$

then

$$\|(Df(x(t)) - Df(\hat{x}(t)))^{\mathrm{T}}Pe(t)\| \le \gamma\|e(t)\|^2\lambda_{\max}(P) \tag{2.14}$$

similarly, the second part of (2.13) becomes

$$\left\| e^{\mathrm{T}}(t)P(Df(x(t)) - Df(\hat{x}(t)))\right\| \le \gamma \left\| e(t)\right\|^2 \lambda_{\max}(P) \tag{2.15}$$

and hence the third part of (2.13) can be simplified as follows:

$$\left\| (g(x(t),t) - g(\hat{x}(t),t)^{\mathrm{T}} Pe(t)\right\| \le \left\| g(x(t),t) - g(\hat{x}(t),t)\right\| \left\| P\right\| \left\| e(t)\right\|$$

Since $g(x(t),t)$ satisfy Lipschitz condition on x with Lipschitz constant β i.e,

$$\left\| g(x(t),t) - g(\hat{x}(t),t)\right\| \le \beta \left\| x(t) - \hat{x}(t)\right\| = \beta \left\| e(t)\right\|$$

then

$$\left\| (g(x(t),t) - g(\hat{x}(t),t))^{\mathrm{T}} Pe(t)\right\| \le \beta \left\| e(t)\right\|^2 \lambda_{\max}(P) \tag{2.16}$$

Similarly, the fourth part of (2.13) becomes

$$\left\| e^{\mathrm{T}}(t)P(g(x(t),t) - g(\hat{x}(t),t))\right\| \le \beta \left\| e(t)\right\|^2 \lambda_{\max}(P) \tag{2.17}$$

where $\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

Substituting (2.14),(2.15),(2.16) and (2.17) into (2.12) gives:

$$\frac{dV(t)}{dt} \le e^{\mathrm{T}}(t)(-Q)e(t) + 2(\gamma + \beta)\left\| e(t)\right\|^2 \lambda_{\max}(P) \tag{2.18}$$

with $e^{\mathrm{T}}(t)\,Q\,e(t) \ge \lambda_{\min}(Q)\left\| e(t)\right\|^2$ (2.19)

One deduces from (2.19):

$$\frac{dV(t)}{dt} \le (-\lambda_{\min}(Q) + 2(\gamma + \beta)\lambda_{\max}(P))\left\| e(t)\right\|^2 \tag{2.20}$$

From (2.20) and (2.9) we have

$$\frac{dV(e(t))}{dt} < 0 \tag{2.21}$$

since P is unique positive definite solution and it is clear that $V(e(t)) = e^{\mathrm{T}}(t)Pe(t) > 0$, $V(0) = 0$ and by (2.21) we have conclude that the error dynamic system (2.4) is asymptotically stable via a single observer gain parameter L. Thus $x(t) \cong \hat{x}(t)$ as $t \to \infty$.

The state observer $\hat{x}(t)$ converges to the actual state of a non-linear system via an observer gain parameter L.

Algorithm (2.1)

The following algorithm is presented in order to design a deterministic observer that estimates the original non-linear dynamical states given in (2.1). Based on the result of the main theorem (2.1).

Step (0): Consider the non-linear dynamical system

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t),t)$$
$$y(t) = C\,x(t)$$
$$x(0) = x_0$$

where $x(t) \in R^n$ is unmeasurable state vector, $u(t) \in R^p$ is the control input and $y(t) \in R^m$ is the output vector. Suppose that the matrices A, B, C and D have a constant entries and appropriate dimensions. The non-linearity functions $Df(x(t)) : R^n \to R^n$ and $g(x(t),t) : R^n \times R \to R^n$ are assumed to be globally Lipschitz in x with a Lipschitz constants γ and β respectively.

Step (1): If (A, C) is observable, go to ***Step(2)***, otherwise, the system should be modified to satisfy the observable condition.

Step (2): Check the following Lipschitz conditions

$$\|Df(x(t)) - Df(\hat{x}(t))\| \le \gamma \|x(t) - \hat{x}(t)\|$$
$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le \beta \|x(t) - \hat{x}(t)\| \quad \text{for } t \in R \ , x \in R^n$$

and design the observer dynamic by

$$\frac{d\hat{x}(t)}{dt} = \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - C\,\hat{x}(t))$$
$$\hat{y}(t) = C\,\hat{x}(t)$$
$$\hat{x}(0) = \hat{x}_0$$

Step (3): Select L that makes $(\mathrm{A} - LC)$ asymptotically stable by using dual of the pole placement. (see subsection 1.5.4).

Step (4): Let the dynamic error $e(t) = x(t) - \hat{x}(t)$ and $e(0) = x(0) - \hat{x}(0)$

$$\frac{de(t)}{dt} = (\mathrm{A} - LC)e(t) + Df(x(t)) - Df(\hat{x}(t)) + g(x(t),t) - g(\hat{x}(t),t)$$
$$e(0) = x(0) - \hat{x}(0)$$

Step (5): Set $V(t) \equiv V(e(t)) = e^{\mathrm{T}}(t)P\,e(t)$

where P is the unique positive definite solution of $(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$ for arbitrary positive definite matrix Q. (Q is designed depending on decision maker).

Step (6): Check $\gamma + \beta < \dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$, where γ and β are found in step (2). and $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q ,$\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

If ***step (6)*** is not satisfied go to ***step (3)*** and choose another L.

2.3 OBSERVER-BASED CONTROL LAW FOR NON-LINEAR DYNAMICAL CONTROL SYSTEM

Consider a non-linear dynamical control system (2.1)

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t),t)$$
$$y(t) = C\,x(t)$$
$$x(0) = x_0$$

is completely state observer and completely state controllable. If the actual state $x(t)$ is not available for a feedback $u = -\mathrm{K}x$, then it is designed state observer for inherently non-linear dynamical control system (2.2) as follows:

$$\frac{d\hat{x}(t)}{dt} = \mathrm{A}\hat{x}(t) + \mathrm{B}u(t) + Df(\hat{x}(t)) + g(\hat{x}(t),t) + L(y(t) - C\,\hat{x}(t))$$
$$\hat{y}(t) = C\,\hat{x}(t)$$
$$\hat{x}(0) = \hat{x}_0$$

(2.22)

where L is the observer gain that has been developed in theorem 2.1

Thus, it is natural to apply the feedback gain parameter K, on the state observer $\hat{x}(t)$ as follows:

$$u(t) = -\mathrm{K}\hat{x}(t) \tag{2.23}$$

as shown in Fig. (2.1)

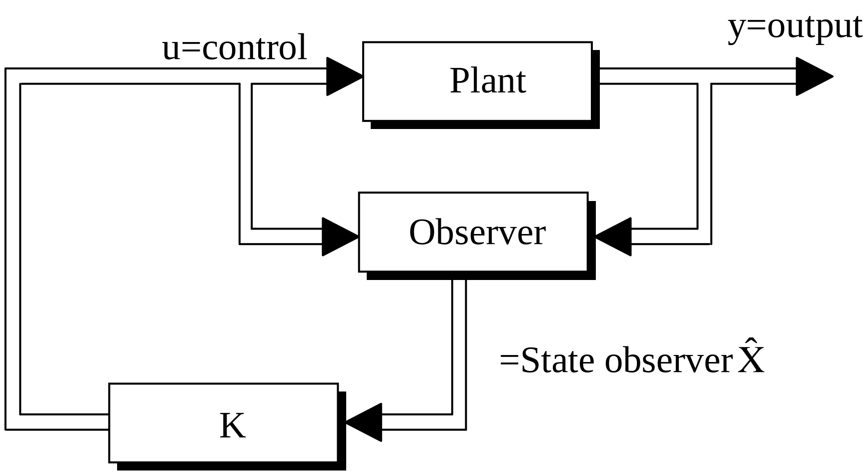


Fig (2.1) Feed-back from observer state

Equation (2.1) with this control (2.23) will be obtained the closed-loop of a non-linear dynamic system as follows:

$$\frac{dx(t)}{dt} = (\mathrm{A} - \mathrm{BK})x(t) + Df(x(t)) + g(x(t), t) + \mathrm{BK}(x(t) - \hat{x}(t)) \tag{2.24}$$

The difference between the actual state $x(t)$ and the state observer $\hat{x}(t)$ of a non-linear dynamical control system (2.1) stands for the dynamical error in state observer:

$$e(t) = x(t) - \hat{x}(t) \tag{2.25}$$

Substituting of dynamical error vector into equation (2.24)

$$\frac{dx(t)}{dt} = (\mathrm{A} - \mathrm{BK})x(t) + \mathrm{BK}e(t) + Df(x(t)) + g(x(t), t) \tag{2.26}$$

Note that, the dynamical error in state observer (2.2) for inherently non-linear dynamical control system is obtained by subtracting (2.2) from (2.1) as follows:

$$\dot{e}(t) = (\mathrm{A} - LC)e(t) + \xi(x(t), \hat{x}(t), t) \tag{2.27}$$

where

$$\xi(x(t), \hat{x}(t), t) = Df(x(t)) - Df(\hat{x}(t)) + g(x(t), t) - g(\hat{x}(t), t)$$

When considering (2.27), it is seems that $e(t)$ converges to zero, independent of initial state, if the observer gain L can be found that makes (2.27) asymptotically stable. As we known from theorem 2.1, such observer gain L often can be found.

Next, we consider (2.26), if (2.26) is verified the sufficient conditions for theorem 2.1 and $e(t) \to 0$ as $t \to \infty$.

Then, the closed loop of a non-linear dynamical system (2.26) is asymptotically stable:

From (2.26) and (2.27), we have:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{A} - \mathrm{BK} & \mathrm{BK} \\ 0 & \mathrm{A} - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} Df(x(t)) + g(x(t), t) \\ \xi(x(t), \hat{x}(t), t) \end{bmatrix} \tag{2.28}$$

equation (2.28) described a dynamic of the state observer feedback control.

Hence, as far as the sufficient conditions gives a theorem 2.1 is concerned, one concludes from (2.28) that the roots of the combined system in Fig (2.1) consist of the sum of the control roots and the estimator roots. The control roots are unchanged from those obtained by assuming state feedback $x(t)$. Hence, the control law and the observer can be designed separately and then used jointly.

The final matter to be settled is the specification of the desired roots of the observer characteristic equation. The estimation error decays at a rate dependent on these roots. In Eq. (2.26), we require the observer error $e(t)$ to decay at fast rate with time constants that are much smaller than the time constants of the controlled system so that the total response is dominated by the slower control roots. Hence, the observer roots should be placed to the left of the control roots in the complex plane. But if the observer roots are placed too far to the left of the control roots, then the observer gains represented by the elements of L will be high. Hence, the measurement noise will not be filtered out and may even be amplified.

Clearly, a compromise is required in selecting the roots of the observer characteristic equation. Optimal estimation theory can be employed for this purpose (Bryson and Ho, 1969) ***[3]***. A rule of thumb is to let $-\eta_i$ be approximately equal to $-4\mu_i$, where $-\eta_i$ are the observer roots, and $-\mu_i$ are the control roots. The control roots are of course chosen to satisfy the performance requirements.

The controller, including the control law and the observer, can be constructed with analog components, such as the operational amplifiers. It is expected that a controller with state feedback would be more expensive than a controller with output feedback. However, digital computer implementation of the controller with state feedback involves software and hence would be cost effective.

Computationally, one can follow the conceptual procedure below to evaluate the single linear state observer feedback control which stabilize the non-linear dynamical control system.

Remark (2.3)

The design a single linear state observer feedback control which stabilize the non-linear dynamical control system (2.1) becomes two stages process as follows:

1. The first stage being the determination feedback gain K.
2. The second stage being the determination of the observer gain L such that the inequality (2.9) is satisfied.

Remark (2.4)

If the second stage of the procedure to obtain linear state observer feedback control does not satisfied then take the nonsingular transformation $z = \mathrm{T}x$ where $\mathrm{T} = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_i \neq 0,$ $i = 1,2,\cdots n$. The transformation used to prove the stability of course, for stability not only do the value of γ and β matter but also how exactly the matrix A gets transformed, in the sense that the structure of the transformed matrix manifests itself in the form of the new Lyapunov solution $\overline{P}$. We shall study the transformed system in the following lemma:

Lemma (2.2)

Consider the non-linear dynamical system (2.1)

$$\frac{dx(t)}{dt} = \mathrm{A}x(t) + \mathrm{B}u(t) + Df(x(t)) + g(x(t), t)$$
$$y(t) = C\,x(t)$$
$$x(0) = x_0$$

where the conditions in theorem (2.1), are satisfied, and consider the following nonsingular state transformation $z = \mathrm{T}x$, with $\mathrm{T} = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_i \neq 0$, $i = 1,2,\ldots,n$. The non-linear dynamical system (2.1), represented in the new coordinates is given by:

$$\frac{dz(t)}{dt} = \overline{\mathrm{A}}z(t) + \overline{\mathrm{B}}u(t) + \mathrm{T}Df(\mathrm{T}^{-1}z(t)) + \mathrm{T}g(\mathrm{T}^{-1}z(t), t)$$
$$Y(t) = \overline{C}\,z(t) \tag{2.29}$$
$$z(0) = z_0$$

where $z(t) \in R^n$ is the state vector, $u(t) \in R^p$ is the input vector, $Y(t) \in R^m$ is the output vector. $\overline{\mathrm{A}} = \mathrm{TAT}^{-1} \in R^{n\times n}$, $\overline{\mathrm{B}} = \mathrm{TB} \in R^{n\times p}$, $\overline{C} = C\mathrm{T}^{-1} \in R^{m\times n}$, $\mathrm{TD} \in R^{n\times n}$. Assume the following conditions are satisfied.

1. The pair $(\bar{A}, \bar{C})$ of a non-linear dynamical control system (2.29), is completely state observer.
2. The non-linearity function $TDf(T^{-1}z(t)): R^n \to R^n$ is assumed to be globally Lipschitz condition with Lipschitz constant $\bar{\gamma}$, i.e.,
$$\left\| TDf(T^{-1}z(t)) - TDf(T^{-1}\hat{z}(t)) \right\| \le \bar{\gamma} \left\| T^{-1}z(t) - T^{-1}\hat{z}(t) \right\| \tag{2.30}$$
3. The non-linearity function $Tg(T^{-1}z(t), t): R^n \times R \to R^n$ is assumed to be globally Lipschitz condition in the first argument with Lipschitz constant $\bar{\beta}$, i.e.,
$$\left\| Tg(T^{-1}z(t), t) - Tg(T^{-1}\hat{z}(t), t) \right\| \le \bar{\beta} \left\| T^{-1}z(t) - T^{-1}\hat{z}(t) \right\| \quad \text{for } t \in R \tag{2.31}$$
4. The suggested non-linear dynamic observer
$$\begin{aligned} \frac{d\hat{z}(t)}{dt} &= \bar{A}\hat{z}(t) + \bar{B}u + TDf(T^{-1}\hat{z}(t)) + Tg(T^{-1}\hat{z}(t), t) + \bar{L}(Y(t) - \bar{C}\hat{z}(t)) \\ \hat{Y}(t) &= \bar{C}\hat{z}(t) \\ \hat{z}(0) &= \hat{z}_0 \end{aligned} \tag{2.32}$$
where the observed state is denoted by $\hat{z}(t)$ and $\bar{L} \in R^{n\times m}$ is the observer gain matrix.
5. The observer gain $\bar{L}$ can be selected such that $(\bar{A} - \bar{L}\bar{C})$ is asymptotically stable matrix.
6. The Riccati equation $(\bar{A} - \bar{L}\bar{C})^T \bar{P} + \bar{P}(\bar{A} - \bar{L}\bar{C}) = -Q$ (2.33)
has a unique positive definite solution $\bar{P}$ for arbitrary positive definite selection matrix Q.
7. On using the Lyapunov function stability $V(E(t)) = E^T(t)\bar{P}E(t)$, where $E(t) = z(t) - \hat{z}(t)$ and $\bar{P}$ satisfy equation (2.33)

If

$$\bar{\gamma}+\bar{\beta}<\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(\bar{P})\lambda_{\max}(\mathrm{T}^{-1})} \tag{2.34}$$

Then the dynamical error is asymptotically stable via a single observer gain parameter $\bar{L}$.

Proof

Let $E(t)=z(t)-\hat{z}(t)$ (2.35)

on simple calculations one can have that

$$\frac{dE(t)}{dt}=(\bar{\mathrm{A}}-\bar{L}\bar{C})E(t)+\mathrm{T}Df(\mathrm{T}^{-1}z(t))-\mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))+\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$$
$$-\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t) \tag{2.36}$$

and

$$\dot{E}^{\mathrm{T}}(t)=E^{\mathrm{T}}(t)(\bar{\mathrm{A}}-\bar{L}\bar{C})^{\mathrm{T}}+(\mathrm{T}Df(\mathrm{T}^{-1}z(t))-\mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)))^{\mathrm{T}}+$$
$$(\mathrm{T}g(\mathrm{T}^{-1}z(t),t)-\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t))^{\mathrm{T}} \tag{2.37}$$

As discussed previously in the main theorem (2.1), let

$$V(t)\equiv V(E(t))=E^{\mathrm{T}}(t)\bar{P}E(t)$$

and thus

$$\frac{dV(t)}{dt}=\dot{E}^{\mathrm{T}}(t)\bar{P}\,E(t)+E^{\mathrm{T}}(t)\bar{P}\,\dot{E}(t)$$

On using (2.36) and (2.37) we have that

$$\frac{dV}{dt}=E^{\mathrm{T}}(t)((\bar{\mathrm{A}}-\bar{L}\bar{C})\bar{P}+\bar{P}(\bar{\mathrm{A}}-\bar{L}\bar{C}))E(t)+(\mathrm{T}Df(\mathrm{T}^{-1}z(t))-\mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)))^{\mathrm{T}}$$
$$\bar{P}E(t)+E^{\mathrm{T}}(t)\bar{P}(\mathrm{T}Df(\mathrm{T}^{-1}z(t))-\mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)))+(\mathrm{T}g(\mathrm{T}^{-1}z(t),t)-$$
$$\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t))^{\mathrm{T}}\bar{P}E(t)+E^{\mathrm{T}}(t)\bar{P}(\mathrm{T}g(\mathrm{T}^{-1}z(t),t)-\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)) \tag{2.38}$$

From (2.33), we have

$$\frac{dV}{dt} = E^{\mathrm{T}}(t)(-Q)E(t) + (\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)))^{\mathrm{T}}\overline{P}E(t) + E^{\mathrm{T}}\overline{P}$$
$$(\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))) + (\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)^{\mathrm{T}}$$
$$\overline{P}E(t) + E^{\mathrm{T}}(t)\overline{P}(\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)) \tag{2.39}$$

From (2.30) and (2.35) one deduces.

$$(\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)))^{\mathrm{T}}\overline{P}E(t) \le \overline{\gamma}\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})\|E(t)\|^2 \tag{2.40}$$

$$E^{\mathrm{T}}(t)\overline{P}(\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))) \le \overline{\gamma}\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})\|E(t)\|^2 \tag{2.41}$$

From (2.31) and (2.35) one deduces.

$$(\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t))^{\mathrm{T}}\overline{P}E(t) \le \overline{\beta}\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})\|E(t)\|^2 \tag{2.42}$$

$$E^{\mathrm{T}}(t)\overline{P}(\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)) \le \overline{\beta}\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})\|E(t)\|^2 \tag{2.43}$$

Substituting (2.40),(2.41),(2.42) and (2.43) into (2.39) gives:

$$\frac{dV(t)}{dt} \le E^{\mathrm{T}}(t)(-Q)E(t) + 2(\overline{\gamma} + \overline{\beta})\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})\|E(t)\|^2 \tag{2.44}$$

with $E^{\mathrm{T}}(t)\,Q\,E(t) \ge \lambda_{\min}(Q)\|E(t)\|^2$ (2.45)

One deduces from (2.45):

$$\frac{dV(t)}{dt} \le (-\lambda_{\min}(Q) + 2(\overline{\gamma} + \overline{\beta})\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1}))\|E(t)\|^2 \tag{2.46}$$

From (2.46) and (2.34) we have

$$\frac{dV(E(t))}{dt} < 0 \tag{2.47}$$

since $\overline{P}$ is unique positive definite solution and its clear that $V(E(t)) > 0$, $V(0) = 0$ and by (2.47) we have conclude that the error dynamic system (2.36) is asymptotically stable via a single observer gain parameter $\overline{L}$. Thus $z(t) \cong \hat{z}(t)$ as $t \to \infty$.

Algorithm (2.2)

Step (0): Consider the non-linear dynamical system

$$\frac{dz(t)}{dt} = \bar{\mathrm{A}}z(t) + \bar{\mathrm{B}}u(t) + \mathrm{T}Df(\mathrm{T}^{-1}z(t)) + \mathrm{T}g(\mathrm{T}^{-1}z(t),t)$$
$$Y(t) = \bar{C}\,z(t)$$
$$z(0) = z_0$$

where $z(t)\in R^n$ is unmeasurable state vector, $u(t)$ is the control input vector and $Y(t)\in R^m$ is the output vector. Suppose that the matrices $\bar{\mathrm{A}} = \mathrm{TAT}^{-1}$, $\bar{\mathrm{B}} = \mathrm{TB}$, $\bar{C} = C\mathrm{T}^{-1}$ and TD have a constant entries and appropriate dimensions. The non-linearity functions $\mathrm{T}Df(\mathrm{T}^{-1}z(t)) : R^n \to R^n$ and $\mathrm{T}g(\mathrm{T}^{-1}z(t),t) : R^n \times R \to R^n$ are assumed to be globally Lipschitz with a Lipschitz constants $\bar{\gamma}$ and $\bar{\beta}$ respectively.

Step (1): Check the following Lipschitz conditions

$$\left\|\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))\right\| \le \bar{\gamma}\left\|\mathrm{T}^{-1}z(t) - \mathrm{T}^{-1}\hat{z}(t)\right\|$$
$$\left\|\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)\right\| \le \bar{\beta}\left\|\mathrm{T}^{-1}z(t) - \mathrm{T}^{-1}\hat{z}(t)\right\| \quad \text{for } t\in R$$

where

$$\frac{d\hat{z}(t)}{dt} = \bar{\mathrm{A}}\hat{z}(t) + \bar{\mathrm{B}}u + \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)) + \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t) + \bar{L}(Y(t) - \bar{C}\hat{z}(t))$$
$$\hat{Y}(t) = \bar{C}\hat{z}(t)$$
$$\hat{z}(0) = \hat{z}_0$$

Step (2): Compute $\bar{L} = \mathrm{T}L$ that makes $(\bar{\mathrm{A}} - \bar{L}\bar{C})$ asymptotically stable matrix.

Step (3): Let the dynamic error $E(t) = z(t) - \hat{z}(t)$ and $E(0) = z(0) - \hat{z}(0)$

$$\frac{dE(t)}{dt} = (\bar{\mathrm{A}} - \bar{L}\bar{C})E(t) + \mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)) + \mathrm{T}g(\mathrm{T}^{-1}z(t),t) -$$
$$\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)$$
$$E(0) = z(0) - \hat{z}(0)$$

Step (4): Set $V(t) \equiv V(E(t)) = E^{\mathrm{T}}(t)\bar{P}\,E(t)$

where $\bar{P}$ is the unique positive definite solution of $(\bar{\mathrm{A}}-\bar{L}\bar{C})^{\mathrm{T}}\bar{P}+\bar{P}(\bar{\mathrm{A}}-\bar{L}\bar{C})=-Q$, for arbitrary selection positive definite matrix Q.

Step (5): Check $\bar{\gamma}+\bar{\beta}<\dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(\bar{P})\lambda_{\max}(\mathrm{T}^{-1})}$

where $\bar{\gamma}$ and $\bar{\beta}$ are found in ***step (1)***, $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q , $\lambda_{\max}(\bar{P})$ denotes the largest eigenvalue of $\bar{P}$ and $\lambda_{\max}(\mathrm{T}^{-1})$ denotes the largest eigenvalue of T^{-1}.

The following illustration has been discussed.

Problem (2.1)

Consider a non-linear dynamical control system described by the following dynamical equations:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 10\end{bmatrix}u+\begin{bmatrix}0.01 & 0 & 0\\ 0 & 0.06 & 0\\ 0 & 0 & 0.01\end{bmatrix}\begin{bmatrix}\cos(x_1+x_2)\\ \sin(x_2)\cos(x_2)\\ \sin^2(x_3)\end{bmatrix}+\begin{bmatrix}0\\ 0.2x_1\sin^2(t)\\ 0.1x_2\cos(2t)\end{bmatrix}$$

$$y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

$$x(0)=\begin{bmatrix}0.5 & 1 & 2\end{bmatrix}^{\mathrm{T}},\ \hat{x}(0)=\begin{bmatrix}0 & 1 & 1.7\end{bmatrix}^{\mathrm{T}} \qquad (2.48)$$

(1) the first stage: the feedback gain K is obtained by algorithm (1.1) as follows:

Step (1): Check the controllability condition for the system

$$M = (B \vdots AB \vdots A^2B)$$

$$= \begin{pmatrix} 0 & 0 & 10 \\ 0 & 10 & -60 \\ 10 & -60 & 250 \end{pmatrix}$$

Hence rank (M) = 3. Therefore (A, B) is completely state controllable.

Step (2): From the characteristic polynomial for matrix A,

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda+6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda - 6$$

$$\equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \text{ , then}$$

$a_1 = 6$, $a_2 = 11$ and $a_3 = -6$

Step (3): Determine the following transformation matrix H

$$H = MW$$

where M is the controllability matrix of ***step (1)***, and using the result of ***step (2)***, W is defined by

$$W = \begin{pmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and hence

$$H = \begin{pmatrix} 0 & 0 & 10 \\ 0 & 10 & -60 \\ 10 & -60 & 250 \end{pmatrix}\begin{pmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

and

$$H^{-1} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Step (4): Let the desired eigenvalues be selected as:

$$\mu_1 = -2 + i2\sqrt{3},\ \mu_2 = -2 - i2\sqrt{3},\ \mu_3 = -1$$

$$(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3) = \lambda^3 + 5\lambda^2 + 20\lambda + 16$$

$$\equiv \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$$

Then

$$\alpha_1 = 5,\ \alpha_2 = 20,\ \alpha_3 = 16$$

Step (5): The state feedback gain matrix K can be determined using the result of ***step (3)*** and ***step (4)*** as follows:

$$\mathrm{K} = [\alpha_3 - a_3 \vdots \alpha_2 - a_2 \vdots \alpha_1 - a_1]\mathrm{H}^{-1}$$

$$\mathrm{K} = (16 + 6 \vdots 20 - 11 \vdots 5 - 6)\mathrm{H}^{-1}$$

$$= (22 \quad 9 \quad -1)\begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$\mathrm{K} = (2.2 \quad 0.9 \quad -0.1). \tag{2.49}$$

(2) the second stage: the observer gain L by algorithm (2.1) is obtained as follows:

Step (1): Check the observability condition for the system

$$\mathrm{N} = (C^* \vdots \mathrm{A}^* C^* \vdots (\mathrm{A}^*)^2 C^*)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence rank (N) = 3. Therefore (A, C) is completely state observable.

Step (2): To verify the non-linearity $Df(x(t))$ and $g(x(t),t)$ satisfy Lipschitz condition:

$$Df(x(t)) = \begin{pmatrix} 0.01\cos(x_1 + x_2) \\ 0.06\sin(x_2)\cos(x_2) \\ 0.01\sin^2(x_3) \end{pmatrix}$$

The Jacobian matrix for $Df(x(t))$ is

$$J_1 = \begin{bmatrix} -0.01\sin(x_1 + x_2) & -0.01\sin(x_1 + x_2) & 0 \\ 0 & 0.06\cos(2x_2) & 0 \\ 0 & 0 & 0.02\sin(x_3)\cos(x_3) \end{bmatrix}$$

where

$$\|J_1\| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_1\| \le 0.0648$$

which implies that:

$$\|Df(x(t)) - Df(\hat{x}(t))\| \le 0.0648\|x(t) - \hat{x}(t)\| \qquad (2.50)$$

Thus, the non-linearity $Df(x(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\gamma = 0.0648$, and,

$$g(x(t),t) = \begin{pmatrix} 0 \\ 0.2x_1\sin^2(t) \\ 0.1x_2\cos(2t) \end{pmatrix}$$

The Jacobian matrix for the function $g(x(t),t)$ is

$$J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.2\sin^2(t) & 0 & 0 \\ 0 & 0.1\cos(2t) & 0 \end{bmatrix}$$

where

$$\|J_2\| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 \right)^{1/2}$$

$$\|J_2\| \le 0.2236$$

hence:

$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le 0.2236\|x(t) - \hat{x}(t)\| \tag{2.51}$$

Thus, the non-linearity function $g(x(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\beta = 0.2236$.

Step (3): Let the desired eigenvalues be selected as:

$$\eta_1 = -8 + i8\sqrt{3}\,,\ \eta_2 = -8 - i8\sqrt{3}\,,\ \eta_3 = -4\,.$$

$$(\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \eta_3) = \lambda^3 + 20\lambda^2 + 320\lambda + 1024$$

$$\equiv \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$$

Then

$$\alpha_1 = 20\,,\ \alpha_2 = 320\,,\ \alpha_3 = 1024\,.$$

The state observer gain matrix L can be determined as follows:

$$L = (\mathrm{W\,N}^*)^{-1}\begin{pmatrix} \alpha_3 - a_3 \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{pmatrix}\begin{pmatrix} 1024 + 6 \\ 320 - 11 \\ 20 - 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 14 \\ 255 \\ -474 \end{pmatrix} \tag{2.52}$$

Step (4): To find P which is the solution of this Riccati equation

$$(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$$

On solving it to get a unique positive definite solution P, on selection of:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.53}$$

hence

$$\begin{bmatrix} -14 & -225 & 480 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$$

$$\begin{bmatrix} -14 & 1 & 0 \\ -225 & 0 & 1 \\ 480 & -11 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 38.6705 & 0.5043 & 1.3633 \\ 0.5043 & 0.2686 & 0.0913 \\ 1.3633 & 0.0913 & 0.0986 \end{bmatrix} \qquad (2.54)$$

$$\lambda_1(P) = 0.0274,\ \lambda_2(P) = 0.2849,\ \lambda_3(P) = 38.7254.$$

Then P is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 1$, $\lambda_{\max}(P) = 38.7254$.

<u>Step (5):</u> From ***step (2)*** of stage (2) it is clear that $\gamma + \beta = 0.2884$.

Now, check $\gamma + \beta < \dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ = 0.0129 (2.55)

Since the observer gain in original coordinates was unsuccessful. Using transformation of coordinates, $z = \mathrm{T}x$ where

$$\mathrm{T} = \mathrm{diag}\ (20 \quad 1 \quad 0.2) \qquad (2.56)$$

Then

$$\mathrm{T}^{-1} = \mathrm{diag}\ (0.05 \quad 1 \quad 5) \qquad (2.57)$$

<u>Step (6):</u> Compute

$$\overline{\mathrm{K}} = \mathrm{K}\mathrm{T}^{-1} = (2.2 \quad 0.9 \quad -0.1)\begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= (0.11 \quad 0.9 \quad -0.5) \qquad (2.58)$$

Step (7): To verify the non-linearity $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ and $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy Lipschitz condition:

$$\mathrm{T}Df(\mathrm{T}^{-1}z(t)) = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0.01\cos\left(\dfrac{z_1}{20} + z_2\right) \\ 0.06\sin(z_2)\cos(z_2) \\ 0.01\sin^2\left(\dfrac{z_3}{0.2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.2\cos\left(\dfrac{z_1}{20} + z_2\right) \\ 0.06\sin(z_2)\cos(z_2) \\ 0.002\sin^2\left(\dfrac{z_3}{0.2}\right) \end{pmatrix}$$

The Jacobian matrix for $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ is

$$\overline{J}_1 = \begin{bmatrix} -\dfrac{0.2}{20}\sin\left(\dfrac{z_1}{20} + z_2\right) & -0.2\sin\left(\dfrac{z_1}{20} + z_2\right) & 0 \\ 0 & 0.06\cos(2z_2) & 0 \\ 0 & 0 & \dfrac{0.004}{0.2}\sin\left(\dfrac{z_3}{0.2}\right)\cos\left(\dfrac{z_3}{0.2}\right) \end{bmatrix}$$

where

$$\left\|\overline{J}_1\right\| \le 0.21$$

hence

$$\left\|\mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t))\right\| \le 0.21\left\|\mathrm{T}^{-1}z - \mathrm{T}^{-1}\hat{z}\right\| \qquad (2.59)$$

Thus, the non-linearity $\mathrm{T}Df(\mathrm{T}^{-1}z(t))$ satisfy the global Lipschitz condition with Lipschitz constant $\overline{\gamma} = 0.21$, and,

$$\mathrm{T}g(\mathrm{T}^{-1}z(t),t) = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 0.01z_1\sin^2(t) \\ 0.1z_2\cos(2t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0.01z_1\sin^2(t) \\ 0.02z_2\cos(2t) \end{pmatrix}$$

The Jacobian matrix for the function $g(x(t),t)$ is

$$J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01\sin^2(t) & 0 & 0 \\ 0 & 0.02\cos(2t) & 0 \end{bmatrix}$$

where

$$\left\|\overline{J}_2\right\| \le 0.02236$$

hence

$$\left\|\mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)\right\| \le 0.02236\left\|\mathrm{T}^{-1}z - \mathrm{T}^{-1}\hat{z}\right\| \qquad (2.60)$$

Thus, the non-linearity function $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\overline{\beta} = 0.02236$.

Step (8): Compute

$$\overline{L} = \mathrm{T}L = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}\begin{pmatrix} 14 \\ 255 \\ -474 \end{pmatrix} = \begin{pmatrix} 280 \\ 225 \\ -94.8 \end{pmatrix} \qquad (2.61)$$

Step (9): To find $\overline{P}$ which is the solution of this Riccati equation:

$$(\overline{\mathrm{A}} - \overline{L}\overline{C})^{\mathrm{T}}\overline{P} + \overline{P}(\overline{\mathrm{A}} - \overline{L}\overline{C}) = -Q$$

On solving it to get a unique positive definite solution $\overline{P}$, on selection of:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore

$$\begin{bmatrix} -14 & -11.25 & 4.8 \\ 20 & 0 & -2.2 \\ 0 & 5 & -6 \end{bmatrix}\begin{bmatrix} \overline{\mathrm{P}}_{11} & \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{13} \\ \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{22} & \overline{\mathrm{P}}_{23} \\ \overline{\mathrm{P}}_{13} & \overline{\mathrm{P}}_{23} & \overline{\mathrm{P}}_{33} \end{bmatrix} + \begin{bmatrix} \overline{\mathrm{P}}_{11} & \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{13} \\ \overline{\mathrm{P}}_{12} & \overline{\mathrm{P}}_{22} & \overline{\mathrm{P}}_{23} \\ \overline{\mathrm{P}}_{13} & \overline{\mathrm{P}}_{23} & \overline{\mathrm{P}}_{33} \end{bmatrix}$$

$$\begin{bmatrix} -14 & 20 & 0 \\ -11.25 & 0 & 5 \\ 4.8 & -2.2 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\rightarrow$

$$\overline{P} = \begin{bmatrix} 0.0497 & -0.0192 & -0.0040 \\ -0.0192 & 0.1357 & 0.0530 \\ -0.0040 & 0.0530 & 0.1275 \end{bmatrix} \tag{2.62}$$

$$\lambda_1(\overline{P}) = 0.0447,\ \lambda_2(\overline{P}) = 0.0814,\ \lambda_3(\overline{P}) = 0.1868.$$

Then $\overline{P}$ is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 1$, $\lambda_{\max}(\overline{P}) = 0.1868$, $\lambda_{\max}(\mathrm{T}^{-1}) = 5$.

Step (10): From ***step(7)*** it is clear that $\overline{\gamma} + \overline{\beta} = 0.23236$.

$$\text{Now, check } \overline{\gamma} + \overline{\beta} < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(\overline{P})\lambda_{\max}(\mathrm{T}^{-1})} = 0.5353. \tag{2.63}$$

Finally, we shall obtain the response of the system to the following initial condition:

$$z(0) = \begin{bmatrix} 10 \\ 1 \\ 0.4 \end{bmatrix},\quad E(0) = \begin{bmatrix} 10 \\ 0 \\ 0.06 \end{bmatrix} \tag{2.64}$$

Referring to Equation (2.28), it becomes

$$\begin{bmatrix} \dot{z}(t) \\ \dot{E}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{T}(\mathrm{A}-\mathrm{BK})\mathrm{T}^{-1} & \mathrm{TBKT}^{-1} \\ 0 & \mathrm{T}(\mathrm{A}-LC)\mathrm{T}^{-1} \end{bmatrix} \begin{bmatrix} z(t) \\ E(t) \end{bmatrix} + \begin{bmatrix} \mathrm{T}Df(\mathrm{T}^{-1}z(t)) + \mathrm{T}g(\mathrm{T}^{-1}z(t),t) \\ \xi(\mathrm{T}^{-1}z(t), \mathrm{T}^{-1}\hat{z}(t), t) \end{bmatrix} \tag{2.65}$$

where

$$\xi(\mathrm{T}^{-1}z(t), \mathrm{T}^{-1}\hat{z}(t), t) = \mathrm{T}Df(\mathrm{T}^{-1}z(t)) - \mathrm{T}Df(\mathrm{T}^{-1}\hat{z}(t)) + \mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)$$

the response to the initial condition can be determined from

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \end{bmatrix} = \begin{bmatrix} 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ -0.16 & -4 & -5 & 0.22 & 1.8 & -1 \\ 0 & 0 & 0 & -14 & 20 & 0 \\ 0 & 0 & 0 & -11.25 & 0 & 5 \\ 0 & 0 & 0 & 4.8 & -2.2 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix} +$$
$$\begin{bmatrix} 0.2\cos\left(\frac{z_1}{20} + z_2\right) \\ 0.06\sin(z_2)\cos(z_2) + 0.01z_1\sin^2(t) \\ 0.002\sin^2\left(\frac{z_3}{0.2}\right) + 0.02z_2\cos(2t) \\ 0.2\cos\left(\frac{z_1}{20} + z_2\right) - 0.2\cos\left(\frac{z_1 - E_1}{20} + (z_2 - E_2)\right) \\ 0.06(\sin(z_2)\cos(z_2) - \sin(z_2 - E_2)\cos(z_2 - E_2)) + 0.01(z_1 - (z_1 - E_1))\sin^2(t) \\ 0.002\left(\sin^2\left(\frac{z_3}{0.2}\right) - \sin^2\left(\frac{z_3 - E_3}{0.2}\right)\right) + 0.02(z_2 - (z_2 - E_2))\cos(2t) \end{bmatrix} \tag{2.66}$$

A MATLAB program using the fourth-order Runge-Kutta method is used to obtain the response is shown in MATLAB program (A1), in Appendix A.

The numerical results and estimators based on the algorithm for problem (2.1) have been shown in the following plotted graphs.

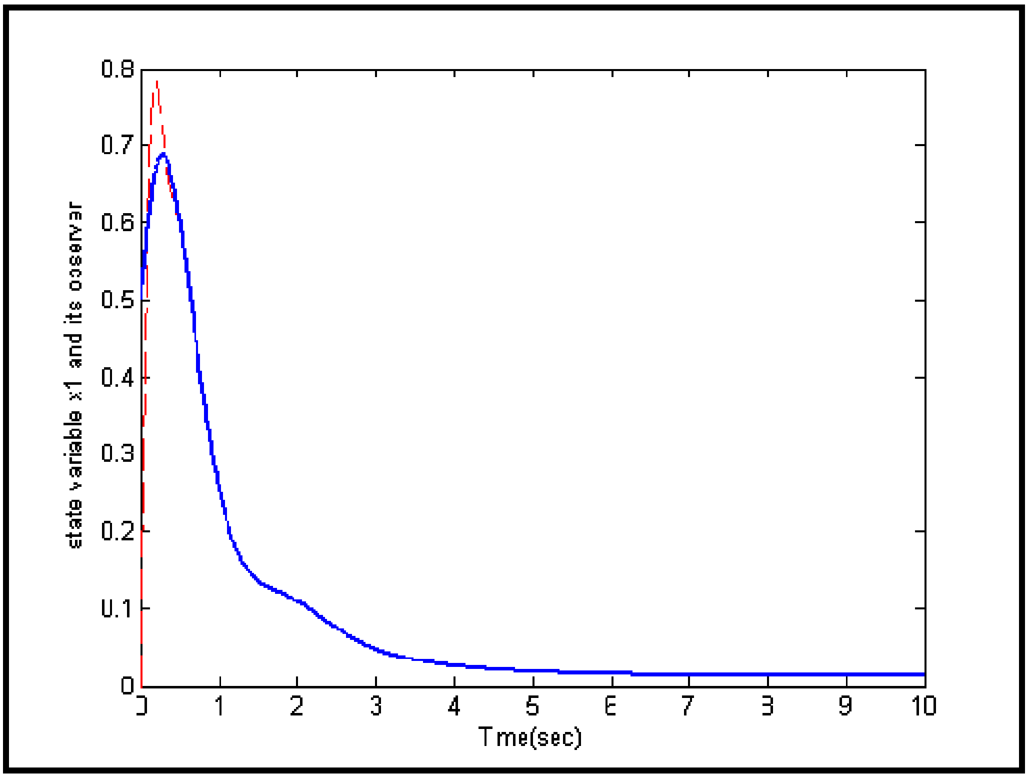


Fig (2.2) Observer performance: state variable x_1 ***(solid curve). and its observer*** $\hat{x}_1$ ***(broken curve) for*** $x_1(0)=0.5$, $\hat{x}_1(0)=0$ ***of problem (2.1).***

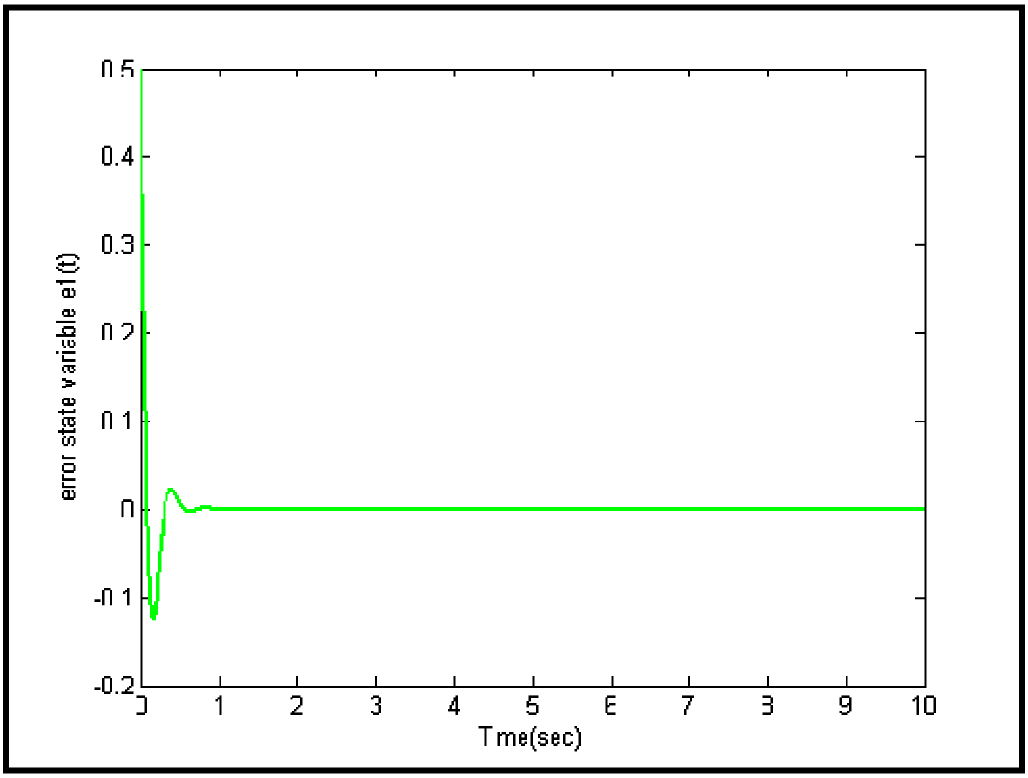


Figure (2.3) error between $x_1(t)$ ***and its observer*** $\hat{x}_1(t)$ ***of problem (2.1).***

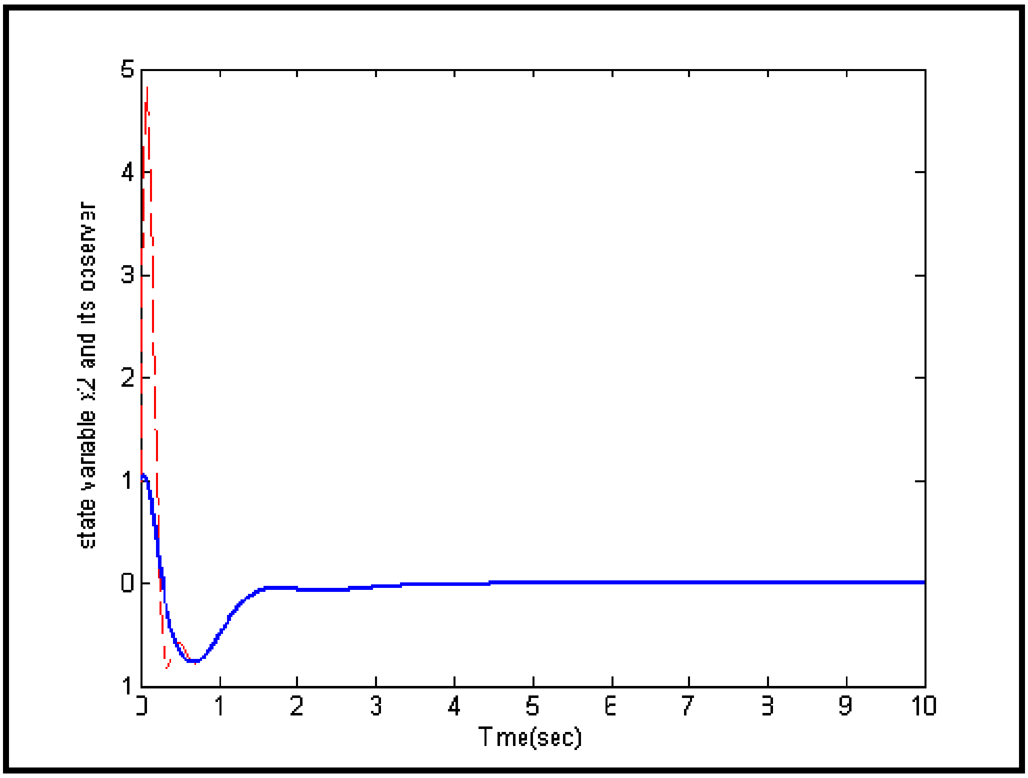


Fig (2.4) Observer performance: state variable x_2 (solid curve). and its observer $\hat{x}_2$ (broken curve) for $x_2(0)=1$, $\hat{x}_2(0)=1$ of problem (2.1).

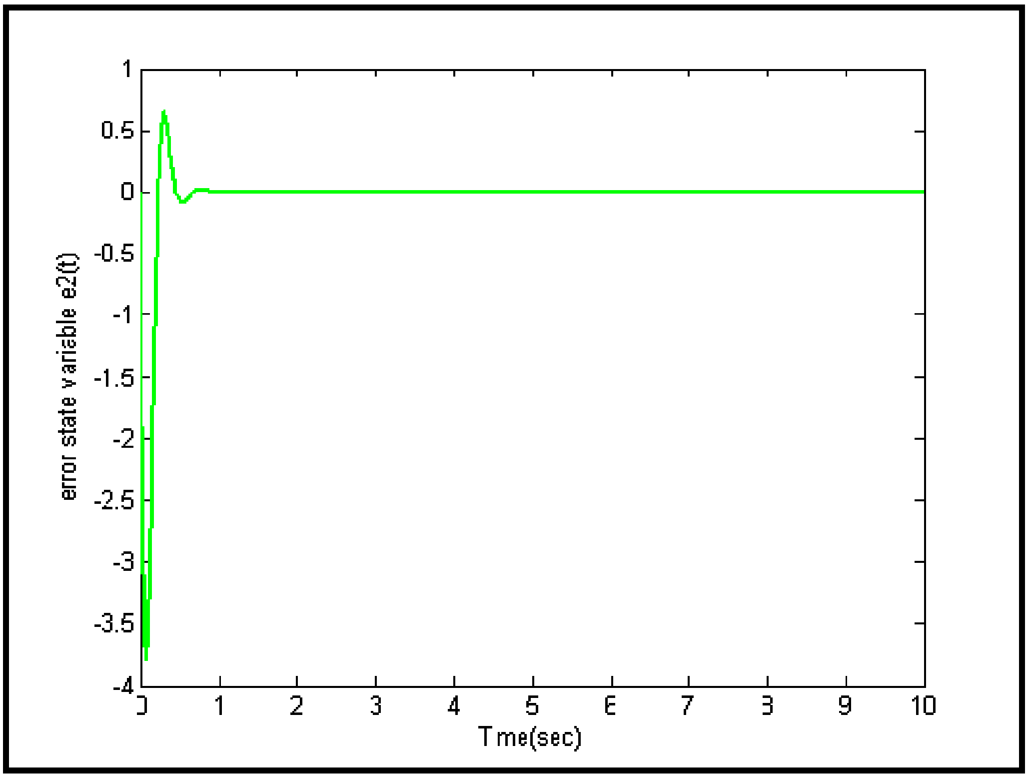


Figure (2.5) error between $x_2(t)$ and its observer $\hat{x}_2(t)$ of problem (2.1).

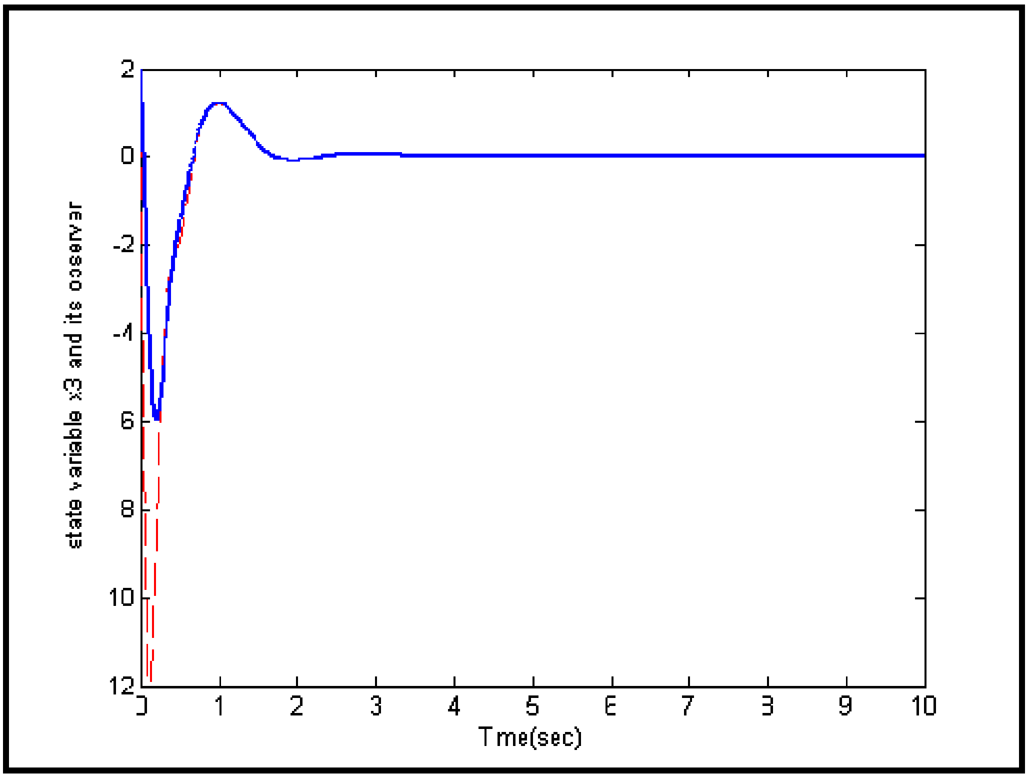


Fig (2.6) Observer performance: state variable x_3 (solid curve). and its observer $\hat{x}_3$ (broken curve) for $x_3(0) = 2$, $\hat{x}_3(0) = 1.7$ of problem (2.1).

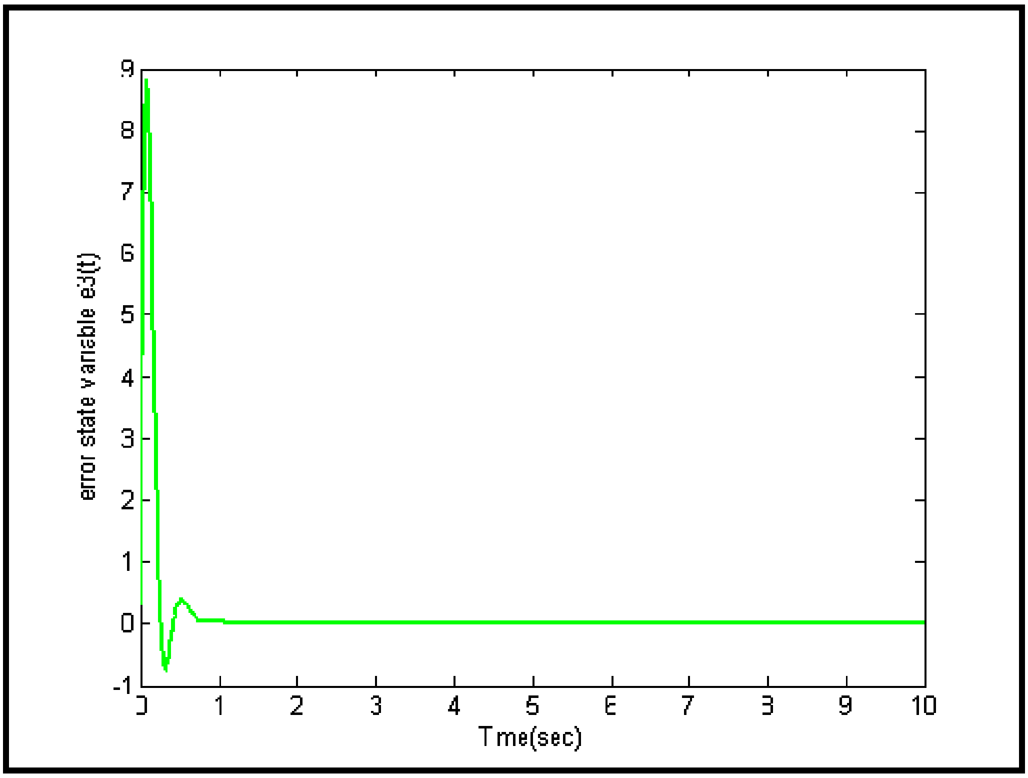


Figure (2.7) error between $x_3(t)$ and its observer $\hat{x}_3(t)$ of problem (2.1).

2.4 AN OBSERVER FOR A SINGLE-LINK FLEXIBLE JOINT ROBOT

Experimental evidence (Sweet and Good 1984) ***[31]*** has revealed that the performance of a large class of robots is severely limited when joint flexibility introduced by their transmission is not considered. The joint flexibility introduces low-frequency resonance effects which, when unaccounted for, limit the robot's performance range. With this in view, recent literature has focused on the control of flexible joint robots, e.g. Spong (1987) ***[30]***. In his work, Spong proposed the use of an I/O linearizing control law for these robots. This control however, uses information on all states-which for a single-link flexible-joint robot are the joint position and velocity, and the link position and velocity. For physical reasons, while one can easily measure the motor position and velocity the measurement of the other states are not trivial.

Figure (2.8), shows the schematic of a laboratory model of single-link flexible joint robot. In the figure, J_m represents the inertia of the actuator, a dc motor, and J_1 represents the inertia of the controlled link. θ_m and θ_1 are angular rotations of the motor and the link respectively, and, ω_m and ω_1 are their angular velocities. In general, these will be different from each other functions of time, due to the torsional compliance, *k*.

A state-space description of this system is given next.

$$\dot{\theta}_m = \omega_m \tag{2.67}$$

$$\dot{\omega}_m = \frac{k}{J_m}(\theta_1 - \theta_m) - \frac{B}{J_m}\omega_m + \frac{k_\tau}{J_m}u \tag{2.68}$$

$$\dot{\theta}_1 = \omega_1 \tag{2.69}$$

$$\dot{\omega}_1 = -\frac{k}{J_1}(\theta_1 - \theta_m) - \frac{mgh}{J_1}\sin(\theta_1) \tag{2.70}$$

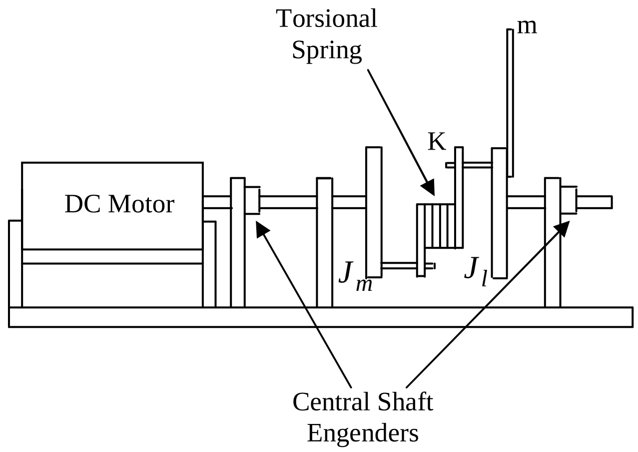


Fig (2.8) Schematic of elastic robot

The length of the link is given by $2b$. B represents the viscous friction in the motor bearing and the back-e .m .f. effects. The following are the simulation parameters used in the simulations. They are representative of a laboratory model that can be used to model a flexible-joint robot.

Table (2.1)

System parameter (Units)	**Value**
Motor inertia. J_m (kg m^2)	3.7×10^{-3}
Link inertia. J_1 (kg m^2)	9.3×10^{-3}
Pointer mass. m (kg)	2.1×10^{-1}
Link length. $2b$ (m)	3.0×10^{-1}
Torsional spring constant. k (Nm rad^{-1})	1.8×10^{-1}
Viscous friction coefficient. B (Nm V^{-1})	4.6×10^{-2}
Amplifier gain. K_τ (Nm V^{-1})	8×10^{-2}

Remark (2.5)

The derivation of mathematical model (2.67), (2.68), (2.69), (2.70) from its mechanical system (single-link flexible joint robot shown in graph

(2.8)) can be found in ***[24]***.

In our work we consider the only mathematical dynamic system (2.67), (2.68), (2.69), (2.70).

The mathematical system model can be represented by the following equations.

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) + g(x(t), t)$$

$$y(t) = C\,x(t)$$

$$\begin{bmatrix} \dot{\theta}_m \\ \dot{\omega}_m \\ \dot{\theta}_1 \\ \dot{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{J_m} & \frac{-B}{J_m} & \frac{k}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_1} & 0 & \frac{-k}{J_1} & 0 \end{bmatrix} \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_1 \\ \omega_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-mgh}{J_1}\sin(\theta_1) \end{bmatrix} \qquad (2.71)$$

On using the value of table (2.1), we get

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -12.4 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.4 & 0 & -19.4 & 0 \end{bmatrix} \quad , \quad \mathrm{B} = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -33.2\sin(\theta_1) \end{bmatrix}$$

Let

$$x(0) = [1 \quad 2 \quad -2 \quad -1]^{\mathrm{T}}, \; \hat{x}(0) = [0.5 \quad 3 \quad -1.5 \quad 2]^{\mathrm{T}}$$

(1) the first stage: the feedback gain K is obtained by algorithm (1.1) as follows:

<u>Step (1):</u> Check the controllability condition for the system

$$\mathrm{M} = (\mathrm{B} \vdots \mathrm{AB} \vdots \mathrm{A}^2\mathrm{B} \vdots \mathrm{A}^3\mathrm{B})$$

$$= \begin{pmatrix} 0 & 22 & -268 & 2271 \\ 22 & -268 & 2271 & -15149 \\ 0 & 0 & 0 & 419 \\ 0 & 0 & 419 & -5196 \end{pmatrix}$$

Hence rank (M) = 4.Therefore (A, B) is completely state controllable.

Step (2): From the characteristic polynomial for matrix A,

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 48.6 & \lambda + 12.4 & -48.6 & 0 \\ 0 & 0 & \lambda & -1 \\ -19.4 & 0 & 19.4 & \lambda \end{vmatrix}$$

$$= \lambda^4 + 12.4\lambda^3 + 68\lambda^2 + 240.56\lambda$$

$$\equiv \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \text{ , then}$$

$$a_1 = 12.4, \quad a_2 = 68, \quad a_3 = 240.56, \quad a_4 = 0.$$

Step (3): Determine the following transformation matrix H

$$H = MW$$

where M is the controllability matrix of ***step (1)***, and using the result of ***step(2)***, W is defined by

$$W = \begin{pmatrix} a_3 & a_2 & a_1 & 1 \\ a_2 & a_1 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 240.56 & 68 & 12.4 & 1 \\ 68 & 12.4 & 1 & 0 \\ 12.4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and hence

$$H = \begin{pmatrix} 0 & 0 & -268 & 2271 \\ 22 & -268 & 2271 & -15149 \\ 0 & 0 & 0 & 419 \\ 0 & 0 & 419 & -5196 \end{pmatrix} \begin{pmatrix} 240.56 & 68 & 12.4 & 1 \\ 68 & 12.4 & 1 & 0 \\ 12.4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 419.04 & 0 & 21.6 & 0 \\ 0 & 419.04 & 0 & 21.6 \\ 419.04 & 0 & 0 & 0 \\ 0 & 419.04 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{H}^{-1} = \begin{pmatrix} 0 & 0 & 0.0024 & 0 \\ 0 & 0 & 0 & 0.0024 \\ 0.0463 & 0 & -0.0463 & 0 \\ 0 & 0.0463 & 0 & -0.0463 \end{pmatrix}$$

Step (4): Let the desired eigenvalues be selected as:

$$\mu_1 = -3.6 + i4.8,\ \mu_2 = -3.6 - i4.8,\ \mu_3 = -4,\ \mu_4 = -5.$$

$$(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)$$

$$= \lambda^4 + 16.2\lambda^3 + 120.8\lambda^2 + 468\lambda + 720$$

$$\equiv \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Then

$$\alpha_1 = 16.2,\ \alpha_2 = 120.8,\ \alpha_3 = 468\ ,\ \alpha_4 = 720.$$

Step (5): The state feedback gain matrix K can be determined using the result of ***step (3)*** and ***step (4)*** as follows:

$$\mathrm{K} = [\alpha_4 - a_4 \vdots \alpha_3 - a_3 \vdots \alpha_2 - a_2 \vdots \alpha_1 - a_1]\mathrm{H}^{-1}$$

$$\mathrm{K} = (720 \vdots 468 - 240.56 \vdots 120.8 - 68 \vdots 16.2 - 12.4)\mathrm{H}^{-1}$$

$$= (720 \quad 227.44 \quad 52.8 \quad 3.8)\begin{pmatrix} 0 & 0 & 0.0024 & 0 \\ 0 & 0 & 0 & 0.0024 \\ 0.0463 & 0 & -0.0463 & 0 \\ 0 & 0.0463 & 0 & -0.0463 \end{pmatrix}$$

$$= (2.4444 \quad 0.1759 \quad -0.7262 \quad 0.3668). \qquad (2.72)$$

(2) the second stage: the observer gain L by algorithm (2.1) is obtained as follows:

Step (1): Check the observability condition for the system

$$N=\begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -48.6 & -12.4 & 48.6 & 0 \\ -48.6 & -12.4 & 48.6 & 0 \\ 602.6 & 105.2 & -602.6 & 48.6 \\ 602.6 & 105.2 & -602.6 & 48.6 \\ -4167.9 & -701.3 & 4167.9 & -602.6 \end{pmatrix}$$

Hence rank (N) = 4. Therefore (A, C) is completely state observable.

Step (2): To verify the non-linearity function g satisfy Lipschitz condition:

$$g=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -33.2\sin(\theta_1) \end{bmatrix}$$

The Jacobian matrix for the function g is

$$J=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -33.2\cos(\theta_1) & 0 \end{bmatrix}$$

where

$$\|J\|=\left(\sum_{i=1}^{4}\sum_{j=1}^{4}|a_{ij}|^2\right)^{1/2}$$

$$\|J\|\le 33.2$$

which implies that:

$$\|g(x(t),t) - g(\hat{x}(t),t)\| \le 33.2\|x(t) - \hat{x}(t)\| \tag{2.73}$$

Thus, the non-linearity function g satisfy the global Lipschitz condition with Lipschitz constant $\beta = 33.2$.

Step (3): Suppose that we use the dual of the pole placement approach to compute observer gain matrix L and the desired poles for this system are selected as:

$$\eta_1 = -14.4 - i19.2\,,\ \eta_2 = -14.4 + i19.2\,,\ \eta_3 = -16\,,\ \eta_4 = -20\,.$$

The state observer gain matrix L can be obtained (by using MATLAB) as shown in program (A2) in Appendix A.

$$L = \begin{pmatrix} 17.9596 & 0.2456 \\ -49.6943 & 34.4404 \\ 0.2981 & 22.0780 \\ -9.1292 & 193.6928 \end{pmatrix} \tag{2.74}$$

Step (4): To find P which is the solution of this Riccati equation:

$$(\mathrm{A} - LC)^{\mathrm{T}} P + P(\mathrm{A} - LC) = -Q$$

On solving it to get a unique positive definite solution P, on selection of:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.75}$$

hence

$$\begin{bmatrix} -17.9596 & 1.0943 & -0.2981 & 28.5292 \\ 0.7544 & -46.8404 & -22.0780 & -193.6928 \\ 0 & 48.6 & 0 & -19.4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \mathrm{P}_{13} & \mathrm{P}_{14} \\ \mathrm{P}_{12} & \mathrm{P}_{22} & \mathrm{P}_{23} & \mathrm{P}_{24} \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} & \mathrm{P}_{34} \\ \mathrm{P}_{14} & \mathrm{P}_{24} & \mathrm{P}_{34} & \mathrm{P}_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \mathrm{P}_{13} & \mathrm{P}_{14} \\ \mathrm{P}_{12} & \mathrm{P}_{22} & \mathrm{P}_{23} & \mathrm{P}_{24} \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} & \mathrm{P}_{34} \\ \mathrm{P}_{14} & \mathrm{P}_{24} & \mathrm{P}_{34} & \mathrm{P}_{44} \end{bmatrix} \begin{bmatrix} -17.9596 & 0.7544 & 0 & 0 \\ 1.0943 & -46.8404 & 48.6 & 0 \\ -0.2981 & -22.0780 & 0 & 1 \\ 28.5292 & -193.6928 & -19.4 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 0.1686 & 0.0393 & -0.8838 & 0.0779 \\ 0.0393 & 0.4877 & -0.2099 & -0.0913 \\ -0.8838 & -0.2099 & 5.9555 & -0.5 \\ 0.0779 & -0.0913 & -0.5000 & 0.0783 \end{bmatrix} \tag{2.76}$$

$$\lambda_1(P) = 0.0099,\ \lambda_2(P) = 0.0375,\ \lambda_3(P) = 0.5055,\ \lambda_4(P) = 6.1372.$$

Then P is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q) = 1,\ \lambda_{\max}(P) = 6.1372.$

Step (5): Now, check

$$\beta < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} = 0.0815. \tag{2.77}$$

Since the observer gain in original coordinates was unsuccessful. Using transformation of coordinates, $z = \mathrm{T}x$ where

$$\mathrm{T} = \mathrm{diag}\ (10 \quad 10 \quad 20 \quad 0.6) \tag{2.78}$$

Then

$$\mathrm{T}^{-1} = \mathrm{diag}\ (0.1 \quad 0.1 \quad 0.05 \quad 1.6667) \tag{2.79}$$

Step (6): Compute

$$\overline{\mathrm{K}} = \mathrm{KT}^{-1}$$

$$= (2.4444 \quad 0.1759 \quad -0.7262 \quad 0.3668)\begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1.6667 \end{pmatrix}$$

$$= (0.24444 \quad 0.01759 \quad -0.03631 \quad 0.61134556) \tag{2.80}$$

Step (7): To verify the non-linearity function $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy Lipschitz condition:

$$\mathrm{T}g(\mathrm{T}^{-1}z(t),t)=\begin{pmatrix}10&0&0&0\\0&10&0&0\\0&0&20&0\\0&0&0&0.6\end{pmatrix}\begin{pmatrix}0\\0\\0\\33.2\sin\left(\frac{z_3}{20}\right)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\19.92\sin\left(\frac{z_3}{20}\right)\end{pmatrix}$$

The Jacobian matrix for the function $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ is

$$\overline{J}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&\frac{19.92}{20}\cos\left(\frac{z_3}{20}\right)&0\end{bmatrix}$$

where

$$\left\|\overline{J}\right\|\le 0.996$$

hence

$$\left\|\mathrm{T}g(\mathrm{T}^{-1}z(t),t)-\mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)\right\|\le 0.996\left\|\mathrm{T}^{-1}z-\mathrm{T}^{-1}\hat{z}\right\| \tag{2.81}$$

Thus, the non-linearity function $\mathrm{T}g(\mathrm{T}^{-1}z(t),t)$ satisfy the global Lipschitz condition with Lipschitz constant $\overline{\beta}=0.996$.

Step (8): Compute

$$\overline{L}=\mathrm{T}L=\begin{pmatrix}10&0&0&0\\0&10&0&0\\0&0&20&0\\0&0&0&0.6\end{pmatrix}\begin{pmatrix}17.9596&0.2456\\-49.6943&34.4404\\0.2981&22.0780\\-9.1292&193.6928\end{pmatrix}$$

$$=\begin{pmatrix}179.5959&2.4559\\-496.4931&344.4041\\5.9626&441.5601\\-5.4775&116.2157\end{pmatrix} \tag{2.82}$$

Step (9): To find $\overline{P}$ which is the solution of this Riccati equation:

$$(\overline{\mathrm{A}}-\overline{L}\overline{C})^{\mathrm{T}}\overline{P}+\overline{P}(\overline{\mathrm{A}}-\overline{L}\overline{C})=-Q$$

On solving it to get a unique positive definite solution $\overline{P}$, on selection of:

$$Q=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

therefore

$$\begin{bmatrix}-17.9596&1.0943&-0.5963&1.7117\\0.7544&-46.8404&-44.1560&-11.6216\\0&24.3&0&-0.5820\\0&0&33.3333&0\end{bmatrix}\begin{bmatrix}\overline{\mathrm{P}}_{11}&\overline{\mathrm{P}}_{12}&\overline{\mathrm{P}}_{13}&\overline{\mathrm{P}}_{14}\\\overline{\mathrm{P}}_{12}&\overline{\mathrm{P}}_{22}&\overline{\mathrm{P}}_{23}&\overline{\mathrm{P}}_{24}\\\overline{\mathrm{P}}_{13}&\overline{\mathrm{P}}_{23}&\overline{\mathrm{P}}_{33}&\overline{\mathrm{P}}_{34}\\\overline{\mathrm{P}}_{14}&\overline{\mathrm{P}}_{24}&\overline{\mathrm{P}}_{34}&\overline{\mathrm{P}}_{44}\end{bmatrix}$$

$$+\begin{bmatrix}\overline{\mathrm{P}}_{11}&\overline{\mathrm{P}}_{12}&\overline{\mathrm{P}}_{13}&\overline{\mathrm{P}}_{14}\\\overline{\mathrm{P}}_{12}&\overline{\mathrm{P}}_{22}&\overline{\mathrm{P}}_{23}&\overline{\mathrm{P}}_{24}\\\overline{\mathrm{P}}_{13}&\overline{\mathrm{P}}_{23}&\overline{\mathrm{P}}_{33}&\overline{\mathrm{P}}_{34}\\\overline{\mathrm{P}}_{14}&\overline{\mathrm{P}}_{24}&\overline{\mathrm{P}}_{34}&\overline{\mathrm{P}}_{44}\end{bmatrix}\begin{bmatrix}-17.9596&0.7544&0&0\\1.0943&-46.8404&24.3&0\\-0.5963&-44.1560&0&33.3333\\1.7117&-11.6216&-0.5820&0\end{bmatrix}=$$

$$\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$\rightarrow$

$$\overline{P}=\begin{bmatrix}0.0293&0.0003&-0.0045&0.0132\\0.0003&0.0468&-0.0209&-0.0660\\-0.0045&-0.0209&0.0527&-0.0150\\0.0132&-0.0660&-0.0150&0.2637\end{bmatrix}\qquad(2.83)$$

$\lambda_1(\overline{P})=0.0132,\ \lambda_2(\overline{P})=0.0280,\ \lambda_3(\overline{P})=0.0681,\ \lambda_4(\overline{P})=0.2832$.

Then $\overline{P}$ is positive definite and also symmetric matrix. Also, it's clear that Q is positive definite matrix. It is clear that $\lambda_{\min}(Q)=1, \lambda_{\max}(\overline{P})=0.2832$, $\lambda_{\max}(\mathrm{T}^{-1})=1.6667$.

Step (10): Now, check

$$\bar{\beta} < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(\bar{P})\lambda_{\max}(\mathrm{T}^{-1})} = 1.0595. \tag{2.84}$$

Finally, we shall obtain the response of the system to the following initial condition:

$$z(0) = \begin{bmatrix} 10 \\ 20 \\ -40 \\ -0.6 \end{bmatrix}, \quad E(0) = \begin{bmatrix} 5 \\ -10 \\ -10 \\ 0.6 \end{bmatrix} \tag{2.85}$$

Referring to Equation (2.28), it becomes

$$\begin{bmatrix} \dot{z}(t) \\ \dot{E}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{T}(\mathrm{A}-\mathrm{BK})\mathrm{T}^{-1} & \mathrm{TBKT}^{-1} \\ 0 & \mathrm{T}(\mathrm{A}-LC)\mathrm{T}^{-1} \end{bmatrix} \begin{bmatrix} z(t) \\ E(t) \end{bmatrix} + \begin{bmatrix} \mathrm{T}g(\mathrm{T}^{-1}z(t),t) \\ \xi(\mathrm{T}^{-1}z(t),\hat{z}(t),t) \end{bmatrix}$$

(2.86)

where

$$\xi(\mathrm{T}^{-1}z(t),\mathrm{T}^{-1}\hat{z}(t),t) = \mathrm{T}g(\mathrm{T}^{-1}z(t),t) - \mathrm{T}g(\mathrm{T}^{-1}\hat{z}(t),t)$$

the response to the initial condition can be determined from

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \\ \dot{E}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -101.4 & -16.2 & 32.14 & -132.06 & 52.8 & 3.8 & -7.84 & 132.06 \\ 0 & 0 & 0 & 33.33 & 0 & 0 & 0 & 0 \\ 1.164 & 0 & -0.58 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -17.95 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.09 & -46.84 & 24.3 & 0 \\ 0 & 0 & 0 & 0 & -0.59 & -44.15 & 0 & 33.33 \\ 0 & 0 & 0 & 0 & 1.71 & -11.62 & -0.58 & 0 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 19.92\sin\left(\frac{z_3}{20}\right) \\ 0 \\ 0 \\ 0 \\ 19.92\left(\sin\left(\frac{z_3}{20}\right) - \sin\left(\frac{z_3 - E_3}{20}\right)\right) \end{bmatrix}$$

(2.87)

A MATLAB program using the fourth-order Runge-kutta method is used to obtain the response is shown in MATLAB program (A2), in Appendix A.

The numerical results and estimators based on the algorithm for problem discussed in (2.4) have been shown in the following plotted graphs.

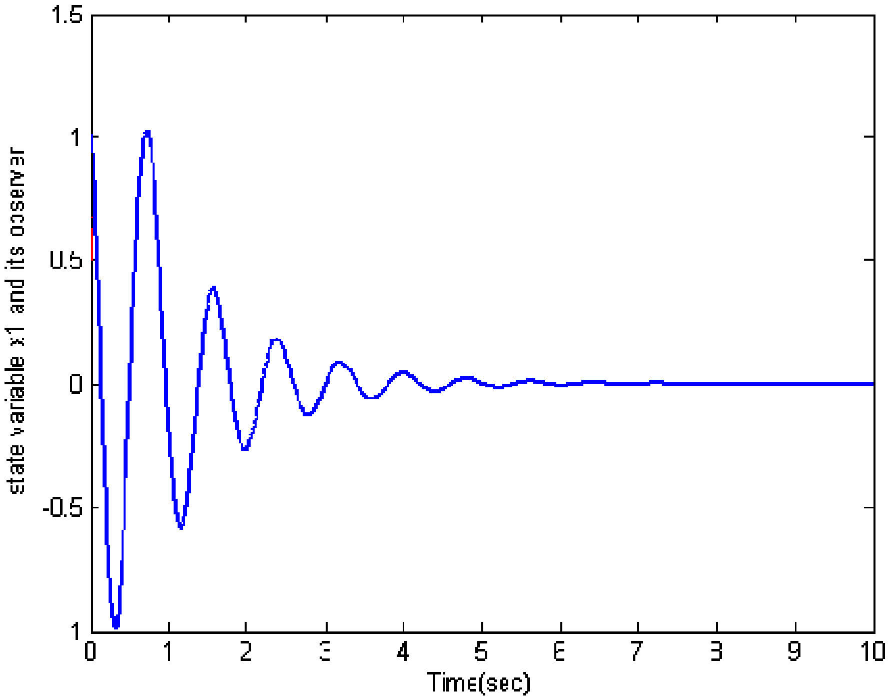


Fig (2.9) Observer performance: state variable x_1 ***(solid curve). and its observer*** $\hat{x}_1$ ***(broken curve) for*** $x_1(0)=1$, $\hat{x}_1(0)=0.5$ ***of problem in (2.4).***

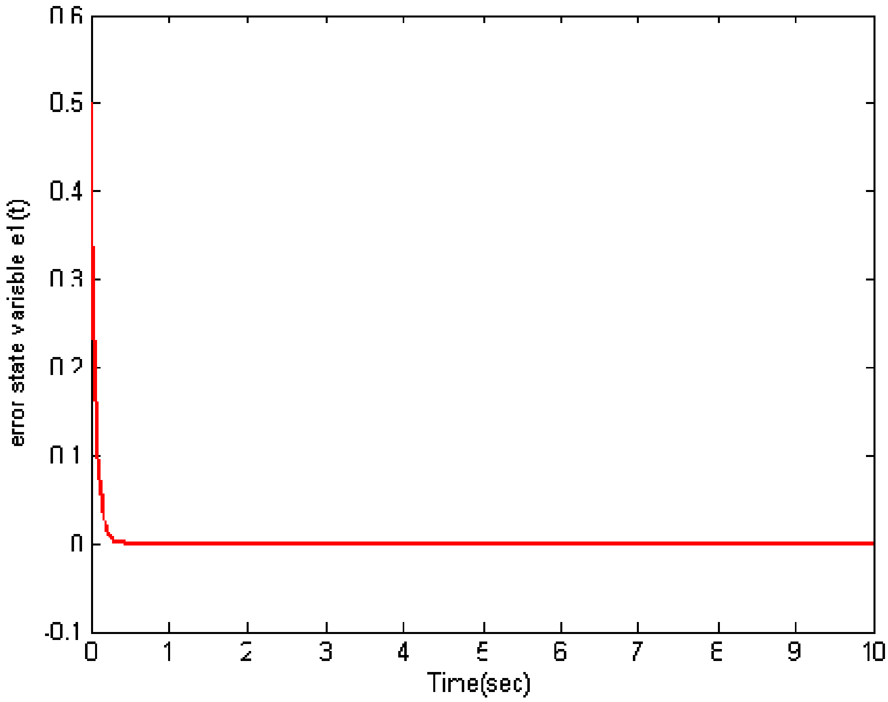


Fig (2.10) error between $x_1(t)$ ***and its observer*** $\hat{x}_1(t)$ ***of problem in (2.4).***

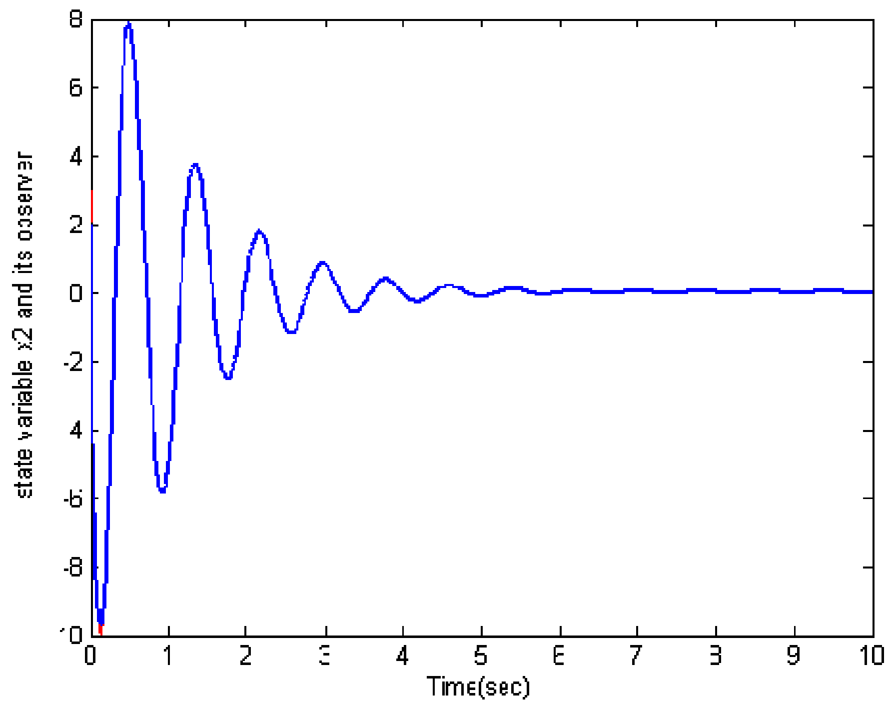


Fig (2.11) Observer performance: state variable x_2 (solid curve). and its observer $\hat{x}_2$ (broken curve) for $x_2(0) = 2$, $\hat{x}_2(0) = 3$ of problem in (2.4).

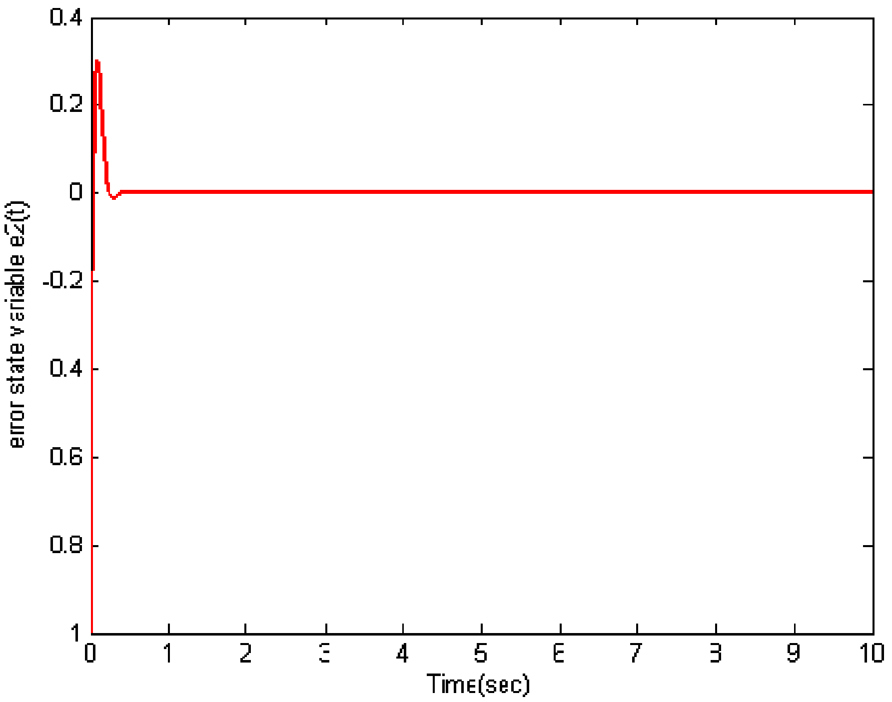
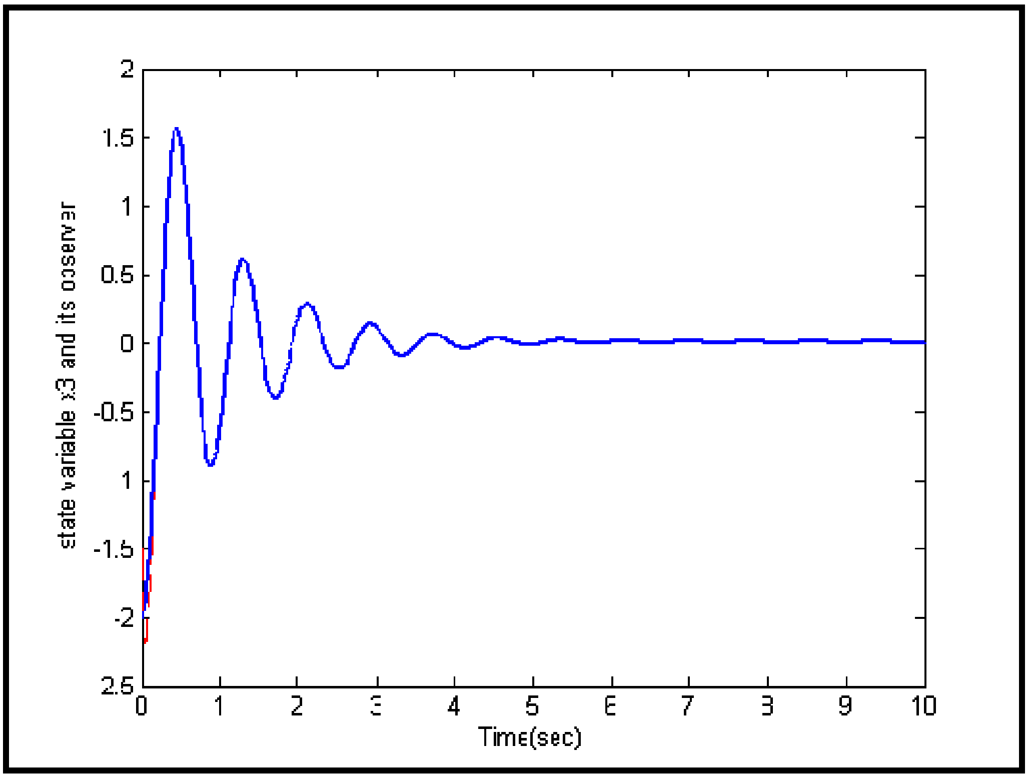


Fig (2.12) error between $x_2(t)$ and its observer $\hat{x}_2(t)$ of problem in (2.4).



Fig(2.13)Observer performance: state variable x_3 ***(solid curve). and its observer*** $\hat{x}_3$ ***(broken curve)for*** $x_3(0) = -2\,,\, \hat{x}_3(0) = -1.5$ ***of problem in(2.4).***

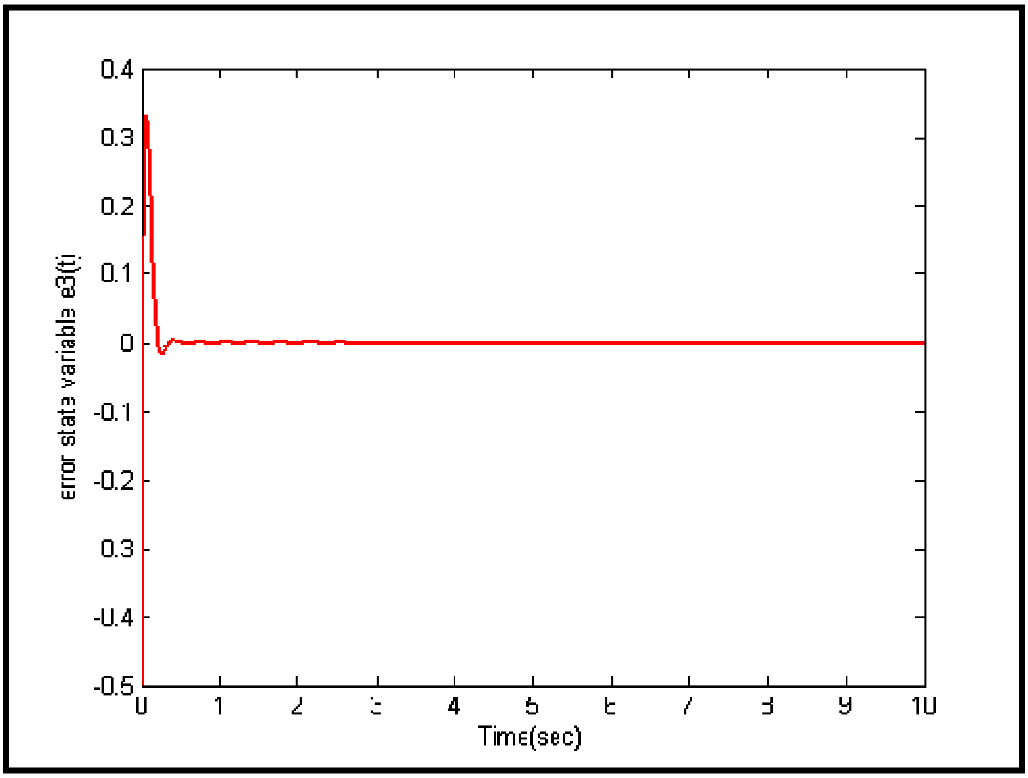


Fig (2.14) error between $x_3(t)$ ***and its observer*** $\hat{x}_3(t)$ ***of problem in (2.4).***

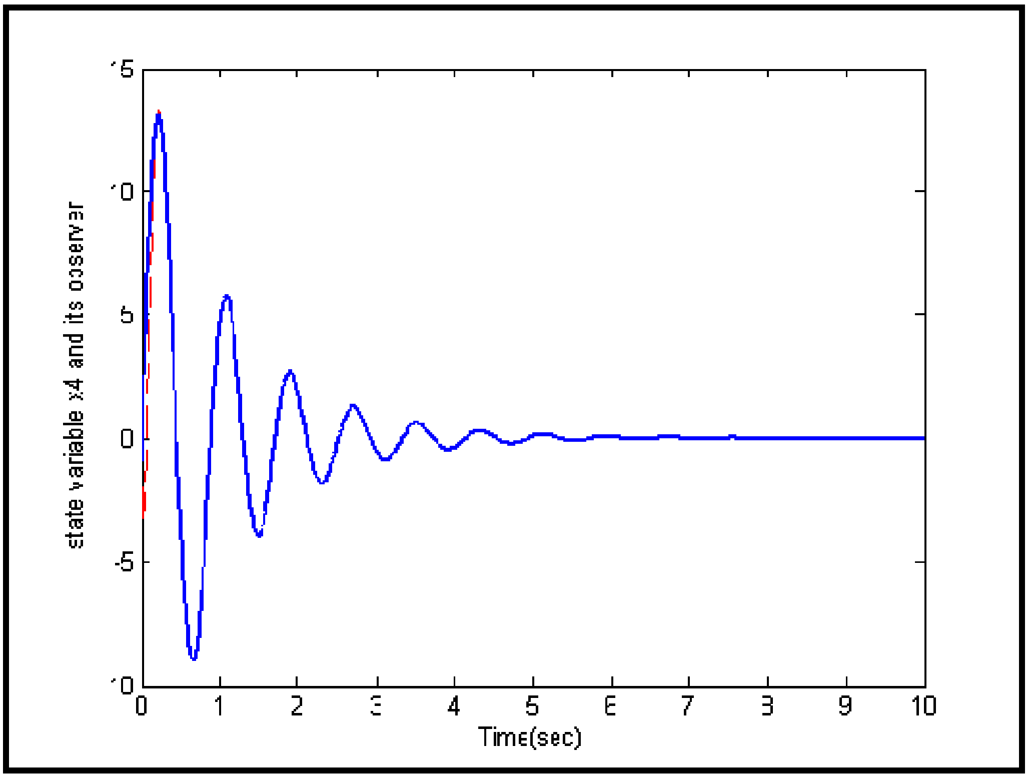


Fig (2.15) Observer performance: state variable x_4 (solid curve). and its observer $\hat{x}_4$(broken curve) for $x_4(0) = -1, \hat{x}_4(0) = -2$ of problem in (2.4).

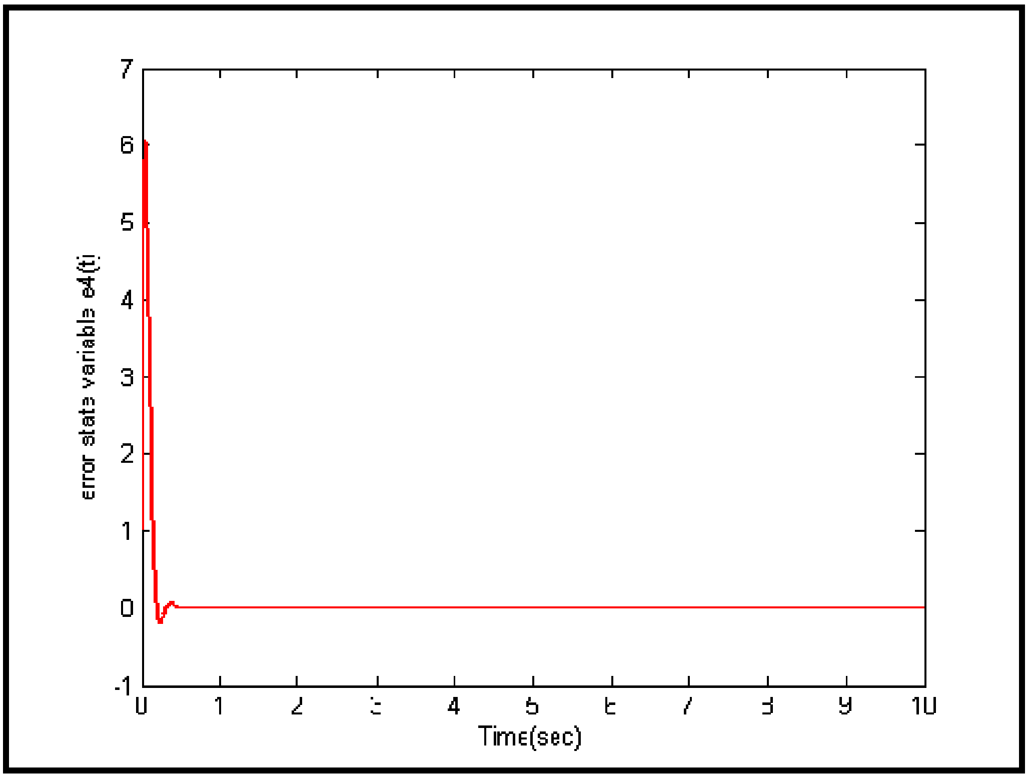


Fig (2.16) error between $x_4(t)$ and its observer $\hat{x}_4(t)$ of problem in (2.4).

Conclusions

From the present study of this thesis, sufficient conditions were given for the design observers for a class of nonlinear systems. These systems are characterized by nonlinear functions which are Lipschitz in nature. Nonlinear observer design is still a filed in its infancy, and we hope this thesis represents a fruitful step forward. The suggested methodology is based on the dual of the results from the theory of stabilization of uncertain systems. Moreover, it has been shown that the separation property holds. Hence, the control and observation algorithms can be joined together to form the observed-state feedback control system, we have tested it on several examples and we proposed in this thesis the design of an observer for the single-link flexible joint robot show the good performances of our method. The computational algorithms, based on the results of proposed theory are found to be very applicable and as one can see this fact from the illustrations. The behavior of the suggested dynamic observers are shown to be very good as one can see this fact from the graphs of the illustrations.

Contents

Future Work

The future work may be considered are the following:

1. Observability full order or reduced are of stochastic dynamic system.
2. Dynamic fuzzy observer for fuzzy control system.
3. Observability of differential inclusion.

(Thau 1973) [33] gave a sufficient condition for estimate convergence of a non-linear dynamical system described by a system of first-order differential equations. The original work was further extended for deterministic problems by (Kou et al. 1975) [11] and (Banks 1981) [1], and for the stochastic case by (Tarn and Rasis 1976) [32].

Exact methods of observing the state of non-linear systems are due to (Krener and Isidori 1983) [12] and to (Krener and Respondek 1985) [13] who, respectively, considered single-output unforced systems and multi-input multi-output systems. The conditions under which these observers can be designed are restrictive and do not apply to many physical systems.

To some extent, approximate observers can be derived on the basis of different techniques. (Zeitz 1987) [34], linearizes the observer error system, expressed in suitable coordinates, and requires the knowledge of the input derivatives. (Nicosia et al. 1989) [18], proposed a method of designing non-linear observers effective near the operating point set, assuming that the first time derivative of the outputs are measurable. (Baumann and Rugh 1986) [2], gave both controller and observer, and proved the convergence of the output feedback controller.

The class of Lipschitz nonlinear systems has been widely investigated, since most physical processes can be described by nonlinear Lipschitz models. (Reif et al. 1999) [29], (Rajamani and Cho 1998) [26], (Rajamani 1998) [25], (Zhu and Han 2002) [35] and (Raghavan and Hedrick 1994) [24], the authors proposed specific solutions to this type of systems where the stability conditions are expressed in terms of the algebraic Riccati equations.

In this thesis, the problem of designing state observers for inherently a non-linear dynamical control system is considered. A sufficient conditions for the jointed design of a linear state feedback control which stabilizes the non-linear dynamical control systems and state observers for inherently a non-linear dynamical control system are given.

This thesis consists of three chapters. The first chapter deals with the basic concepts of modern dynamical control system theory.

In chapter two, the problem of designing state observers for inherently non-linear dynamical control systems is discussed. A sufficient theorem for the design state observers for a non-linear dynamical control systems is stated and proved. A computational algorithm to construct the state observers for non-linear dynamical control system is presented. Observer-based control law of non-linear dynamical control system is studied. Some useful transformations to simplify the dynamic deterministic state space observer have been developed. This transformation helps the designer to overcome the some difficulties in the nature of nonlinearity (Lipschitz condition). Several problems are demonstrated to justify the validity of our results.

In chapter three, the generalization theorem to the results of chapter two has been developed. Illustrations using an open loop controller and closed loop controller have been presented and developed. An approximate state observer for some non-linear dynamical control system has also been given.

Concluding remarks, future work, list of references, appendix of MATLAB programs have also been presented.

Introduction

Modern control theory, which is based on state space concepts, is extremely useful not only for designing a specific dynamic control system. But also, for improving the principle on which the system will operate. By using the state space approach the control engineering may able to design dynamical control system with performance characteristic that can not be achieved by the classical approach by means of the frequency response method or the root locus method.

Dynamic control has played a vital role in the advance of engineering and science. In addition to its extreme importance in space-vehicle systems, missile-guidance systems, robotic systems, and the like, dynamic control has become an important and integral part of modern manufacturing and industrial processes.

The problem of designing nonlinear observers has been studied for a long time. Much of the effort has resulted in extensions of the linear Luenberger observer (Kailath 1980) [7] examples of this are the extended Kalman filter psuedo linearization techniques etc (see Misawa and Hedrick 1989 for a survey) [17]. These techniques are valid in a small range around the operating point. They also frequently require heavy real-time computation. A geometric techniques were proposed to build exact observers for a general description of nonlinear systems (Krener and Isidori 1983 [12], and Keller 1987 [8]). However, the conditions that are required to be satisfied by these observers are extremely stringent, making the applicable class very small. Recently, the quadratic stabilization of uncertain systems of (Khargonekar et al. 1990) [9], (Peterson 1987) [23] and (Peterson and Hollot 1986) [22] is used to construct observers for a class of non-linear systems.

References

1. Banks, S. P., " A Note on Non-Linear Observer", Int. J. Control, Vol. 34, No. 1, pp. 185-190, 1981.
2. Baumann, W. T., and Rugh, W. J., "Feedback Control of Nonlinear Systems by Extended Linearization", IEEE, Transactions on Automatic Control, Vol. 31, pp. 40-46, 1986.
3. Bryson, A. E., and Ho, Y. C. "Applied Optimal Control", Waltham, MA: Blaisdell, 1969.
4. Chen, C. T., "Linear System Theory and Design", CBS College Publishing, New York, 1984.
5. Hus, J. C., and Meyer, A. U., "Modern Control Principle and Applications", McGraw-Hill Book Company, Inc., USA, 1968.
6. Jayc, H., and Andrew, U. M., "Modren Control Principles and Applications", NewYork, McGraw Hill, 1968.
7. Kailath, T., "Linear Systems", Englewood cliffs, Prentic Hall, 1980.
8. Keller, H., "Non-Linear Observer Design by Transformation into a Generalized Observer Canonical Form", Int. J. of Control, Vol. 46, pp. 1915-1930, 1987.
9. Kharonekar, P. P., Peterson, I. R., and Zhow, K., "Robust Stabilization of Uncertainty Linear Systems: Quadratic Stabilizability and H_∞ Control Theory" , IEEE Transaction on Automatic Control, Vol. 35, pp. 356-361, 1990.
10. Kolman, B., "Introductory Linear Algebra with Applications", 1984.
11. Kou, S. R., Elliot, D.L., and Tarn, T.J., "Exponential Observers for Nonlinear Dynamic Systems", Information and Control, Vol. 29, pp. 204-216, 1975.

12. Krener, A. J., and Isidori, A., "Linearization by Output Injection and Non-Linear Observers', Systems and Control Leuers, Vol. 3, pp. 47- 52, 1983.
13. Krener, A. J., and Respondek, W., "Nonlinear Observers with Linearizable Error Dynamics", SIAM Journal on Control Optimization, Vol. 23, pp.197-216, 1985.
14. Kwakernaak, H., and Sivan, R., "Linear Optimal Control Systems", JohnWiley and Sons, Inc., 1972.
15. Lee, E. B., and Markus, L., "Foundation of Optimal Control Theory", Wiley and Sons Inc., New York, 1967.
16. Lu, Q., Sun, Y., and Mei, S., "Non-linear Control Systems and Power System Dynamics", Kluwer Academic Publishers, 2001.
17. Misawa, E. A., and Hedrick, J. K., " Nonlinear Observers: A State of the Art Survey", ASME J. of Dynamic Systems and Measurement, Vol. 11, pp. 344 – 352, 1989.
18. Nicosia, S., Tomei, B., and Tornambe, A., " An approximate Observer for a Class of Nonlinear Systems", Systems and Control Letters, Vol. 12, pp. 43-51, 1989.
19. Ogata, K., "State Space Analysis of Control Systems", Prentice – Hall, Inc., Englewood Cliffs, USA, 1967.
20. Ogata, K., "Modern Control Engineering", Prentice – Hall, Inc., India, 1996.
21. Ogata, K., "Modern Control Engineering", Fourth Edition, Prentice – Hall, Inc., India, 2002.
22. Peterson, I. H., and Hollot, C.V., "A Riccati Equation Approach to the Stabilization of Uncertain Linear System", Automatica, Vol. 22, pp. 397-411, 1986.
23. Peterson, I. R., "A stabilization Algorithm for a Class of Uncertain Linear Systems", Systems and Control Letters, Vol.7, pp.351-257,1987.

24. Raghavan, S., and Hedrick, K. J., "Observer Design for a Class of Nonlinear Systems", Int. J. of Control, Vol. 59, No. 2, pp. 515-528, 1994.
25. Rajamani, A., "Observers for Lipschitz Nonlinear Systems", IEEE Trans. Aulomatic Control, Vol. 43, No. 3, 1998.
26. Rajamani, R., and Cho, M. Y., "Existence and Design of Observer for Nonlinear Systems: Relation to Distance to un Observer Vability", Int. J. of Control, Vol. 69, No. 5, pp. 717-731, 1998.
27. Rao, M. R., "Ordinary Differential Equations", Edward Arnold, Affiliated East-West Private Limited, 1980.
28. Rao, M., and Qiu, H., "Process Control Engineering", Gordon and Breach Science Publishers S. A., 1993.
29. Reif, K., Sonnemann, F., and Unbehauen, R., "Nonlinear State Observation using h_{∞}- Filtring Riccati Design", IEEE Trans. Automatic Control, Vol. 44, No. 8, 1999.
30. Spong, M., "Modeling and Control of Elastic Joint Robots", ASME Journal of Dynamic Systems, Measurement and Control, Vol. 109, pp. 310-319, 1987.
31. Sweet, L., and Good, M. C., "Re-Definition of the Robot Motion Control Problem: Effects of Plant Dynamics, Drive System Constraints, and user Requirements", Proceedings of 23 rd IEEE Conference on Decision and Control, 1984.
32. Tarn, T. J., and Rasis, Y., "Observers for Nonlinear Stochastic Systems", IEEE Trans. Aut. Control, Vol. AC-21, No.4, Aug 1976.
33. Thau, F. E., "Observing the State of Nonlinear Dynamical Systems", Int. J.Control, Vol. 17, No. 3, 1973.
34. Zeitz, M., "The Extended Luenberger Observer for Nonlinear Systems", Systems and Control Letters, Vol. 9, pp. 149-156,1987.

35. Zhu, F., and Han, Z., "A note on Observers for Lipschitz Nonlinear Systems", IEEE Trans. Automatic Control, Vol. 47, No. 10, pp. 1751-1754, 2002.

Abstract

The main theme of this thesis is the design of a full order nonlinear deterministic dynamic observer for estimation of state space from its nonlinear input-output dynamic control system.

The quadratic Lyapunov function stabilization approach has been adapted and developed.

The sufficient conditions for existence of the dynamic observer for some class of nonlinear input-output dynamic system have been presented and discussed. Useful linear transformations have also been adapted to design a stable controller based on the suggested dynamic (deterministic) observer system. Computational algorithms based on the presented theorems for design a deterministic stable controller have also been discussed and developed. Illustrations are presented to demonstrate the validity of the presented procedure.

Dedication

To whom I hope he can feel me . . . my father (Allah bless him)

IBTISAM

Examining Committee's Certification

We certify that we read this thesis entitled ***"SYSTEM OBSERVER DESIGN FOR NONLINEAR DYNAMIC CONTROL SYSTEM AND ITS CONTROLLER"*** and as examining committee examined the student, ***Ibtisam Kamil Hanan*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

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Notations and Symbols

Glossary of Symbols:

$x^{\mathrm{T}}, \mathrm{A}^{\mathrm{T}}, \cdots$	The transpose of the vector *x* and the matrix A.
$x^*, \mathrm{A}^*, \cdots$	The complex conjugate transpose of the vector *x* and the matrix A.
$\lVert x \rVert, \lVert \mathrm{A} \rVert, \cdots$	The norm of the vector *x* and the matrix A.
A^{-1}	The inverse of the square matrix A.
$\lambda_i(\mathrm{A})$	The i-th eigenvalue of the square matrix A.
R	The field of real numbers.
C	The field of complex numbers.
$diag(x_1, x_2, \ldots, x_n)$	The diagonal matrix with diagonal elements $x_1, x_2, \ldots, x_n$.
$\dot{x}(t), \dfrac{dx(t)}{dt}$	The first time derivative of the time-varying vector $x(t)$.
$\det(\mathrm{A}), \lvert \mathrm{A} \rvert$	The determinant of the square matrix A.
$\lambda_{\min}(\mathrm{A})$	The smallest eigenvalue of the square matrix A.
$\lambda_{\max}(\mathrm{A})$	The largest eigenvalue of the square matrix A.
exp(.)	The exponential function.
e.m.f	Electromotive force
Sec.	Seconds.

Supervisors Certification

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science, Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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In view of the available recommendations; I forward this thesis for debate by examination committee.

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Ministry of Higher Education
and Scientific Research
Al-Nahrain University
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System Observer Design for Nonlinear Dynamic Control System and its Controller

A Thesis

Submitted to the Department of Mathematics, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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(B.Sc. 2004)

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April — *Rabè AL-Awal*

2007 — *1428*

وزارة التعليم العالي والبحث العلمي
جامعــة النهرين
كلية العلوم

تصميم نظام مخمن لنظام سيطرة ديناميكي لا خطي و مُسيطرهُ

رسالة
مقدمه إلى كلية العلوم في جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير
علوم في الرياضيات

من قبل
ابتسام كامل حنّان

(بكالوريوس علوم ٢٠٠٤)

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نيسان ٢٠٠٧

ربيع الاول ١٤٢٨

المستخلص

الهدف الرئيسي لهذه الرسالة هو تصميم مخمن دينامي كامل الرتبة غير خطي غير عشوائي (deterministic) لتخمين فضاء دالةالحالة (state space)من خلال نظام سيطرة مدخل-مخرج-(input-output)الدينامي غير الخطي.
تم تبني وتطوير دالة ليابانوف التربيعية المستقره.
لقدنوقشت وعرضت الشروط الكافية (sufficient conditions) لوجوديةالنظام المخمن الدينامي غير الخطي الخاص بنظام المدخل-المخرج الدينامي غير الخطي.استعملت بعض التحويلات الخطية المهمة لتصميم المسيطر المستقر معتمدين على المخمن الدينامي المقترح. نوقشت وطورت الخوارزميات الحسابية لتصميم المسيطر المستقر غير العشوائي معتمدين على النظريات المقدمة. قدمت كذلك امثلة توضيحية لتحديد ودراسة صحة الاسلوبية المقدمة.