Ministry of Higher Education
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Al-Nahrain University
College of Science

On the Multi-Dimensional Integral Equations

A Thesis
Submitted to the Department of Mathematics, Collage of Science, Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
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[اللّهُ لاَ إِلَـهَ إِلاَّ هُوَ الْحَيُّ الْقَيُّومُ لاَ تَأْخُذُهُ سِنَةٌ
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مَن ذَا الَّذِي يَشْفَعُ عِنْدَهُ إِلاَّ بِإِذْنِهِ يَعْلَمُ مَا بَيْنَ
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وَلاَ يَؤُودُهُ حِفْظُهُمَا وَهُوَ الْعَلِيُّ الْعَظِيمُ ْ%
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الأهــداء

الى المعلم الأول رسول المحبة والأنسانيه

محمد رسـول الله (صلى الله عليه وسلـم)

الى التي لم تبخل علي بسنين عمرهـا

وجادت لي بروحها و يستنير دربي بدعائها

(أمـي الحنونـه)

الى من توج أسمي بأسمه ومضــى...

وكنت أتمنـــى أن يكون معـــي...

(أبـي الحبيب(رحمه الله))

الى من شجعني وغمرنـي بحنانــه

وهـم أعـــز النـاس على قلبـــي

(أخوتي وأخواتي وعموري ومريومه)

الى من منحتني حبها وعطفها وحنانها

ورسمت على شفتي أبتسامة أمـــل

(د. أحـلام)

الى أساتذتي ألأعزاء الذين غمرونـي بحبهم...

أهـــدي أليهـــم ثمــرة جهـــدي هــذا

لمـى النعيمـي

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics / College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics.

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Luma Luay
2006

Examining committee certification

We certify that we have read this thesis entitled ***"On the Multi-Dimensional Integral Equations"*** an as examining committee examined the student ***(Luma Luay Abdul-Lateef Al-Niamey)*** in its contents and in what it connected with, and that is in our opinion it meet the standards of a thesis for the degree of Master of Science in Mathematics.

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Abstract

The main aim of this work is to generalize the study of the one-dimensional integral equations to include the multi-dimensional integral equations.

This study includes the classification of the multi-dimensional integral and integro-differential equations.

Also, some extended theorems for the existence and uniqueness of solution for the multi-dimensional integral equations are given.

Moreover, some generalized methods are used to solve the multi-dimensional integral equations, with some illustrative examples.

Appendices

Appendix A:

The following remark whish be useful have:

Remark:

1. $L_2[a,b]=\left\{ f \,\middle|\, f : [a,b] \longrightarrow D, \int_a^b |f(x)|^2 dx < \infty \right\}.$

2. $L_2[D]$ set of square lebsque integral function.

3. $L_2[D]$ is Hilbert space.

4. $(f,g)=\int_c^d \int_a^b f(x,y)\overline{g(x,y)}dxdy$.

5. $\|f\|^2=\int_c^d \int_a^b f(x,y)\overline{g(x,y)}dxdy$.

6. $\|f\|^2=\int_c^d \int_a^b |f(x,y)|^2 dxdy$.

7. $\|K\|=\sup_{\|u\|\neq 0} \frac{\|Ku\|}{\|u\|}, K \text{ is operator}$.

8. $u : D \longrightarrow L_2[D]$.

Appendix B (Some basic concepts of operator theory):

B.1 Linear operator [Taylor, 1958]:

An operator $\mathrm{T}:\mathrm{H} \longrightarrow \mathrm{H}$, {H is a Hilbert space} is called a linear operator if it satisfies.

$(i)\, T(x+y) = Tx + Ty,\ \forall x \in X$.

$(ii)\, Y(\alpha x) = \alpha x, \forall x \in X$ and $\alpha \in R$ or C.

B.2 Bounded linear operator [Taylor, 1958]:

Let H be a Hilbert space and $\mathrm{T}:\mathrm{H} \longrightarrow \mathrm{H}$ a linear operator. The operator T is said to be bounded if there is a real number M such that

$$\|Tx\| \leq M\,\|u\|,\ \forall x \in X.$$

The following are some useful examples.

B.2.1 Examples [Taylor, 1958]:

(i) Identity operator : Let H is a Hilbert space, $\mathrm{I}:\mathrm{H} \longrightarrow \mathrm{H}$ is a bounded.

(ii) Integral operator :

We define an integral operator, $T: C[0,1] \longrightarrow C[0,1]$, by:

$$Tx(t) = \int_0^1 k(t,\tau)x(\tau)d\tau,$$

Here k is given function, which is called the kernel of T and is assumed to be continuous on the closed domain. This operator is linear and bounded.

B.3 Bounded operator [Hochstadt H., 1973]:

An operator K is said to be bounded if for some M we have

$$\|Kf\| \le M\|f\|$$

for all f in Hilbert space.

B.4 Compact operator [Hochstadt H., 1973]:

Let K be a bounded, linear operator on a Hilbert space H. Let $\{f_n\}$ be an infinite uniformly bounded sequence in H; that is for some M we have $\|f_n\| \le M$ for all n. K is said to be compact operator if from the sequence $\{Kf_n\}$ one can extract a subsequence $\{Kf_{n_k}\}$ that is a Cauchy sequence. The sequence $\{Kf_{n_k}\}$ converge, of course, since H is a Hilbert space.

B.4.1 Definition Cauchy sequence [Hochstadt H., 1973]:

Let H be an inner product space and $\{f_n\}$ a Cauchy sequence in H. Such a sequence has the property that for every $\varepsilon > 0$ we can find an $N(\varepsilon)$ such that

$$\|f_n - f_m\| < \varepsilon \quad \text{for} \quad n, m > N(\varepsilon).$$

in other words

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0.$$

H is said to be a Hilbert space if every Cauchy sequence is converges to an element in H.

Appendix C:

C.1 Equicontinuous set [Marsden, 1995]:

A subset S of $C[a,b]$ is said to be equicontinuous, for each $\varepsilon > 0$, there is a $\delta > 0$, such that :

$$|x - y| < \delta \quad \text{and} \quad u \in M \quad \text{imply} \quad \|u(x) - u(y)\|_{C[a,b]} < \varepsilon.$$

C.2 Schauder fixed point theorems [Hochstadt H., 1973]:

Let S be closed convex and compact set in Hilbert space, and K a continuous mapping of S into itself. Then K has at least fixed point.

C.3 Banach fixed point theorems [Hochstadt H., 1973]:

Let T be a contraction operator on Hilbert space, $T : H \longrightarrow H$ then T has a unique fixed point.

C.4 Arzela□'s theorem [Hochstadt H., 1973]:

Let $\{g_n(x)\}$ be a set of continuous function defined on [0,1], that is uniformly bounded and equicontinuous. That is

$$|g_n(x)| \le M \quad \text{for all } n$$

$$|g_n(x_1) - g_n(x_2)| < \varepsilon \ \text{ for } \ |x_1 - x_2| < \delta(\varepsilon)$$

where M and $\delta(\varepsilon)$ are independent of n. One can then extract a subsequence $\{g_{n_k}(x)\}$ that converge uniformly to a continuous function $g(x)$.

C.5 Convex set [Hochstadt H., 1973]:

A set S in a linear space is said to be convex, if for any two points X, Y in S the set points $tX + (1-t)Y$, $0 \le t \le 1$, also belong to S.

C.6 Compact set [Hochstadt H., 1973]:

A set S in a Hilbert space is said to be compact if every infinite sequence of points in S, say $\{X_n\}$, has a convergent subsequence, that converges to a point in S.

Appendix D:

D.1 Theorem (Cauchy-Schwarz inequality)[Hochstadt H., 1973]:

Let f and g belong to an inner product space, then:

$$|(f, g)| \leq \|f\| \, \|g\|$$

Equality is achieved if and only if f and g are linearly independent .

1.1 Introduction:

Recall that the one-dimensional (denoted by 1-D) Integral equation is an equation in which the integration is carried out with respect to one variable. In this case, the unknown function depends only on one independent variable, [Delves L., Walsh J., 1974].

Moreover the mathematicians categorize the 1-D integral equation into linear, nonlinear, Volterra, Fredholm, homogeneous, nonhomogeneous and first kind, second kind, etc., [Chambers L., 1976].

The main aim of this chapter is to generalize the one-dimensional integral and integro-differential equations to the multi-dimensional integral and integro-differential equations.

Also the relation between the partial differential equation and the multi-dimensional integral equations is studied. Moreover, some basic concepts for the partial integro-differential equations are given.

This chapter consists of three sections.

In section one and two, the classification of the one-dimensional integral and integro-differential equations are extended to include the multi-dimensional integral and integro-differential equations.

In section three, the special types of related to partial differential equations and the multi-dimensional integral equations are studied.

In section four, a simple classification of partial integro-differential equations is devoted.

1.2 The Multi-Dimensional Integral Equations:

It is known that the two-dimensional (denoted by 2-D) integral equation is an integral equation in which the integration is carried out with respect to two variables, [David K., 1999].

On the other hand; the m-dimensional (denoted by m-D) integral equation is an integral equation in which the integration is carried out with respect to m variables.

An example of the 2-D integral equations:

$$u(x,y)=3xy+4\int_0^1\int_0^3 (3xy+z^2m+ze^x)\,u(z,m)\,dzdm$$

An example of the 3-D integral equations:

$$u(x,y,z)=4x+5e^z+y^2+5\int_1^2\int_3^4\int_0^5 (3xm+2\sin k+zy^2+n^3)u(m,n,k)\,dmdndk$$

In this section, we extend the classification of the 1-D integral equations to include the m-D integral equations.

A general form of the m-D linear integral equation is:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$

$$\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}\cdots\int_{\alpha_1}^{\beta_1(x_1)} k(x_1,x_2,...,x_m,z_1,z_2,...,z_m)\,u(z_1,z_2,...,z_m)\,dz_1dz_2\ldots dz_m \qquad (1.1)$$

where h and f are known functions of $x_1,x_2,...,x_m$, k is a known function of $x_1,x_2,...,x_m,z_1,z_2,...,z_m$, β_i is a known function of x_i for each $i=1,2,...,m$, α_i is a known constant for each $i=1,2,...,m$, λ is a scalar parameter and u is the unknown function that must be determined.

This equation is said to be m-D Volterra linear integral equation when $\beta_i(x_i) = x_i$, $i = 1,2,\ldots,m$. In this case, eq.(1.1) becomes:

$$h(x_1,x_2,\ldots,x_m)u(x_1,x_2,\ldots,x_m) = f(x_1,x_2,\ldots,x_m) +$$
$$\lambda \int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1} k(x_1,x_2,\ldots,x_m,z_1,z_2,\ldots,z_m)\, u(z_1,z_2,\ldots,z_m)\, dz_1 dz_2 \ldots dz_m \tag{1.2}$$

where $\alpha_i \le x_i < \infty$ for each $i = 1,2,\ldots,m$.

When $h(x_1,x_2,\ldots,x_m) = 0$, eq.(1.2) is said to be m-D Volterra linear integral equation of the first kind and takes the form:

$$f(x_1,\ldots,x_m) = -\lambda \int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)dz_1\ldots dz_m \tag{1.3}$$

Also if $h(x_1,x_2,\ldots,x_m) = 1$, then eq.(1.2) reduces to:

$$u(x_1,x_2,\ldots,x_m) = f(x_1,x_2,\ldots,x_m) +$$
$$\lambda \int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)\, dz_1\ldots dz_m \tag{1.4}$$

this equation is said to be m-D Volterra linear integral equation of the second kind. Moreover, if $f(x_1,x_2,\ldots,x_m) = 0$, then eq.(1.2) becomes:

$$h(x_1,x_2,\ldots,x_m)\, u(x_1,x_2,\ldots,x_m) =$$
$$\lambda \int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)\, dz_1\ldots dz_m \tag{1.5}$$

this homogeneous equation is said to be the m-D Volterra generalized linear integral eigenvalue problem. Also if $h(x_1,x_2,...,x_m)=1$ in eq.(1.5) then this equation is said to be m-D Volterra standard linear integral eigenvalue problem.

It is clear that $u=0$ is a solution of eq.(1.5) for any values of λ. Therefore the problem here is to determine the nontrivial solution u, which satisfy eq.(1.5).

On the other hand, the m-D Fredholm linear integral equation is the m-D linear integral equation in which the upper limits of integrations are constants. In this case, eq.(1.1) becomes

$$h(x_1,x_2,...,x_m)\,u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$

$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1}k(x_1,...,x_m,z_1,...,z_m)u(z_1,...,z_m)\,dz_1..dz_m \quad (1.6)$$

The following integral equations are:

$$f(x_1,...,x_m)=-\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1}k(x_1,...,x_m,z_1,...,z_m)u(z_1,...,z_m)\,dz_1..dz_m \quad (1.7)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$

$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1}k(x_1,...,x_m,z_1,...,z_m)u(z_1,...,z_m)dz_1...dz_m \quad (1.8)$$

are the m-D Fredholm linear integral equations of the first and second kinds respectively.

Moreover the integral equations:

$$h(x_1,\ldots,x_m)\,u(x_1,\ldots,x_m)=$$
$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)\,dz_1\ldots dz_m \tag{1.9}$$

and

$$u(x_1,\ldots,x_m)=\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)\,dz_1\ldots dz_m \tag{1.10}$$

are the m-D Fredholm generalized and standard linear integral eigenvalue problems respectively.

More generally, we can recognize the m-D Fredholm and Volterra non-linear integral equations as:

(1) The m-D Volterra non-linear integral equations of the:

A. First kind:

$$f(x_1,\ldots,x_m)=-\lambda\int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m,u(z_1,\ldots,z_m))dz_1\ldots dz_m \tag{1.11}$$

B. Second kind:

$$u(x_1,\ldots,x_m)=f(x_1,\ldots,x_m)+\lambda\int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1}\{k(x_1,\ldots,x_m,$$
$$z_1,\ldots,z_m,u(z_1,\ldots,z_m))\}dz_1\ldots dz_m \tag{1.12}$$

Also, the m-D Volterra generalized and standard non-linear integral eigenvalue problems are:

$$h(x_1,x_2,\ldots,x_m)\,u(x_1,x_2,\ldots,x_m)=\lambda\int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1}\{k(x_1,x_2,\ldots,$$

$$x_m,z_1,z_2,\ldots,z_m,u(z_1,z_2,\ldots,z_m))\}\,dz_1dz_2\ldots dz_m \qquad (1.13)$$

and

$$u(x_1,x_2,\ldots,x_m)=\lambda\int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}}\cdots\int_{\alpha_1}^{x_1}\{k(x_1,x_2,\ldots,x_m,z_1,z_2,\ldots,$$

$$z_m,u(z_1,z_2,\ldots,z_m))\}\,dz_1dz_2\ldots dz_m \qquad (1.14)$$

respectively.

(2) The m-D Fredholm non-linear integral equations can be easily obtained form eq.(1.11)-(1.14) by replacing x_i by the constants β_i for each $i=1,2,\ldots,m$ to get the same previous types.

1.3 The Multi-Dimensional Integro-Differential Equations:

It is known that, the one-dimensional integro-differential equation is a 1-D integral equation in which the unknown function (depends only on one independent variable) appears under the ordinary derivative sign, [Delves L. and Mohamed J., 1985].

Also, the two-dimensional integro-differential equation is a 2-D integral equation in which the unknown function (depends only on two independent variables) appears under the partial derivative signs, [David K., 1999].

So, the multi-dimensional integro-differential equation is an m-D integral equation in which the unknown function (depends only on m independent variables) appears under the partial derivative signs, [Volterra V., 1959].

The general form for the n-th order 1-D linear integro-differential equation is:

$$\sum_{i=0}^{n} a_i(x)\, u^{(i)}(x) = f(x) + \lambda \int_{\alpha}^{\beta(x)} k(x,y)\, u(y)dy \qquad (1.15)$$

where a_i, $i = 1,2,...,m$ is a known function of x such that $a_n \neq 0$, f and β are known functions of *x*, *k* is a known function of *x* and *y*, α is a known constant, λ is a scalar parameter and u is the unknown function that must be determined, [Delves L. and Mohamed J., 1985].

An example of the second order 1-D linear integro-differential equation is:

$$2u''(x) + 3x^2 u(x) + 4u(x) = 3x + 2\int_0^x (3x + y^2 \sin x)u(y)dy$$

The general form for the n-th order 2-D linear Volterra integro-differential equation is:

$$\sum_{\substack{i,j=0 \\ i+j\leq n}}^{n} a_{ij}(x,y)\frac{\partial^{i+j} u(x,y)}{\partial x^i \partial y^j} = f(x,y) +$$

$$\lambda \int_{\alpha_2}^{\beta_2(y)} \int_{\alpha_1}^{\beta_1(x)} k(x,y,z,m)u(z,m)\, dzdm \qquad (1.16)$$

where a_{ij} is a known function of x and y for each $i, j = 0,1,...,n$ such that $i + j \leq n$ and $a_{ij} \neq 0$ for some *i*, *j* such that $i + j = n$, f is a known function of x and y, k is a known function of x, y, z and m, β_1 and β_2 are known functions of x and y respectively, α_1 and α_2 are known constants, λ is a scalar parameter and u is the unknown function that must be determined.

An example of the third order 2-D linear integro-differential equation is:

$$x\,\frac{\partial^3 u}{\partial x\,\partial y^2}+3y^3\sin x\,\frac{\partial^2 u}{\partial x^2}=e^{-x}y+2\int_0^y\int_0^x (z\sin m+xe^{-z})\,u(z,m)\,dzdm$$

So, the general form for the n-th order m-D linear integro-differential equation is:

$$\sum_{\substack{i_1,i_2,\ldots,i_m=0\\ 0\le\sum_{k=1}^{m} i_k\le n}}^{n} a_{i_1 i_2\ldots i_m}(x_1,\ldots,x_m)\frac{\partial^{\sum_{k=1}^{m} i_k} u(x_1,\ldots,x_m)}{\partial x_1^{i_1}\partial x_2^{i_2}\ldots\partial x_m^{i_m}}=f(x_1,\ldots,x_m)+$$

$$\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}\cdots\int_{\alpha_1}^{\beta_1(x_1)} k(x_1,\ldots,x_m,z_1,\ldots,z_m)\,u(z_1,\ldots,z_m)\,dz_1\ldots dz_m \qquad (1.17)$$

where $a_{i_1 i_2\ldots i_m}$ is a known functions of $x_1,x_2,\ldots,x_m$ for each $i_1,i_2,\ldots,i_m=0,1,\ldots,n$ such that $0\le\sum_{k=1}^{m} i_k\le n$ and $a_{i_1 i_2\ldots i_m}\ne 0$ for some $i_1,i_2,\ldots,i_m$ such that $\sum_{k=1}^{m} i_k=n$, f is a known function of $x_1,x_2,\ldots,x_m$, k is a known function of $x_1,x_2,\ldots,x_m,z_1,z_2,\ldots,z_m$, β_i is a known function of x_i, α_i is a known constant for each $i=1,2,\ldots,m$, λ is a scalar parameter and u is the unknown function that must be determined.

If $\beta_i(x_i)=x_i$ for each $i=1,2,...,m$ then eq.(1.17) is of Volterra type. On the other hand, eq.(1.17) is of Fredholm type if $\beta_i(x_i)=\beta_i$ for each $i=1,2,...,m$, where β_i is a known constant.

Moreover if $a_{i_1 i_2 \ldots i_m}(x_1,x_2,\ldots,x_m)=a_{i_1 i_2 \ldots i_m}$, where $a_{i_1 i_2 \ldots i_m}$ is a known constant for each $i_1,i_2,...,i_m=0,1,...,n$ such that $0 \le \sum_{k=1}^{m} i_k \le n$ then eq.(1.17) is said to be the n-th order m-D integro-differential equation with constant coefficients, otherwise it is with non-constant coefficients.

Also, if $f(x_1,x_2,...,x_m)=0$ in eq.(1.17) then the homogeneous equation:

$$\sum_{\substack{i_1,i_2,...,i_m=0 \\ 0\le \sum_{k=1}^{m} i_k \le n}}^{n} a_{i_1 i_2 \ldots i_m}(x_1,x_2,..., x_m)\frac{\partial^{\sum_{k=1}^{m} i_k} u(x_1,x_2,..., x_m)}{\partial x_1^{i_1}\partial x_2^{i_2}\ldots \partial x_m^{i_m}} =$$

$$\lambda \int_{\alpha_m}^{\beta_m(x_m)} \int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})} \cdots \int_{\alpha_1}^{\beta_1(x_1)} \left\{k(x_1,...,x_m,z_1,...,z_m)\, u(z_1,...,z_m)\right\} dz_1..dz_m \tag{1.18}$$

is the n-th order m-D generalized linear intgro-differential eigenvalue problem.

The n-th order 1-D non-linear integro-differential equation may take the following form:

$$F\left(x, u(x), u'(x),...,u^{(n)}(x), Ku(x)\right)=0$$

where:

$$K\ u(x)=\int_{\alpha}^{\beta(x)} k\left(x,y,u(x),u(y),u'(x),u'(y),\ldots,\ u^{(n)}(x),\ u^{(n)}(y)\right) dy$$

where α and $\beta(x)$ are defined similar to the previous, k is known function of $x,y,u(x),u(y),u'(x),u'(y),\ldots,\ u^{(n)}(x)$ and $u^{(n)}(y)$ and u is the unknown function that must be determined.

As an example of the second order 1-D nonlinear integro-differential equation is:

$$u''(x)+2u(x)\ u'(x)=3x^2+\int_1^x [u'(y)e^{yu(y)+u'(y)}+\sin(yu''(y))]\ dy$$

Also, the n-th order 2-D non-linear integro-differential equation may take the following form:

$$F\left(x,y,u(x,y),\frac{\partial^{i_1+i_2}u(x,y)}{\partial x^{i_1}\partial y^{i_2}},Ku(x,y)\right)=0$$

where

$$K\ u(x,y)=\int_{\alpha_2}^{\beta_2(y)}\int_{\alpha_1}^{\beta_1(x)} k\left(x,y,z,m,\frac{\partial^{i+j}u(x,y)}{\partial x^i\partial y^j},\frac{\partial^{k+\ell}u(z,m)}{\partial x^k\partial y^\ell}\right) dzdm$$

α_1 and α_2 are known constants, β_1 and β_2 are known functions of x and y respectively and i_1, i_2, $i,j,k,\ell=0,1,\ldots,n$ such that $0\le i_1+i_2\le n$, $0\le i+j\le n$ and $0\le k+\ell\le n$.

As an example of the third order 2-D nonlinear integro-differential equation is:

$$u_{xyy}+u_{xxy}+u_{yxy}+u_x u_{yy}=\int_1^{6y}\int_2^{4x}\left(u_z+u_m u_{z\ m}+z^2 u_{zz}\right) dz\ dm$$

where $u=u(x,y)$.

Thus, the n-th order m-D non-linear integro-differential equation may take the following form:

$$F\left(x_1,x_2,\dots,x_m,z_1,z_2,\dots,z_m,\frac{\partial^{\sum_{k=1}^{m} i_k} u(x_1,x_2,\dots,x_m)}{\partial x_1^{i_1}\partial x_2^{i_2}\dots\partial x_m^{i_m}},K\,u\right)=0$$

where:

$$K\,u=\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}\dots\int_{\alpha_1}^{\beta_1} k\left(x_1,\dots,x_m,z_1,\dots,z_m,\frac{\partial^{\sum_{k=1}^{m}\ell_k} u(x_1,\dots,x_m)}{\partial x_1^{\ell_1}\dots\partial x_m^{\ell_m}},\frac{\partial^{\sum_{k=1}^{m} j_k} u(z_1,\dots,z_m)}{\partial z_1^{j_1}\dots\partial z_m^{j_m}}\right)dz_1\dots dz_m$$

β_i is a known function of x_i, α_i is a known constant for each $i=1,2,\dots,m$, $0\le\sum_{k=1}^{m} i_k\le n$. k is a known function of $x_1,\dots,x_m,z_1,\dots,z_m$, $\frac{\partial^{\sum_{k=1}^{m}\ell_k} u(x_1,\dots,x_m)}{\partial x_1^{\ell_1}\dots\partial x_m^{\ell_m}}$ and $\frac{\partial^{\sum_{k=1}^{m} j_k} u(z_1,\dots,z_m)}{\partial z_1^{j_1}\dots\partial z_m^{j_m}}$ for each $\ell_1,\ell_1,\dots,\ell_m$, $j_1,j_2,\dots,j_m=0,1,2,,\dots,n$ such that $0\le\sum_{k=1}^{m}\ell_k\le n$ and $0\le\sum_{k=1}^{m} j_k\le n$.

Moreover $i_1,i_2,\dots,i_m=0,1,2,\dots,n$ such that $0\le\sum_{k=1}^{m} i_k\le n$. $\beta_i(x_i)$, α_i are defined similar to the previous for each $i_1, i_2,\dots, i_m = 0, 1, 2,\dots, n$; such that:

As an example of the second order 3-D nonlinear integro-differential equation is:

$$u_z + u_{xx} + yu_{xy} = (3x^2 + y + z^2) + \int_0^z \int_0^y \int_0^x u^2(z_1, z_2, z_3)\, dz_1 dz_2 dz_3$$

In this work we restrict our discussion to the special types of the n-th order m-D non-linear integro-differential equation of the form:

$$\sum_{\substack{i_1, i_2, \ldots, i_m = 0 \\ 0 \le \sum_{k=1}^{m} i_k \le n}}^{n} a_{i_1 i_2 \ldots i_m}(x_1, x_2, \ldots, x_m) \frac{\partial^{\sum_{k=1}^{m} i_k} u(x_1, x_2, \ldots, x_m)}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_m^{i_m}} = f(x_1, x_2, \ldots, x_m) +$$

$$\lambda \int_{\alpha_m}^{\beta_m(x_m)} \int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})} \cdots \int_{\alpha_1}^{\beta_1(x_1)} k\left(x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_m, \right.$$

$$\left. \frac{\partial^{\sum_{k=1}^{m} j_k} u(z_1, z_2, \ldots, z_m)}{\partial z_1^{j_1} \partial z_2^{j_2} \ldots \partial z_m^{j_m}} \right) dz_1 dz_2 \ldots dz_m$$

where α_i, β_i, λ and f are defined similar to the previous, k is a known function of $x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_m, u(z_1, z_2, \ldots, z_m)$ and its partial derivatives up to n times and either $a_{i_1 i_2 \ldots i_m} \neq 0$ for some $i_1, i_2, \ldots, i_m =$ $0,1,\ldots,n$; such that $\sum_{k=1}^{m} i_k = n$ or $\sum_{k=1}^{m} i_k = n$ for some $j_1, j_2, \ldots, j_m =$ $0,1,\ldots,n$.

As an example of the second order 2-D nonlinear integro-differential equation is:

$$u_{yy} + 3xu_x = x + y + \int_0^y \int_0^x u^2(z, m) dz dm$$

1.4 The Relation Between the m-D Integral and Integro-differential Equations and The Partial Differential Equations:

Many problems in the field of ordinary differential equations can be recast as a 1-D integral and integro-differential equations, [Delves L. and Mohamed J., 1985].

In this section, we give some partial differential equations that can be expressed as m-D integral and integro-differential equations. To do this, first consider:

$$\left.\begin{array}{l} u_{xy} = f(x,y,u), \qquad a < x < \infty,\ b < y < \infty \\ \text{with the initial and boundary conditions:} \\ u(a,y) = h(y),\ b \le y < \infty \\ u(x,b) = g(x),\ a \le x < \infty \\ \text{where } h(b) = g(a). \end{array}\right\} \qquad (1.19)$$

Then by integrating the above partial differential equation with respect to *x* one can get:

$$u_y(x,y) - u_y(a,y) = \int_a^x f(z,y,u(z,y))\,dz$$

Again by integrating the above integral equation with respect to *y*, one can obtain:

$$u(x,y) - u(x,b) - u(a,y) + u(a,b) = \int_b^y \int_a^x f(z,m,u(z,m))\,dzdm$$

Thus:

$$u(x,y)=g(x)+h(y)-g(a)+\int_b^y\int_a^x f(z,m,u(z,m))\,dzdm$$

which is a 2-D Volterra integral equation. It is easy to check that the solution of the above integral equation satisfy eq.(1.19). Moreover if f is a linear function with respect to u in eq.(1.19) then the above 2-D integral equation is linear. Otherwise it is nonlinear.

Second, consider the non-linear hyperbolic partial differential equation**:**

$$u_{xy}=f(x,y,u,u_x,u_y),\; a<x<\infty\;,\; b<y<\infty$$

with the initial and boundary conditions: (1.20)

$$u(x,b)=g(x),\; a\le x<\infty;\; u(a,y)=h(y),\; b\le y<\infty$$

where; $h(b)=g(a)$.

Then by integrating the above partial differential equation first with respect to x and second with respect to y one can obtain:

$$u(x,y)=g(x)+h(y)-g(a)+\int_b^y\int_a^x f(z,m,u(z,m),u_z(z,m),u_m(z,m))\,dzdm$$

which is a 2-D Volterra integro-differential equation. It is easy to check that the solution of the above integro-differential equation satisfy eq.(1.20). Moreover if f is a linear function with respect to u, u_x and u_y in eq.(1.20) then the above 2-D integro-differential equation is linear. Otherwise it is nonlinear.

1.5 The Partial Integro-Differential Equations:

It is known that, partial integro-differential equation is an integro-differential equation in which the unknown function appears under the partial derivatives as well as the unknown function or it's partial derivatives appear under an integral sign, [Volterra V., 1959].

In this section, we concerns with the partial integro-differential equations in which the integration is carried out with respect to one variable.

The general form of the first order linear partial integro-differential equation is:

$$\sum_{i=1}^{m} a_i(x_1,\dots,x_m)\frac{\partial u(x_1,\dots,x_m)}{\partial x_i} + b(x_1,\dots,x_m)u(x_1,\dots,x_m) = f(x_1,\dots,x_m) +$$

$$\lambda \int_{\alpha}^{\beta(x_j)} \Bigg\{ \sum_{i=1}^{m} k(x_1,x_2,\dots,x_m,t)\frac{\partial u(x_1,x_2,\dots,x_{j-1},t,x_{j+1},\dots,x_m)}{\partial x_i} +$$

$$\ell_1(x_1,x_2,\dots,x_m,t)\frac{\partial u(x_1,x_2,\dots,x_{j-1},t,x_{j+1},\dots,x_m)}{\partial t} +$$

$$\ell_2(x_1,x_2,\dots,x_m,t)u(x_1,x_2,\dots,x_{j-1},t,x_{j+1},\dots,x_m)\Bigg\} dt$$

where a_i and b are known functions of $x_1,x_2,\dots,x_m$, k_i, ℓ_1 and ℓ_2 are known functions of $x_1,x_2,\dots,x_m$ and t, f is a known of $x_1,x_2,\dots,x_m$, β is a known function of x_j for some $1 \le j \le m$, λ is a scalar parameter, α is a known constant and u is the unknown function that must be determined.

As an example of the first order linear partial integro-differential equation is:

$$x^2 y \frac{\partial u(x,y)}{\partial x} = \int_1^2 \big(\sin(xt) + \cos(yt)\big)u(x,t)dt$$

A general form for the second order linear partial integro-differential equation is:

$$\sum_{i_1,i_2=1}^{m}\sum_{\substack{i,j=0\\ i+j\le 2}}^{2} a_{i_1i_2ij}(x_1,\ldots,x_m)\frac{\partial^{i+j}u(x_1,\ldots,x_m)}{\partial x_{i_1}^{i}\partial x_{i_2}^{j}}=f(x_1,\ldots,x_m)+$$

$$\lambda\int_{\alpha}^{\beta(x_k)}\left\{\sum_{i_1,i_2=1}^{m}\sum_{\substack{i,j=0\\ i+j\le 2}}^{2} k_{i_1i_2ij}(x_1,\ldots,x_m,t)\frac{\partial^{i+j}u(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial x_{i_1}^{i}\partial x_{i_2}^{j}}+\right.$$

$$\left.\sum_{i=0}^{2}\ell_i(x_1,x_2,\ldots,x_m,t)\frac{\partial^{i}u(x_1,x_2,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial t^{i}}\right\}dt$$

where $a_{i_1i_2ij}$ and f are known functions of $x_1,x_2,\ldots,x_m$, $k_{i_1i_2ij}$ and ℓ_i are known functions of $x_1, x_2,\ldots, x_m$ and t, λ is a scalar parameter, α is a known constant, β is a known function of x_k, $1\le k\le m$ and u is the unknown function that must be determined.

As an example of the second order linear partial integro-differential equation is:

$$(x^2+y)u_{xx}+2yu_{xz}=(3x^2y+z)+\int_0^2(3x+y+ze^t)u(x,y,t)dt$$

Therefore, the general form for the n-th order partial integro-differential equation is:

$$\sum_{i_1,\ldots,i_n=1;}^{m}\ \sum_{\substack{j_1,\ldots,j_n=0 \\ \sum_{i=1}^{n} j_i \le n}}^{n} a_{i_1\ldots i_n j_1\ldots j_n}(x_1,\ldots,x_m)\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}} =$$

$$f(x_1,\ldots,x_m)+\lambda\int_{\alpha}^{\beta(x_k)}\left\{\sum_{i_1,\ldots,i_n=1;}^{m}\ \sum_{\substack{j_1,\ldots,j_n=0 \\ \sum_{i=1}^{n} j_i \le n}}^{n} k_{i_1\ldots i_n j_1\ldots j_m}(x_1,\ldots,x_m,t)\right.$$

$$\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}}+\sum_{i=0}^{n}\ell_i(x_1,\ldots,x_m,t)$$

$$\left.\frac{\partial^{i} u(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial t^{i}}\right\}dt$$

where $a_{i_1 i_2\ldots i_n j_1 j_2\ldots j_n}$ and f are known functions of $x_1, x_2,\ldots,x_m$, $k_{i_1 i_2\ldots i_n, j_1 j_2\ldots j_m}$ is a known function of $x_1, x_2,\ldots, x_m$ and t, λ is a scalar parameter, α is a known constant, β is a known function of x_k, $1 \le k \le m$ and u is the unknown function that must be determined.

As an example of the third order partial integro-differential equation is:

$$u_{xxx}(x,y)=xy+\int_0^1 e^{yt^2}u(x,t)dt$$

Thus, the n-th order non-linear partial integro-differential equation may takes the following form

$$F\left(x_1,x_2,\ldots,x_m,t,\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,x_2,\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}},Ku(x_1,x_2,\ldots,x_n \right)=0$$

where:

$$Ku(x_1,\ldots,x_n)=\int_{\alpha}^{\beta(x_k)}\sum_{i_1,i_2,\ldots,i_n=1}^{m}\sum_{\substack{j_1,j_2,\ldots,j_n=0\\ \sum_{i=1}^{n} j_i\le n\\ j_1,j_2,\ldots,j_n\ne k}}^{n} k_{i_1i_2\ldots i_n j_1j_2\ldots j_n}\Bigg(x_1,\ldots,x_n,$$

$$t\,\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}}\Bigg)dt$$

here f is a non-linear function with respect to $\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}}$ or *Ku*,

$k_{i_1i_2\ldots i_n j_1j_2\ldots j_n}$ is a known of $x_1,\ldots,x_n,t,\frac{\partial^{\sum_{i=1}^{n} j_i} u(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_m)}{\partial x_{i_1}^{j_1}\partial x_{i_2}^{j_2}\ldots\partial x_{i_n}^{j_m}}$,

α and β are defined similar to the previous.

As an example of the second order nonlinear partial integro-differential equation is:

$$u_{xx}^2(x,y,z)=3xyz+\int_0^1 e^{zt}u(x,y,t)dt$$

2.1 Introduction:

It is known that, the existence and the uniqueness theorems of the solution for the one-dimensional Fredholm and Volterra linear and nonlinear integral equations due to fixed point theorems are devoted by Hochstadt H. in 1973.

Moreover, some of these theorems are generalized to include system of the one-dimensional integral equations, [Mustafa M., 2004]. On the other hand, these theorems are extended to include the one-dimensional Fredholm and Volterra fuzzy integral equations, [Najieb, S., 2002].

The purpose of this chapter is to modify some of these theorems to ensure the existence and uniqueness solution for the multi-dimensional Fredholm and Volterra integral equations. This chapter consists of the main part of this work.

This chapter consists of three sections

In section one and two, we give some extended theorems for the existence and uniqueness of the solution for the multi-dimensional uniqueness Fredholm and Volterra linear integral equations.

In section three, the existence and uniqueness theorems for special type of the multi-dimensional Fredholm non-linear integral equations are introduced.

2.2 Existence and Uniqueness Theorems for the Multi-Dimensional Fredholm linear Integral Equations:

In This section we shall develop some existence and uniqueness theorems for the 1-D Fredholm linear integral equations to be valid for the m-D Fredholm linear integral equations.

We start this section by recalling the following definition and lemma that we needed later.

Definition (2.1), [Hochstadt H.,1973]:

Let H be a Hilbert space and T be is a bounded operator on H. T is said to be a contraction operator if there exists a positive constant $\alpha < 1$ such that:

$$\|Tf_1 - Tf_2\| \leq \alpha \, \|f_1 - f_2\|$$

for all f_1, f_2 in H.

Lemma (2.1), [Hochstadt H.,1973]:

Let T be a contraction operator defined on a Hilbert space H. The equation

$$Tf = f \qquad (2.1)$$

has a unique solution f in H. Such a solution is said to be a fixed point of T.

Proof:

Suppose there are two fixed points f and g so that

$$Tf = f$$

and

$$Tg = g$$

Then

$$\| f - g \| = \|Tf - Tg\| \leq \alpha \| f - g \|$$

and

$$(1-\alpha)\| f - g \| \leq 0$$

Since $\|f-g\|$ is necessarily non-negative we see that

$$\|f-g\|=0$$

so that $f=g$. It follows that if eq.(2.1) have a solution it must be unique.

To show that eq.(2.1) has a solution we shall set up an iteration procedure. Select any f_0 and then construct a sequence $\{f_n\}$ defined by:

$$f_{n+1}=T\, f_n,\ n=0,1,2,...$$

We shall first show that this sequence is a Cauchy sequence, and then that its limit is indeed a solution of eq.(2.1). That it has a limit will follow from the fact that a Cauchy sequence must have a unique limit in a Hilbert space. The limit will be independent of the initial choice f_0, since it will be a solution of eq.(2.1), which must be unique,

First we note that:

$$\|f_{n+1}-f_n\|=\|Tf_n-Tf_{n-1}\|\leq\alpha\|f_n-f_{n-1}\|$$

By a successive application of the above we have

$$\|f_{n+1}-f_n\|\leq\alpha\,\|f_n-f_{n-1}\|\leq\alpha^2\|f_{n-1}-f_{n-2}\|\leq\cdots\leq\alpha^n\,\|f_1-f_0\|$$

More generally we have, if $n>m$,

$$\|f_n-f_m\|=\|\,(f_n-f_{n-1})+(f_{n-1}-f_{n-2})+\cdots+(f_{m+1}-f_m)\,\|$$

$$\leq\|f_n-f_{n-1}\|+\|f_{n-1}-f_{n-2}\|+\cdots+\|f_{m+1}-f_m\|$$

$$\leq\left(\alpha^{n-1}+\alpha^{n-2}+\cdots+\alpha^m\right)\,\|f_1-f_0\|$$

$$\leq\left(\alpha^m+\alpha^{m+1}+\cdots\right)\,\|f_1-f_0\|=\frac{\alpha^m}{1-\alpha}\|f_1-f_0\|$$

so that:

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0$$

It follows that $\{f_n\}$ is a Cauchy sequence, and we denote its limit by f.

We shall have to show that the limit f is a solution of eq.(2.1). In view of the fact that T is a continuous operator, we have:

$$Tf = T\left(\lim_{n\to\infty} f_n\right)$$

$$= \lim_{n\to\infty} Tf_n$$

$$= \lim_{n\to\infty} f_{n+1} = f \ . \quad \blacksquare$$

Now, recall that the 1-D Fredholm linear integral operator K defined by:

$$Ku = \int_a^b k(x,y)\,u(y)dy \qquad (2.2)$$

is bounded in case $k(x,y)$ is continuous function for all $x,y \in [a,b]$, [Hohstadt H., 1979].

The following theorem is a modification of the above fact to include the 2-D and hence the m-D Fredholm linear integral operator.

<u>Theorem (2.1):</u>

Consider $L_2[D]$ where $D = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$. Suppose $k(x,y,z,m)$ is continuous for all x, z in $[a,b]$ and for all y, m in $[c,d]$. Then the operator

$$Ku = \int_c^d \int_a^b k(x,y,z,m)u(z,m)dzdm$$

is bounded.

Proof:

Since $k(x,y,z,m)$ is continuous on a closed and bounded set, it must be bounded. Then, there exists $M > 0$ such that $|k(x,y,z,m)| \le M$. Hence:

$$|Ku| = \left| \int_c^d \int_a^b k(x,y,z,m)u(z,m)dzdm \right|$$

$$\le \int_c^d \int_a^b |k(x,y,z,m)| \ |u(z,m)|dzdm$$

$$\le M \int_c^d \int_a^b |u(z,m)|dzdm \ ,$$

and y using Cauchy Schwarz inequality

$$|Ku| \le M \left(\int_c^d \int_a^b |u(z,m)|^2 dzdm \right)^{1/2} \left(\int_c^d \int_a^b dzdm \right)^{1/2}$$

$$= M \| u \| (b-a)^{1/2} (d-c)^{1/2}$$

Thus:

$$\|Ku\| = \left[\int_c^d \int_a^b |Ku|^2 dxdy \right]^{1/2}$$

$$\le \left[\int_c^d \int_a^b M^2 \|u\|^2 (b-a) \ (d-c) \ dxdy \right]^{1/2}$$

$$= M \|u\| \, (b-a)^{1/2} (d-c)^{1/2} \left[\int_c^d \int_a^b dx dy \right]^{1/2}$$

$$= M \|u\| (b-a) \, (d-c)$$

Therefore:

$$\|Ku\| \le M \|u\| (b-a) \, (d-c)$$

and hence

$$\|K\| \le M \, (b-a) \, (d-c) \; . \quad \blacksquare$$

Theorem (2.2):

Consider $L_2[D]$, where $D = \{(x_1, x_2, ..., x_m) | \, \alpha_i \le x_i \le \beta_i, \, i = 1,2,...,m\}$. Suppose $k(x_1, x_2, ..., x_m, z_1, z_2, ..., z_m)$ is continuous for all $\alpha_i \le x_i, \, z_i \le \beta_i, \, i = 1,2,...,m.$ Then the operator:

$$Ku = \int_{\alpha_m}^{\beta_m} \int_{\alpha_{m-1}}^{\beta_{m-1}} ... \int_{\alpha_1}^{\beta_1} k \, (x_1,..,x_m, z_1,..,z_m) u(z_1,..,z_m) dz_1 ... dz_m$$

is bounded.

Next, remember that a sufficient condition for the 1-D Fredholm linear integral operator given by eq.(2.2) to be bounded is:

$$\int_a^b \int_a^b |k(x,y)|^2 dx dy = M^2 < \infty$$

[Hochstadt H., 1973].

The following theorem is an extension of the above fact to include the 2-D and hence the m-D Fredholm linear integral operators.

Theorem (2.3):

Consider $L_2[D]$ where If $D=\{(x,y)\mid a\le x\le b,\ c\le y\le d\}$. If

$$\int_c^d\int_a^b\int_c^d\int_a^b |k(x,y,z,m)|^2 dzdmdxdy = M^2 < \infty$$

then the operator:

$$Ku=\int_c^d\int_a^b k(x,y,z,m)u(z,m)\ dzdm$$

is bounded.

Proof:

By Cauchy-Schwarz inequality:

$$|Ku|\le\left[\int_c^d\int_a^b |k(x,y,z,m)|^2\ dzdm\right]^{1/2}\left[\int_c^d\int_a^b |u(z,m)|^2 dzdm\right]^{1/2}$$

$$\le\left[\int_c^d\int_a^b |k(x,y,z,m)|^2 dzdm\right]^{1/2}\|u\|$$

Then

$$\|Ku\|=\left[\int_c^d\int_a^b |Ku|^2\ dzdm\right]^{1/2}$$

$$\le\left[\int_c^d\int_a^b \left|\left(\int_c^d\int_a^b |k(x,y,z,m)|^2 dzdm\right)\|u\|^2\right| dxdy\right]^{1/2}$$

$$=\|u\|\left[\int_c^d\int_a^b\int_c^d\int_a^b |k(x,y,z,m)|^2 dzdmdxdy\right]^{1/2}$$

Therefore $\|Ku\|\le M\|u\|$. ■

Theorem (2.4):

Consider $L_2[D]$ where $D = \{(x_1,...,x_m) \mid \alpha_i \le x_i \le \beta_i,\ i = 1,...,m\}$. If:

$$\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}|k(x_1,\ldots,x_m,z_1,\ldots,z_m)|^2 dz_1\ldots dz_m, dx_1\ldots dx_m = M^2 < \infty$$

then the operator:

$$Ku = \int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,..,x_m,z_1,..,z_m)u(z_1,..,z_m)dz_1...dz_m$$

is bounded.

Now, recall that the 1-D Fredholm linear integral equation:

$$u(x) = f(x) + \lambda\int_a^b k(x,y)u(y)dy$$

has a unique solution for all f and sufficiently small $|\lambda|$, provided K is a bounded operator, where K is defined by eq.(2.2), [Hochstadt H., 1973].

The following theorem generalize the above fact to be valid for the 2-D Fredholm linear integral equations and hence for the m-D Fredholm linear integral equations.

Theorem (2.5):

Consider the 2-D Fredholm linear integral equation

$$u(x,y) = f(x,y) + \lambda\int_c^d\int_a^b k(x,y,z,m)u(z,m)dzdm \qquad (2.3)$$

If the operator K defined by:

$$Ku = \int_c^d \int_a^b k(x, y, z, m) u(z, m) dz dm$$

is a bounded linear operator, then eq.(2.3) has a unique solution for all f in $L_2[D]$ and for sufficiently small $|\lambda|$, where $D = \{(x, y) | a \le x \le b, c \le x \le d\}$.

Proof:

Rewrite eq.(2.3) in the form

$$Tu = u$$

where

$$Tu = f + \lambda Ku$$

Then

$$\|Tu_1 - Tu_2\| = \|f + \lambda Ku_1 - f - \lambda Ku_2\|$$

$$= \|\lambda Ku_1 - \lambda Ku_2\|$$

$$= |\lambda| \, \|Ku_1 - Ku_2\|$$

But K is a linear operator then

$$\|Tu_1 - Tu_2\| = |\lambda| \, \|K(u_1 - u_2)\|$$

Since K is a bounded operator then there exists a constant $M > 0$ such that

$$\|Ku\| \le M \, \|u\|$$

for all u in the Hilbert space $L_2[D]$. Therefore:

$$\|Tu_1 - Tu_2\| \le |\lambda| \, M \, \|u_1 - u_2\|$$

For $|\lambda| \, M < 1$, T is a contraction operator, so by using lemma (2.1), T has a unique fixed point u in $L_2[D]$ which is the unique solution of eq.(2.3). ■

The proof of the following corollary is easy, so we omitted it.

Corollary (2.1):

Consider eq.(2.3). If

$$\int_c^d\int_a^b\int_c^d\int_a^b |k(x,y,z,m)|^2 dzdmdxdy < \infty$$

then eq.(2.3) has a unique solution for all f in $L_2[D]$ and for sufficiently small $|\lambda|$.

Next, the proofs of the following generalized theorem and its corollary are easy, thus we omitted them.

Theorem (2.6):

Consider the m-D Fredholm linear integral equation:

$$u(x_1,x_2,\ldots,x_m) = f(x_1,x_2,\ldots,x_m) +$$

$$\lambda \int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)\, dz_1\ldots dz_m \quad (2.4)$$

If the operator K defined by:

$$Ku = \int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)dz_1\ldots\, dz_m$$

is bounded operator then eq.(2.4) has a unique solution for all f in $L_2[D]$ and for sufficiently small $|\lambda|$, where $D = \{(x_1,\ldots,x_m) | \alpha_i \le x_i \le \beta_i, i = 1,\ldots,m\}$.

Corollary (2.2):

Consider eq.(2.4). If:

$$\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}\left|k(x_1,\ldots,x_m,z_1,\ldots,z_m)\right|^2 dz_1\ldots dz_m, dx_1\ldots dx_m < \infty$$

then eq.(2.4) has a unique solution for all f in $L_2[D]$ and for sufficiently small $|\lambda|$.

Next, recall that the 1-D Fredholm linear integral operator defined by

$$Ku = \int_0^1 k(x,y)u(y)dy$$

has at least one positive eigenvalue, corresponding to a positive eigenfunction in case $k(x,y)$ be a continuous, positive function for all $0 \le x, y \le 1$.

The following theorem generalize the above fact to be valid for the 2-D Fredholm linear integral equations and hence for the m-D Fredholm linear integral equations.

Theorem (2.7):

Let $k(x,y,z,m)$ be a continuous, positive function for all $0 \le x, y,$ z,$m \le 1$. Consider the operator

$$K\,u = \int_0^1\int_0^1 k(x,y,z,m)\,u(z,m)\,dzdm$$

K has at least one positive eigenvalue, corresponding to a positive eigenfunction.

Proof:

Since $k(x,y,z,m)$ is positive and continuous, then we can assume that $0 < m \le k(x,y,z,m) \le M$. Let:

$$S = \left\{ u(x,y) \middle| u(x,y) \geq 0, \int_0^1\int_0^1 |u(x,y)|^2 dxdy \leq 1 \right\}$$

We show that S is convex subset of $L_2[D]$. To do this, let u_1 and u_2 in S, $0 \leq t \leq 1$ then $t\ u_1 + (1-t)\ u_2 \geq 0$

$$\int_0^1\int_0^1 \left| t\ u_1(x,y) + (1-t)\ u_2(x,y) \right|^2 dxdy \leq t^2 \int_0^1\int_0^1 |u_1(x,y)|^2 dxdy +$$

$$2t(1-t)\int_0^1\int_0^1 |u_1(x,y)||u_2(x,y)| dxdy + (1-t)^2 \int_0^1\int_0^1 |u_2(x,y)|^2 dxdy$$

$$\int_0^1\int_0^1 \left| tu_1 + (1-t)u_2 \right|^2 dxdy \leq t^2 + 2t(1-t)\left[\int_0^1\int_0^1 |u_1(x,y)|^2 dxdy \int_0^1\int_0^1 |u_2(x,y)|^2 dxdy \right]^{1/2} + (1-t)^2$$

$$\leq t^2 + 2t(1-t) + (1-t)^2 = 1$$

Therefore S is convex subset of $L_2[D]$.

Also, we show that S is closed, to do this, let $\{u_n\}$ be a sequence in S such that $u_n \longrightarrow u$. Then $\| u_n \| \longrightarrow \| u \|$. But $\| u_n \| \leq 1$, thus $\| u \| \leq 1$ and hence $u \in S$. Therefore S is closed subset of $L_2[D]$.

Since $k(x,y,z,m) \geq m > 0$, we have:

$$K\left[u + \frac{1}{n} \right] = \int_0^1\int_0^1 k(x,y,z,m) \left(u(z,m) + \frac{1}{n} \right) dzdm$$

if $u \in S$, then $u(z,m) \geq 0$, hence:

$$u(z,m)+\frac{1}{n}\geq\frac{1}{n},$$

But

$$k(x,y,z,m)\geq m \quad \text{then} \quad k(x,y,z,m)\left[u(z,m)+\frac{1}{n}\right]\geq\frac{m}{n}$$

then

$$\int_0^1\int_0^1 k(x,y,z,m)\left[u(z,m)+\frac{1}{n}\right]dzdm \geq \int_0^1\int_0^1 \frac{m}{n}dzdm = \frac{m}{n}$$

thus $k\left[u+\frac{1}{n}\right]\geq\frac{m}{n},\ u\in S$

where n is a positive integer. From the above inequality we see that

$$\left\| K\left[u+\frac{1}{n}\right]\right\| = \left\{\int_0^1\int_0^1\left(\int_0^1\int_0^1 k(x,y,z,m)\ \left(u(z,m)+\frac{1}{n}\right)dzdm\right)^2 dxdy\right\}^{1/2}$$

$$\geq \frac{m}{n} \tag{2.5}$$

Consider now the mapping of S into itself defined by

$$T_n u = \frac{K\left[u+\frac{1}{n}\right]}{\left\| K\left[u+\frac{1}{n}\right]\right\|} \tag{2.6}$$

Evidently the denominator is bounded away from 0 by use of ineq.(2.5). Also $\| T_n u \| = 1$ and for u in S we clearly have $T_n u \geq 0$, so that T_n maps S into itself. It is also easy to see that T_n is continuous mapping.

Since $k(x,y,z,m)\leq M$ and u is in S, we observe that

$$\left| K\left[u+\frac{1}{n}\right] \right| \le \left\{ \int_0^1\int_0^1 k^2(x,y,z,m)dzdm \int_0^1\int_0^1 \left(u(z,m)+\frac{1}{n}\right)^2 dzdm \right\}^{1/2}$$

$$\le 2M$$

so that the numerator of eq.(2.6) is uniformly bounded. Similarly:

$$\left| K\left[u+\frac{1}{n}\right] \right| \ge \left| K\frac{1}{n} \right| \ge \frac{m}{n}$$

so that

$$|T_n u| \le \frac{2nM}{m}$$

so that the set of functions $\{T_n u\}$, $u \in S$ is uniformly bounded, and also continuous. Similarly they are equicontinuous so that $T_n(S)$ is compact and T_n has fixed point so that for some u_n we have

$$T_n u_n = u_n \qquad (2.7)$$

We shall now show that we can select a subsequence of $\{u_n\}$ that converges to an eigenfunction of k, corresponding to a positive eigenvalue.

By the above equation and eq.(2.6) we obtain

$$K\left[u_n+\frac{1}{n}\right] = \lambda_n u_n \qquad (2.8)$$

where:

$$\lambda_n = \left\| K\left[u_n+\frac{1}{n}\right] \right\| \qquad (2.9)$$

Since $\| u_n + (1/n) \| \le \| u_n \| + \| 1/n \| \le 2$. we see that the set $\{u_n + 1/n\}$ is uniformly bounded in $L_2[D]$. The linear integral operator K is a compact operator, and therefore we can extract a subsequence $\{u_{n_k}+1/n_k\}$ such that the sequence $\{K\ [u_{n_k}+1/n_k]\}$ converges to some function, say ψ.

We shall now show that the sequence $\{\lambda_{n_k}\}$ also converges to some positive limit. We see that:

$$u_n=\frac{1}{\lambda_n}K\left[u_n+\frac{1}{n}\right]$$

$$\geq\frac{1}{\lambda_n}K\left[\frac{1}{\mathrm{n}}\right]\geq\frac{m}{n\lambda_n}$$

From eq.(2.9) and also that $u_n(x,y)$ is a positive continuous function on D. Let

$$\min_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}} u_n(x,y)=\beta_n>0$$

so that

$$\beta_n\geq\frac{m}{n\ \lambda_n}$$

We can refine these estimates by suing eq.(2.9) again so that

$$u_n=\frac{1}{\lambda_n}K\left[u_n+\frac{1}{n}\right]\geq\frac{1}{\lambda_n}\left[m\beta_n+\frac{m}{n}\right]$$

and it follows that

$$\beta_n\geq\frac{1}{\lambda_n}\left[m\beta_n+\frac{m}{n}\right]$$

The last estimate shows that $\lambda_n\geq m$ and is therefore bounded away from the origin. Eq.(2.9) shows that $\lambda_{n_k}=\left\|K\left[u_{n_k}+\frac{1}{n_k}\right]\right\|\longrightarrow\|\ \psi\ \|$, and we shall denote that limiting value by $\lambda_0\geq m$.

Finally we see from eq.(2.8) that $u_{n_k} \longrightarrow \frac{1}{\lambda_0}\psi$. Accordingly:

$$K\left[u_{n_k} + \frac{1}{n_k}\right] \longrightarrow K\frac{1}{\lambda_o}\psi \quad \text{and also} \quad K\left[u_{n_k} + \frac{1}{n_k}\right] \longrightarrow \psi,$$

so that $K\left[\frac{1}{\lambda_0}\psi\right] = \psi$, or more explicitly:

$$\int_0^1\int_0^1 k(x,y,z,m)\ \psi(z,m)\ dzdm = \lambda_0\psi(x,y)$$

The left side of the above equation is continuous and positive so that $\psi(x,y)$ is continuous and positive. ■

Theorem (2.8):

Let $k(x_1,x_2,...,x_m,z_1,z_2,...,z_m)$ be a continuous positive for all $0 \le x_i, z_i \le 1$. Consider operator K defined by:

$$Ku = \int_0^1\int_0^1\cdots\int_0^1 k(x_1,\ldots,x_m,z_1,\ldots,z_m)\ u(z_1,\ldots,z_m)dz_1\ldots\ dz_m$$

K has at least one positive eigenvalue, corresponding to a positive eigenfunction.

2.3 Existence and Uniqueness Theorems for The Multi-Dimensional Volterra Linear Integral Equations:

In this section we modify some existence and uniqueness theorems for the 1-D Volterra linear integral equations to be valid for the m-D Volterra linear integral equations.

We start this section by the following lemma which we need it later.

Lemma (2.2), [Hochstadt H.,1973]:

Let T be an operator defined on a Hilbert space H, such that the nth power of T, namely T^n is a contraction operator. Then the equation:

$$Tf = f$$

has a unique solution f in H.

Proof:

By lemma (2.1) we can assert that the equation:

$$T^n f = f$$

has a unique solution. In fact, we can obtain the solution by finding

$$\lim_{k\to\infty} T^{kn} f_0 = f$$

for an arbitrary initial function f_0 In particular, we see that, by letting $f_0 = Tf$

$$\lim_{k\to\infty} T^{kn} Tf = f$$

But since $T^n f = f$ we also have $T^{kn} f = f$, so that:

$$\lim_{k\to\infty} T^{kn} Tf = \lim_{k\to\infty} T\, T^{kn} f$$
$$= \lim_{k\to\infty} Tf = Tf$$

so that $Tf = f$.

To show that this solution is unique we note that if:

$$Tf = f, \quad Tg = g$$

then we also have:

$$T^n f = f, \quad T^n g = g$$

and since T^n is a contraction operator with a unique fixed point then $f = g$. ■

Now, recall that, the 1-D Volterra linear integral equation:

$$u(x) = f(x) + \lambda \int_0^x k(x,y)u(y)dy \tag{2.10}$$

has a unique solution $u(x)$ for all λ and $f(x)$ in $L_2[0,1]$ in case $k(x,y)$ is a continuous function for $x, y \in [0,1]$ and therefore uniformly bounded, say $|k(x,y)| \leq M$, [Hochstadt H.,1973].

The following theorem is a generalization of the above fact to be hold for the 2-D and hence for the m-D Volterra linear integral equations.

Theorem (2.9):

Consider the 2-D Volterra linear integral equation:

$$u(x,y) = f(x,y) + \lambda \int_0^y \int_0^x k(x,y,z,m)u(z,m)dzdm \tag{2.11}$$

where $f(x,y) \in L_2[D]$ and suppose that $k(x, y, z, m)$ is a continuous function for $x, y, z, m \in [0,1]$ and therefore uniformly bounded, say $|k(x,y,z,m)| \leq M$. Then eq.(2.10) has a unique solution, for all λ and f in $L_2[D]$, where $D = \{(x,y) | \ 0 \leq x \leq 1, \ \ 0 \leq y \leq 1\}$

Proof:

Rewrite eq.(2.11) in the form:

$Tu = u$

where:

$$Tu = f + \lambda Ku \quad \text{and} \quad Ku = \int_0^y \int_0^x k(x,y,z,m)u(z,m)dzdm$$

Then, we use the mathematical induction to prove

$$T^n u = f + \lambda Ku + \cdots + \lambda^{n-1}K^{n-1}f + \lambda^n K^n f \qquad (2.12)$$

For $n = 1$, $Tu = f + \lambda Ku$

For $n=2$, one can get

$$T^2 u = T(Tu) = T(f + \lambda Ku) = f + \lambda(f + \lambda Ku) = f + \lambda f + \lambda^2 Ku$$

Assume eq.(2.11) hold for $n = \ell$, then:

$$T^{\ell+1}u = T\left(T^{\ell}u\right)$$

$$= T\left(f + \lambda Kf + \cdots + \lambda^{\ell-1}K^{\ell-1}f + \lambda^{\ell}K^{\ell}u\right)$$

$$= f + \lambda K\left(f + \lambda Kf + \cdots + \lambda^{\ell-1}K^{\ell-1}f + \lambda^{\ell}K^{\ell}u\right)$$

$$= f + \lambda Kf + \lambda^2 K^2 f + \cdots + \lambda^{\ell}K^{\ell}f + \lambda^{\ell+1}K^{\ell+1}u$$

Therefore

$$T^n u = f + \lambda Kf + \cdots + \lambda^{n-1}K^{n-1}f + \lambda^n K^n u$$

Next, we also use the mathematical induction to prove

$$K^n u(x,y) = \int_0^y \int_0^x k_n(x,y,s,t)\, u(s,t)\, ds\, dt$$

where

$$k_1(x,y,s,t) = k(x,y,s,t)$$

and

$$k_n(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_{n-1}(z,m,s,t)dzdm,\ \ n=2,3,\ldots$$

For $n=1$, one can get

$$K^1u(x,y)=\int_0^y\int_0^x k_1(x,y,s,t)u(s,t)dsdt=\int_0^y\int_0^x k(x,y,s,t)u(s,t)dsdt$$

For $n=2$, one can get

$$K^2u(x,y)=K\left[\int_0^y\int_0^x k(x,y,z,m)u(z,m)dzdm\right]$$

$$=\int_0^y\int_0^x k(x,y,z,m)\left[\int_0^m\int_0^z k(z,m,s,t)u(s,t)dsdt\right]dzdm$$

where:

$$k_2(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k(z,m,s,t)dzdm$$

hence

$$K^2u(x,y)=\int_0^y\int_0^x k_2(x,y,s,t)\ u(s,t)\ dsdt$$

Assume

$$K^\ell u(x,y)=\int_0^y\int_0^x k_\ell(x,y,s,t)\ u(s,t)\ dsdt$$

where

$$k_\ell(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_{\ell-1}(z,m,s,t)\ dzdm$$

then

$$K^{\ell+1}u(x,y)=K\left(K^{\ell}u(x,y)\right)$$

$$=\int_0^y\int_0^x k(x,y,z,m)\,K^{\ell}u(z,m)dzdm$$

$$=\int_0^y\int_0^x k(x,y,z,m)\left[\int_0^m\int_0^z k_{\ell}(z,m,s,t)\,u(s,t)dsdt\right]dzdm$$

$$=\int_0^y\int_0^x\left[\int_t^y\int_s^x k(x,y,z,m)k_{\ell}(z,m,s,t)dzdm\right]u(s,t)dsdt$$

$$=\int_0^y\int_0^x k_{\ell+1}(x,y,s,t)u(s,t)\,dsdt$$

since $k(x,y,s,t)$ is bounded, then there exists a constant M such that $\left|k(x,y,s,t)\right|\le M$ for all $(x,y),\ (s,t)\in \mathrm{D}$. Then we shall show that

$$\left|k_n(x,y,s,t)\right|\le\frac{M^n\left(x-s\right)^{n-1}\left(y-t\right)^{n-1}}{\left[(n-1)\ !\right]^2},0\le s\le x,0\le t\le y \quad (2.13)$$

To do this, we use the mathematical induction.

For n=1, the above inequality is obviously valid. For n=2,

$$\left|k_2(x,y,s,t)\right|=\left|\int_t^y\int_s^x k(x,y,z,m)k(z,m,s,t)dzdm\right|$$

$$\le\int_t^y\int_s^x\left|k(x,y,z,m)\right|\ \left|k(z,m,s,t)\right|dzdm$$

$$\le\int_t^y\int_s^x M^2dzdm\ =M^2(x-s)(y-t)$$

For $n = 3$

$$k_3(x,y,s,t) = \left| \int_t^y \int_s^x k(x,y,z,m) k_2(z,m,s,t) dz dm \right|$$

$$\leq \int_t^y \int_s^x |k(x,y,z,m)| |k_2(x,y,z,m)| dz dm$$

$$\leq \int_t^x \int_s^y MM^2 (z-s)(m-t) dz dm \;= M^3 \frac{(x-s)^2 (y-t)^2}{(2!)^2}$$

Assume

$$|k_\ell(x,y,s,t)| \leq \frac{M^\ell (x-s)^{\ell-1} (y-t)^{\ell-1}}{\left[(\ell-1)\,!\right]^2}, \;\; 0 \leq s \leq x, \;\; 0 \leq t \leq y$$

then

$$|k_{\ell+1}(x,y,s,t)| = \left| \int_t^y \int_s^x k(x,y,z,m) k_{\ell-1}(z,m,s,t) dz dm \right|$$

$$\leq \int_t^y \int_s^x |k(x,y,z,m)| \; |k_{\ell-1}(z,m,s,t)| dz dm$$

$$\leq \int_t^y \int_s^x \frac{MM^\ell (z-s)^{\ell-1} (m-t)^{\ell-1}}{\left[(\ell-1)\,!\right]^2} dz dm$$

$$= \frac{M^{\ell+1}}{\left[(\ell-1)\,!\right]^2} \left[\left. \frac{(z-s)^\ell}{\ell} \right|_s^x \right] \left[\left. \frac{(m-t)^\ell}{\ell} \right|_t^y \right]$$

$$= \frac{M^{\ell+1}}{\left[(\ell-1)\,!\right]^2} \; \frac{(x-s)^\ell}{\ell} \; \frac{(y-t)^\ell}{\ell}$$

$$= \frac{M^{\ell+1} (x-s)^\ell (y-t)^\ell}{[\ell\,!]^2}$$

Thus ineq.(1.13) holds. Therefore, we have:

$$\|T^n u_1 - T^n u_2\| = \|f + \lambda Kf + \cdots + \lambda^{n-1}K^{n-1}f + \lambda^n K^n u_1 - f - \lambda Kf - \cdots - \lambda^{n-1}K^{n-1}f - \lambda^n K^n u_2\|$$

$$= \|\lambda^n K^n u_1 - \lambda^n K^n u_2\|$$

$$= |\lambda|^n \ \|K^n (u_1 - u_2)\|$$

$$= |\lambda|^n \left\| \int_0^y \int_0^x k_n(x,y,s,t)\big(u_1(s,t) - u_2(s,t)\big)dsdt \right\|$$

$$\leq |\lambda|^n \left\| \int_0^y \int_0^x |k_n(x,y,s,t)| \ \big(u_1(s,t) - u_2(s,t)\big)dsdt \right\|$$

$$\leq |\lambda|^n \left\| \int_0^y \int_0^x \frac{M^n (x-s)^{n-1}(y-t)^{n-1}}{\left[(n-1)\,!\right]^2} \ \big(u_1(s,t) - u_2(s,t)\big)dsdt \right\|$$

But since $0 \leq s \leq x \leq 1$, then $0 \leq x - s \leq 1 - s \leq 1$ and hence $(x-s)^n \leq 1$. Similarly, one can get $(y-t)^n \leq 1$, therefore:

$$\|T^n u_1 - T^n u_2\| \leq \frac{|\lambda|^n M^n}{\left[(n-1)\,!\right]^2} \left\| \int_0^y \int_0^x \big(u_1(s,t) - u_2(s,t)\big) \ dsdt \right\|$$

$$\leq \frac{|\lambda|^n M^n}{[(n-1)!]^2} \| u_1 - u_2 \|$$

For n sufficiently large

$$\frac{|\lambda|^n M^n}{\left[(n-1)\,!\right]^2} < 1$$

so that T^n is a contraction operator and hence by using lemma (2.2), eq.(2.11) has a unique solution. ■

Now, the proof of the following theorem is easy to be satisfied, thus we omitted it.

Theorem (2.10):

Consider the m-D Volterra linear integral equation:

$$u(x_1,x_2,\ldots,x_m)=f(x_1,x_2,\ldots,x_m)+$$

$$\lambda\int_0^{x_m}\int_0^{x_{m-1}}\cdots\int_0^{x_1}k(x_1,\ldots,x_m,z_1,\ldots,z_m)u(z_1,\ldots,z_m)dz_1\ldots dz_m$$

where $f(x_1,x_2,\ldots,x_m)\in L_2[D]$ and $k(x_1,x_2,\ldots,x_m,z_1,z_2,\ldots,z_m)$ is a continuous function for all $x_1,x_2,\ldots,x_m,z_1,z_2,\ldots,z_m$ in [0,1] and therefore uniformly bounded, say $|k(x_1,x_2,...,x_m,z_1,z_2,...,z_m)|\le M$. Then this integral equation has a unique solution for all λ and $f(x_1,x_2,\ldots,x_m)$ in $L_2[D]$, where $D=\{x_1,x_2,\ldots,x_m \mid 0\le x_i \le 1,\ i=1,2,\ldots,\mathrm{m}\}$.

2.4 Existence and Uniqueness Theorems For The Multi-Dimensional Fredholm non-linear Integral Equations:

In This section we shall develop some existence and uniqueness theorems for the 1-D Fredholm non-linear integral equations to be valid for the m-D Fredholm non-linear integral equations.

We start this section by recalling that, the 1-D non-linear integral equation:

$$u(x)=f(x)+\lambda\int_a^b k(x,y,u(y))dy$$

has a unique solution in $L_2[a,b]$ if the following conditions are satisfied

(i) Suppose $\left\| \int_a^b k(x,y,u(y))dy \right\| \leq M \|u\|$.

(ii) $|k(x,y,z,m) - k(x,y,z,m)| \leq N(x,y)|z_1 - z_2|$.

(iii) $\int_a^b \int_a^b |N(x,y)|^2 dxdy = P^2 < \infty$.

(iv) $|\lambda| P < 1$, [Hochstadt H., 1973].

The following theorem is a generalization of the above fact to include the 2-D and hence the m-D Fredholm non-linear integral equations.

Theorem (2.11), [Hochstadt H., 1973]:

Consider the 2-D Fredholm non-linear integral equation

$$u(x,y) = f(x,y) + \lambda \int_c^d \int_a^b k(x,y,z,m,u(z,m))dzdm \qquad (2.14)$$

Suppose that:

$$\left\| \int_a^b \int_a^b k(x,y,z,m,u(z,m))dzdm \right\| \leq M \|u\|$$

and

$$|k(x,y,z,m,u_1) - k(x,y,z,m,u_2)| \leq N(x,y,z,m) \; |u_1 - u_2|$$

where:

$$\int_c^d \int_a^b \int_c^d \int_a^b |N(x,y,z,m)|^2 dzdmdxdy = P^2 < \infty$$

If $|\lambda| P < 1$, then eq.(2.14) has a unique solution for all f in $L_2[D]$, where

Proof:

Rewrite eq.(2.5) in the form

$$Tu = u$$

where

$$Tu = f + \lambda K u$$

and

$$Ku(x,y) = \int_c^d \int_a^b k(x,y,z,m,u(z,m))dzdm$$

Then

$$\|Tu_1 - Tu_2\| = \left[\int_c^d \int_a^b |Tu_1 - Tu_2|^2 dxdy\right]^{1/2}$$

$$= \left[\int_c^d \int_a^b \left|\lambda \int_c^d \int_a^b k(x,y,z,m,u_1(z,m))dzdm - \lambda \int_c^d \int_0^b k(x,y,z,m,u_2(z,m))\, dzdm\right|^2 dxdy\right]^{1/2}$$

$$= |\lambda| \left\{\int_c^d \int_a^b \left[\int_c^d \int_a^b |k(x,y,z,m,u_1(z,m)) - k(x,y,z,m,u_2(z,m))|\, dzdm\right]^2 dxdy\right\}^{1/2}$$

$$\leq |\lambda| \left\{\int_c^d \int_a^b \left[\int_c^d \int_a^b N(x,y,z,m)|u_1(z,m) - u_2(z,m)|dzdm\right]^2 dxdy\right\}^{1/2}$$

$$\leq |\lambda| \left\{\int_0^d \int_0^b \left[\int_c^d \int_a^b |N(x,y,z,m)|^2 dzdm \int_c^d \int_a^b |u_1(z,m) - u_2(z,m)|^2 dzdm\right] dxdy\right\}^{1/2}$$

$$\leq |\lambda| \left\{ \int_c^d \int_a^b \left[\int_c^d \int_a^b |N(x,y,z,m)|^2 dz dm \right] \| u_1 - u_2 \|^2 dx dy \right\}^{1/2}$$

$$= |\lambda| \; \| u_1 - u_2 \| \left[\int_c^d \int_a^b \int_c^d \int_a^b |N(x,y,z,m)|^2 dz dm \; dx dy \right]^{1/2}$$

$$\leq |\lambda| \; P \; \| u_1 - u_2 \|$$

It follows that for $|\lambda| P < 1$, T is a contraction operator so that it has a unique fixed point. By using lemma (2.1) this fixed point is a solution of eq.(2.14). ■

In a similar manner, one can prove the following theorem.

Theorem (2.12):

Consider the m-D Fredholm non-linear integral equation,

$$u(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m) +$$

$$\lambda \int_{\alpha_m}^{\beta_m} \int_{\alpha_{m-1}}^{\beta_{m-1}} \cdots \int_{\alpha_1}^{\beta_1} k(x_1, \ldots, x_m, z_1, \ldots, z_m, u(z_1, \ldots, z_m)) dz_1 \ldots dz_m \quad (2.15)$$

Suppose:

$$\left\| \int_{\alpha_m}^{\beta_m} \int_{\alpha_{m-1}}^{\beta_{m-1}} \cdots \int_{\alpha_1}^{\beta_1} k(x_1, \ldots, x_m, z_1, \ldots, z_m, u(z_1, \ldots, z_m)) dz_1 \ldots dz_m \right\| \leq M \; \|u\|$$

and that

$$|k(x_1, \ldots, x_m, z_1, \ldots, z_m, u_1) - k(x_1, \ldots, x_m, z_1, \ldots, z_m, u_2)| \leq$$

$$N(x_1, \ldots, x_m, z_1, \ldots, z_m) | \; u_1 - u_2 |$$

where:

$$\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1}|N(x_1,\ldots,x_m,z_1,\ldots,z_m)|^2 dz_1\ldots dz_m dx_1\ldots dx_m = P^2<\infty$$

If $|\lambda|P<1$, then eq.(2.15) has a unique solution for all f in $L_2[D]$ where $D=\{(x_1,x_2,\ldots,x_m) \mid \alpha_i \le x_i \le \beta_i,\ i=1,2,...,m\}$.

Next, recall that the 1-D non-linear integral equation:

$$u(x)=\lambda\int_0^1 k(x,y)\psi(y,u(y))dy \qquad (2.16)$$

has a unique solution in $L_2[0,1]$ if the following condition:

(i) $k(x,y)$ is continuous for all x, y in [0,1] and that $\psi(y,t)$ is continuous for all y in [0,1] and all t.

(ii) $\int_0^1 |\psi(y,u(y))|^2 \le A^2\|u\|^2$.

(iii) $\psi(y,t)$ satisfies the Lipschitz condition

$$|\psi(y,t_1)-\psi(y,t_2)| \le B\,|t_1-t_2|$$

where B is independent of y.

(iv) $|k(x,y)| \le C$.

(v) $|\lambda| < \dfrac{1}{BC}$, [Hochstadt H., 1973].

The following theorem is a generalization of the above fact to include the 2-D and hence the m-D Fredholm non-linear integral equations.

Theorem (2.13):

Consider the 2-D Fredholm non-linear integral equation

$$u(x,y)=\lambda\int_0^1\int_0^1 k(x,y,z,m)\ \psi\big(z,m,u(z,m)\big)\ dz\ dm \qquad (2.17)$$

and that x,y,z,m in [0,1] is continuous for all $k(x,y,z,m)$ suppose that is continuous for all z, m in [0,1] and all t and that $\psi(z,m,t)$

$$\int_0^1\int_0^1 \left|\ \psi\ \big(z,m,u(z,m)\big)\ \right|^2 dzdm \le A^2\|\ u\|^2$$

Suppose that $\psi(z,m,t)$ also satisfies the Lipschitz condition

$$\left|\ \psi(z,m,t_1)-\ \psi(z,m,t_2)\ \right|\le B\ \left|\ t_1-t_2\right|$$

where B is independent of z and m, and let

$$\left|\ k(x,y,z,m)\ \right|\le C$$

Then eq.(2.17) has a unique solution in $L_2[D]$ provided $\left|\ \lambda\ \right|<\dfrac{1}{BC}$ where $D=\big\{(x,y)\ \big|\ 0\le x\le 1,\ 0\le y\le\ 1\big\}$.

Proof:

Define the operator T by

$$T\ u=\lambda\int_0^1\int_0^1 k(x,y,z,m)\ \psi\big(z,m,u(z,m)\big)\ dzdm$$

so that a solution of eq.(2.17) is a fixed point. T is a bounded operator since:

$$\|\ T\ u\|=\left[\ |\lambda|^2\int_0^1\int_0^1 |Tu|^2 dxdy\ \right]^{1/2}$$

$$=|\lambda|\left[\int_0^1\int_0^1 |Tu|^2\,dxdy\right]^{1/2}$$

$$=|\lambda|\left[\int_0^1\int_0^1\left|\int_0^1\int_0^1 k(x,y,z,m)\ \psi\big(z,m,u(z,m)\big)\,dzdm\right|^2 dxdy\right]^{1/2}$$

$$\le|\lambda|\left[\int_0^1\int_0^1\left\{\left(\int_0^1\int_0^1 |k(x,y,z,m)|^2 dzdm\right)^{1/2}\left(\int_0^1\int_0^1 |\ \psi\big(z,m,u(z,m)\big)\ |^2 dzdm\right)^{1/2}\right\}^2 dxdy\right]^{1/2}$$

$$\le|\lambda|\left[\int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m)|^2\,dzdmdxdy\int_0^1\int_0^1 |\ \psi\big(z,m,u(z,m)\big)\ |^2 dzdm\right]^{1/2}$$

$$\le|\lambda|\left[\ C^2A^2\|u\|^2\right]^{1/2}=|\lambda|\ C\ A\ \|u\|.$$

Therefore T is a bounded operator. Now:

$$\|Tu_1-Tu_2\|=\left\|\lambda\int_0^1\int_0^1 k(x,y,z,m)\ \psi\big(z,m,u_1(z,m)\big)\,dzdm-\lambda\int_0^1\int_0^1 k(x,y,z,m)\ \psi\big(z,m,u_2(z,m)\big)\,dzdm\right\|$$

$$\le|\lambda|\ \left\{\int_0^1\int_0^1\left|\int_0^1\int_0^1 k(x,y,z,m)\big[\psi\big(z,m,u_1(z,m)\big)-\psi\big(z,m,u_2(z,m)\big)\big]dzdm\right|^2 dxdy\right\}^{1/2}$$

Thus:

$$\|Tu_1-Tu_2\|\le|\lambda|\left\{\int_0^1\int_0^1\left[\int_0^1\int_0^1 |k(x,y,z,m)|\,\big|\psi\big(z,m,u_1(z,m)\big)-\psi\big(z,m,u_2(z,m)\big)\big|\,dzdm\right]^2 dxdy\right\}^{1/2}$$

$$\le|\lambda|\left\{\int_0^1\int_0^1\left[\left(\int_0^1\int_0^1 |k(x,y,z,m)|^2 dzdm\right)^{1/2}\left(\int_0^1\int_0^1 \big|\psi\big(z,m,\ u_1(z,m)\big)-\psi\big(z,m,\ u_2(z,m)\big)\big|^2 dzdm\right)^{1/2}\right]^2 dxdy\right\}^{1/2}$$

$$\leq |\lambda| \left\{ \int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m)|^2 dzdmdxdy \int_0^1\int_0^1 |\psi(z,m,\ u_1(z,m)) - \psi(z,m,\ u_2(z,m))|^2 dzdm \right\}^{1/2}$$

$$\leq |\lambda| \left\{ \int_0^1\int_0^1\int_0^1\int_0^1 C^2 dzdmdxdy \ \int_0^1\int_0^1 B^2 |u_1(z,m) - u_2(z,m)|^2 dzdm \right\}^{1/2}$$

$$\leq |\lambda| CB \left[\int_0^1\int_0^1 | u_1(z,m) - u_2(z,m)|^2 dzdm \right]^{1/2} \leq |\lambda| \ CB \ \| u_1 - u_2 \|$$

But $|\lambda| CB < 1$, thus T is contraction operator and by using lemma (2.1), thus a unique fixed point of T exists which is the solution of eq.(2.17). ■

Now, recall that eq.(2.16) has at least one solution in $L_2[0,1]$ if the following conditions are satisfied:

(i) $k(x,y)$ is continuous for all x, y in [0,1] and that $\psi(y,t)$ is continuous for all y in [0, 1] and all t.

(ii) $\psi(y,t)$ satisfies the Lipschitz condition

$$|\psi(y,t_1) - \psi(y,t_2)| \leq B\, |t_1 - t_2|$$

where B is independent of y.

(iii) $|k(x,y)| \leq C$, $|\psi(y,t)| \leq B$.

(iv) $|\lambda| \leq \dfrac{1}{BC}$.

(v) for every $\varepsilon < 0$ we can find a $\delta(\varepsilon)$, such that:

$$\int_0^1 |\psi(y,u_1(y)) - \psi(y,u_2(y))|^2 dy < \varepsilon \quad \text{if} \quad \|u_1 - u_2\| < \delta(\varepsilon)$$

where u_1 and u_2 are in $L_2[D]$, [Hochstadt H., 1973].

The following theorem is an extension of the previous fact to include the 2-D and hence the m-D Fredholm non-linear integral equations. But before that we need the following lemma.

Lemma (2.3):

Let S be a closed and convex set in a Hilbert space and T is a continuous mapping of S into itself. Suppose that $T(S)$ is compact. that T has at least one fixed point in S.

Theorem (2.14):

Consider eq.(2.17) with the same previous conditions on $k(x,y,z,m)$ and $\psi(z,m,t)$ in theorem (2.13). Let $k(x,y,z,m)$ be continuous for all (x,y,z,m) in $[0, 1]$, where:

$$|\ \psi(z,m,t)\ | \leq B$$

and for every $\varepsilon > 0$ we can find a $\delta(\varepsilon)$, such that:

$$\int_0^1\int_0^1 \left|\psi\left(z,m,u_1(z,m)\right) - \psi\left(z,m,u_2(z,m)\right)\right|^2 dz dm < \varepsilon \ \text{ if } \ \| u_1 - u_2 \| < \delta(\varepsilon)$$

where u_1 and u_2 are in $L_2[D]$. Then eq.(2.17) has at least one solution in $L_2[D]$ provided

$$|\lambda| < \frac{1}{BC}$$

Proof:

Let $S = \{u \in L_2[D] \mid \|u\| \leq 1\}$.

We prove S is convex, to do this Then let $u_1, u_2 \in S$, then:

$$\| t\ u_1 + (1-t)\ u_2 \| \ \le t\ \| u_1 \| + (1-t)\ \| u_2 \| \ \le t + (1-t) = 1 \quad \text{for} \quad 0 \le t \le 1$$

since $t\ u_1 + (1-t)\ u_2 \in S$. Therefore S is convex.

Moreover, we show that S is closed. To do this, let $\{ u_n \}$ be a sequence in S such that $u_n \longrightarrow u$.

Then $\| u_n \| \longrightarrow \| u \|$. But $\| u_n \| \le 1$, thus $\| u \| \le 1$ and hence $u \in S$. Therefore S is closed subset of $L_2[D]$.

Define the operator T by:

$$T\ u = \lambda \int_0^1 \int_0^1 k(x,y,z,m) \psi\big(z,m,u(z,m)\big)\ dzdm$$

if $u \in S$, then:

$$\|Tu\| \le \left[\int_0^1 \int_0^1 \left| \lambda \int_0^1 \int_0^1 k(x,y,z,m) \psi\big(z,m,u(z,m)\big) dzdm \right|^2 dxdy \right]^{1/2}$$

$$\le |\lambda| \left[\int_0^1 \int_0^1 \left[\int_0^1 \int_0^1 |k(x,y,z,m)| \left|\psi\big(z,m,u(z,m)\big)\right| dzdm \right]^2 dxdy \right]^{1/2}$$

$$\le |\lambda| \left[\int_0^1 \int_0^1 \int_0^1 \int_0^1 |k(x,y,z,m)|^2 dzdmdxdy \left|\psi\big(z,m,u(z,m)\big)\right|^2 dzdm \right]^{1/2}$$

$$\le |\ \lambda\ |\ BC \le 1$$

So T maps S into itself. Also we show that T is a continuous mapping. To do this, consider:

$$\|Tu_1 - Tu_2\| \le \left[\int_0^1\int_0^1 |Tu_1 - Tu_2|^2 \, dxdy \right]^{1/2}$$

$$\le \left[\int_0^1\int_0^1 \left| \lambda \int_0^1\int_0^1 k(x,y,z,m)\psi\big(z,m,u_1(z,m)\big)\, dzdm - \right.\right.$$

$$\left.\left. \lambda \int_0^1\int_0^1 k(x,y,z,m)\psi\big(z,m,u_2(z,m)\big)\, dzdm \right|^2 dxdy \right]^{1/2}$$

$$\le |\lambda| \left[\int_0^1\int_0^1 \left[\int_0^1\int_0^1 |k(x,y,z,m)| \big|\psi\big(z,m,u_1(z,m)\big) - \right.\right.$$

$$\left.\left. \psi\big(z,m,u_2(z,m)\big)\big| dzdm \right]^2 dxdy \right]^{1/2}$$

$$\le |\lambda| \left[\int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m)|^2 dzdmdxdy \int_0^1\int_0^1 \big|\psi\big(z,m,u_1(z,m)\big) \right.$$

$$\left. - \psi\big(z,m,u_1(z,m)\big)\big|^2 \, dzdm \right]^{1/2}$$

$$\|Tu_1 - Tu_2\| \le |\lambda| C \left[\int_0^1\int_0^1 \big|\psi\big(z,m,u_1(z,m)\big) - \psi\big(z,m,u_1(z,m)\big)\big|^2 dzdm \right]^{1/2}$$

then $\|Tu_1 - Tu_2\| \le |\lambda| C \varepsilon^{1/2}$ if $\| u_1 - u_2\| < \delta(\varepsilon)$

let $\varepsilon^* = |\lambda| C \varepsilon^{1/2}$, then:

$$\|Tu_1 - Tu_2\| \le \varepsilon^* \text{ if } \| u_1 - u_2\| < \delta(\varepsilon^*)$$

Therefore T is a continuous mapping. We note that Tu is a continuous function of x, y and that

$$|\,T\ u\,| = |\,\lambda\,| \int_0^1\int_0^1 |k(x,y,z,m)|\,|\psi(z,m,u(z,m))|\,dzdm \le |\,\lambda\,|\,B\ \mathrm{C}$$

so that the set $T(S)$ is uniformly bounded.

Next, consider

$$\left|\lambda\int_0^1\int_0^1 k(x_1,y_1,z,m)\,\psi(z,m,u(z,m))\,dzdm - \lambda\int_0^1\int_0^1 k(x_2,y_2,z,m)\,\psi(z,m,u(z,m))\,dzdm\right|$$

If:

$$|\,(x_1,y_1)-(x_2,y_2)\,| < \delta(\varepsilon)$$

$$\le |\lambda|\,B\int_0^1\int_0^1 |\,k(x_1,y_1,z,m)-k(x_2,y_2,z,m)\,|\,dzdm < \varepsilon$$

Therefore $T(S)$ is equicontinuous. By using Arzelà's theorem, $T(S)$ is compact.

Hence, by using lemma (2.3) T has at least fixed point in S and therefore eq.(2.17) has at least one solution in $L_2[D]$. ■

Now, recall that the eq.(2.16) has at least one solution in S if the following conditions are satisfied:

(i) $k(x,y)$ and $\psi(x,y)$ are continuous functions of their variables.

(ii) $|\lambda| \le \dfrac{M}{BC}$.

(iii) S be the set of functions u in $L_2[0,1]$ for which $\|u\| \le M$.

(iv) $|k(x,y)| \le C$, $0 \le x,\ y \le 1$.

(v) $\int_0^1 |\psi(y,u_1(y)) - \psi(y,u_2(y))|^2 dy < \varepsilon$ if $\|u_1 - u_2\| < \delta(\varepsilon)$.

The following theorem is a generalization of the above fact to include the 2-D and hence the m-D Fredholm non-linear integral equations.

Theorem (2.15):

Consider the 2-D Fredholm non-linear integral equation given by where $k(x,y,z,m)$ and $\psi(z,m,u(z,m))$ are continuous functions of their variables. Let S be the set of functions u in $L_2[D]$ for which $\| u \| \le M$. Suppose that:

$$| k(x,y,z,m) | \le C , \quad 0 \le x , y , \quad z , m \le 1$$

$$\int_0^1\int_0^1 |\psi(z,m,u(z,m))|^2 dzdm \le B^2 \quad \text{for all } u \text{ such that } \| u \| \le M$$

and for every positive ε we can find $\delta(\varepsilon)$ such that

$$\int_0^1\int_0^1 |\psi(z,m,u_1(z,m)) - \psi(z,m,u_2(z,m))|^2 dzdm \le \varepsilon \quad \text{if} \quad \| u_1 - u_2 \| < \delta(\varepsilon)$$

then eq.(2.17) has at least one solution in S for $| \lambda | \le \dfrac{M}{CB}$

Proof:

Define the operator T by

$$T\ u = \lambda \int_0^1\int_0^1 k(x,y,z,m)\ \psi(z,m,u(z,m))\ dzdm$$

for $u \in S$, $S = \{u \in L_2[D] \mid \|u\| \le 1\}$.

$$|Tu| = \left| \lambda \int_0^1 \int_0^1 k(x,y,z,m)\ \psi\big(z,m,u(z,m)\big)\ dzdm \right|$$

$$\leq |\lambda| \int_0^1 \int_0^1 |k(x,y,z,m)|\ \big|\psi\big(z,m,u(z,m)\big)\big|\ dzdm$$

by using Cauchy Schwarz inequality

$$|Tu| = |\lambda| \left(\int_0^1 \int_0^1 |k(x,y,z.m)|^2 dzdm \right)^{1/2} \left(\int_0^1 \int_0^1 \big|\psi\big(z,m,u(z,m)\big)\big|^2 dzdm \right)^{1/2}$$

from the condition $|\ k(x,y,z,m)\ | \leq C$ and from the other condition

$$\int_0^1 \int_0^1 \big|\psi\big(z,m,u(z,m)\big)\big|^2 dzdm \leq B^2 \quad \text{for all } u \text{ such that } \|\ u\ \| \leq M$$

Thus $|Tu| \leq |\lambda| CB$, and

$$|Tu(x_1,y_1) - Tu(x_2,y_2)| = \left| \lambda \int_0^1 \int_0^1 k(x_1,y_1,z,m) \psi(z,m,u(z,m)) dzdm \right.$$

$$\left. -\lambda \int_0^1 \int_0^1 k(x_2,y_2,z,m) \psi(z,m,u(z,m)) dzdm \right|$$

$$\leq |\lambda| B \int_0^1 \int_0^1 |k(x_1,y_1,z,m) - k(x_2,y_2,z,m)|$$

$$\leq |\lambda| B \left[\int_0^1 \int_0^1 |k(x_1,y_1,z,m) - k(x_2,y_2,z,m)|^2 dzdm \right]^{1/2}$$

since k is continuous, then:

$$|k(x_1,y_1,z,m) - k(x_2,y_2,z,m)|^2 < \varepsilon \quad \text{if} \quad |(x_1,y_1) - (x_2,y_2)| < \delta(\varepsilon)$$

Thus

$$|Tu(x_1,y_1)-Tu(x_2,y_2)|<|\lambda|B\varepsilon \quad \text{if} \quad |(x_1,y_{1)}-(x_2,y_2)|<\delta(\varepsilon)$$

$$|Tu(x_1,y_1)-Tu(x_2,y_2)|<\varepsilon^* \quad \text{if} \quad |(x_1,y_{1)}-(x_2,y_2)|<\delta(\varepsilon^*)$$

We prove S is convex, to do this Then let $u_1,u_2 \in S$, then:

$$\|t\ u_1+(1-t)\ u_2\| \le t\ \|u_1\|+(1-t)\ \|u_2\|$$

$$\le t\,M + (1-t)M = M \quad \text{for} \quad 0 \le t \le 1$$

since $t\ u_1+(1-t)\ u_2 \in S$. Therefore, S is convex.

Moreover, we show that S is closed. To do this, let $\{u_n\}$ be a sequence in S such that $u_n \longrightarrow u$

Then $\|u_n\| \longrightarrow \|u\|$. But $\|u_n\| \le M$, thus $\|u\| \le M$ and hence $u \in S$. Therefore S is closed subset of $L_2[D]$.

To show that T is continuous

$$\|Tu_1-Tu_2\| \le |\lambda|C\left[\int_0^1\int_0^1 |\psi(z,m,u_1(z,m))-\psi(z,m,u_1(z,m))|^2 dzdm\right]^{1/2}$$

by the conditions $|\,k(x,y,z,m)\,| \le C$ and for every $\varepsilon>0$ we can find a $\delta(\varepsilon)$ such that

$$\int_0^1\int_0^1 |\psi(z,m,u_1(z,m))-\psi(z,m,u_2(z,m))|^2 dzdm<\varepsilon \quad \text{if} \quad \|u_1-u_2\|<\delta(\varepsilon)$$

then $\|Tu_1-Tu_2\| \le |\lambda|C\,\varepsilon^{1/2}$ if $\|u_1-u_2\|<\delta(\varepsilon)$

let $\varepsilon^*=|\lambda|C\,\varepsilon^{1/2}$, then:

$$\|Tu_1 - Tu_2\| \le \varepsilon^* \quad \text{if} \quad \| u_1 - u_2 \| < \delta(\varepsilon^*)$$

Therefore T is a continuous mapping. We note that Tu is a continuous function of x, y and that.

If $u \in S$, then:

$$\|Tu\| = \left(\int_0^1 \int_0^1 |Tu|^2 \, dxdy \right)^{1/2}$$

$$\le |\ \lambda\ |\ BC \le M \quad \text{for all } u \in s$$

So T maps S into itself. So that the set $T(S)$ is uniformly bounded.

Therefore $T(S)$ is equicontinuous and bounded. By using Arzelà's theorem, $T(S)$ is compact. Hence, by using lemma (2.3) T has at least fixed point in S and therefore eq.(2.17) has at least one solution in $L_2[D]$. ■

Now, recall that the eq.(2.16) has at least one solution in S if the following conditions are satisfied:

(i) $k(x, y)$ and $\psi(x, y)$ are continuous functions of their variables.

(ii) $|\lambda| \le \dfrac{M}{BC}$.

(iii) $\int_0^1 \int_0^1 |k(x, y)|^2 dxdy < C^2 < \infty$.

(iv) $|\ k(x, y, z, m)\ | \le C$, $\quad 0 \le x,\ y, z, m \le 1$

(v) $\int_0^1 \int_0^1 \left|\psi\left(z, m, u(z, m)\right)\right|^2 dzdm \le B^2$ for all u such that $\| u \| \le M$.

The following theorem is a generalization of the above fact to be hold for the 2-D and hence for the m-D Fredholm non-linear integral equations.

Theorem (2.16):

Consider eq.(2.17) with the same conditions on $\psi(z,m,u(z,m))$ as in the theorem (2.15). Let $k(x, y, z, m)$ be such that

$$\int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m)|^2 dxdydzdm < C^2 < \infty \qquad (2.18)$$

Then eq.(2.17) has at least one solution in S for

$$|\lambda| \le \frac{M}{CB}.$$

Proof:

Let $k_n(x,y,z,m)$ be a sequence of continuous kernels, such that:

$$\lim_{n\to\infty}\int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m) - k_n(x,y,z,m)|^2 dxdydzdm = 0 \qquad (2.19)$$

since ineq.(2.18) holds, then we can assume

$$\int_0^1\int_0^1\int_0^1\int_0^1 |k_n(x,y,z,m)|^2 dxdydzdm \le C^2$$

We can define a sequence of integral operators

$$T_n u = \lambda\int_0^1\int_0^1 k_n(x,y,z,m)\ \psi(z,m,u(z,m))\ dzdm$$

We also see that:

$$\|Tu\| = \left\| \lambda \int_0^1 \int_0^1 k(x,y,z,m)\ \psi\big(z,m,u(z,m)\big)\ dzdm \right\|$$

$$\le |\lambda| \left[\int_0^1\int_0^1\int_0^1\int_0^1 |k_n(x,y,z,m)|^2 dzdmdxdy \int_0^1\int_0^1 |\psi\big(z,m,u(z,m)\big)|^2\ dzdm \right]^{1/2}$$

$$\le |\lambda| CB \le M \ \text{ if } \ |\lambda| \le \frac{M}{CB}$$

and also $\| T_n u \| \le M$, so that T and all T_n map S into itself. These mapping are also continuous, and if $T(S)$ is compact we would be done. On the other hand, from the proof of theorem (2.15), all $T_n(S)$ are compact.

Consider:

$$\| T u - T_n u \| \le |\lambda| \left\{ \int_0^1\int_0^1\int_0^1\int_0^1 | k(x,y,z,m) - k_n(x,y,z,m) |^2 dzdmdxdy \int_0^1\int_0^1 | \psi\big(z,m,u(z,m)\big) |^2 dzdm \right\}^{1/2}$$

$$\le \frac{MB}{CB} \left\{ \int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,m) - k_n(x,y,z,m)|^2 dydzdxdy \right\}^{1/2}$$

$$< \varepsilon, \quad \text{if } \ n > N(\varepsilon) \qquad (2.20)$$

Then ineq.(2.20) is clearly uniform for all u in S.

Let $\{u_n\}$ be any sequence in S. We can select a subsequence $\{u_{n(1)}\}$ such that $\{T_1 u_{n(1)}\}$ converges. From that subsequence we can extract a new subsequence $\{u_{n(2)}\}$ so that $\{T_2 u_{n(2)}\}$ converges, and so on. In this fashion we obtain a chain of subsequences

$$\{u_n\} \supset \{u_{n(1)}\} \supset \{u_{n(2)}\} \supset \cdots \supset \{u_{n(k)}\} \supset \cdots$$

such that the sequence $\{T_i u_{n(k)}\}$ converges for all $i = 1, 2, ..., k$. Finally, we take the sequence $\{u_{n(n)}\}$ which is a subsequence of every $\{u_{n(k)}\}$ except for a finite number of elements and clearly $\{T_k u_{n(n)}\}$ converges for every *k*.

Now:

$$\| Tu_{n(n)} - Tu_{m(m)} \| = \| Tu_{n(n)} - T_k u_{n(n)} + T_k u_{n(n)} - T_k u_{m(m)} + T_k u_{m(m)} - Tu_{m(m)} \|$$

$$\leq \| Tu_{n(n)} - T_k u_{n(n)} \| + \| T_k u_{n(n)} - T_k u_{m(m)} \| + \| T_k u_{m(m)} - Tu_{m(m)} \|$$

The first and third terms on the right of above inequality can be made small for large k by use of eq.(2.18). The middle term becomes small for large *n* and *m* since

$$\| T\ u_{n(n)} - T\ u_{m(m)} \| < 2\varepsilon, \ \ n, \ m > M(\varepsilon)$$

and this implies that the sequence $\{Tu_{n(n)}\}$ is a Cauchy sequence so that $T(S)$ is compact and by using lemma (2.3), T must have a fixed point. ■

3.1 Introduction:

Recall that there are many methods for solving the one-dimensional integral equations. These methods depend on the structure of the one-dimensional integral equations, [Golberg A., 1979].

On the other hand, some of these methods are extended to solve the multi-dimensional integral equations like the variational method and the Taylor´s expansion method, [Hasson H., 2005] and the expansion methods, [Al-Bayati, B., 2005].

This chapter concerned with modifying another methods for solving the multi-dimensional Fredholm and Volterra linear integral equations.

This chapter consists of two sections.

In section one, we generalize three methods for solving the multi-dimensional Fredholm linear integral equations, namely the degenerate kernel method, the method of iterated kernels and the method of Fredholm resolvent kernel.

In section two, we modify two methods for solving the multi-dimensional Volterra linear integral equations, namely the resolvent kernel method: Neumann series and the method of successive approximation.

3.2 Some Methods for Solving the Multi-Dimensional Fredholm Linear Integral Equations:

It is known, there are many methods for solving the 1-D Fredholm linear integral equations, say the Taylor expansion method, [kanwal R. and Liv k., 1989], the degenerate kernel method, [Delves L. and Walsh J., 1974], the expansion methods, [Delves L. and Mohamed J., 1985], the quadrature methods, [Chombers L., 1976], the variational method, [Zaboon A., 1993], the method of iterated kernels and the method of Fredholm resolvent kernel, [Jerri J., 1985] and etc.

In this section, we generalize some of there methods to solve the m-D Fredholm linear integral equations, say the degenerate kernel method, the method of iterated kernels and the method of Fredholm resolvent kernel.

3.2.1 The Degenerate Kernal Method:

It is known that, the degenerate kernel method is a method for solving the homogeneous and the nonhomogeneous 1-D Fredholm integral equations when the kernels of them are degenerate. Also, this method can be also modified to solve the homogeneous and the nonhomogeneous 1-D Fredholm integral equations of nondegenerate kernels, [Jerri A, 1985].

In this section, we generalize this method to include the homogeneous and the nonhomogeneous m-D Fredholm integral equations when the kernels of them are also degenerate. Also a modification of this method for solving the homogeneous and nonhomogeneous m-D Fredholm integral equation with nondegenerate kernels is presented. For simplicity, we use this method to solve the homogeneous and the nonhomogeneous 2-D Fredholm integral equations.

First, consider the nonhomogeneous 2-D Fredholm linear integral equation of the second kind with degenerate kernel

$$k(x,y,z,m)=\sum_{k=1}^{n} a_k(x,y)b_k(z,m) \tag{3.1}$$

that is, consider

$$u(x,y)=f(x,y)+\lambda\int_{c}^{d}\int_{a}^{b} k(x,y,z,m)u(z,m)dzdm \tag{3.2}$$

after using eq.(3.1) and exchanging summation with integration, one can get:

$$u(x,y) = f(x,y) + \lambda \int_c^d \int_a^b \sum_{k=1}^{n} a_k(x,y) b_k(z,m) u(z,m) dzdm$$

$$= f(x,y) + \lambda \sum_{k=1}^{n} a_k(x,y) \int_c^d \int_a^b b_k(z,m) u(z,m) dzdm \qquad (3.3)$$

In the following we show how the solution of this 2-D Fredholm integral equation with degenerate kernel reduces to solving a system of linear equations. If we define c_k as the integrals in the above equation,

$$c_k = \int_c^d \int_a^b b_k(z,m) u(z,m) dzdm \qquad (3.4)$$

Then eq.(3.3) becomes

$$u(x,y) = f(x,y) + \lambda \sum_{k=1}^{n} a_k(x,y) c_k \qquad (3.5)$$

If we multiply both sides of eq.(3.5) by $b_m(z,m)$ and integrate the resulting equation first from a to b and second from c to d, we produce c_m on the left hand side,

$$\int_c^d \int_a^b b_m(x,y) u(x,y) dxdy = \int_c^d \int_a^b b_m(x,y) f(x,y) dxdy + \lambda \sum_{k=1}^{n} c_k \int_c^d \int_a^b b_m(x,y) a_k(x,y) dxdy$$

(3.6)

If we define the integrals in eq.(3.6) as:

$$f_m = \int_c^d \int_a^b b_m(x,y) f(x,y) dxdy \qquad (3.7)$$

and

$$a_{mk} = \int_c^d \int_a^b b_m(x,y) a_k(x,y) dx dy \quad (3.8)$$

then eq.(3.6) becomes:

$$c_m = f_m + \lambda \sum_{k=1}^{n} a_{mk} c_k \ , \quad m = 1, 2, ..., n \quad (3.9)$$

which is a set of n linear equations in $c_1, c_2, \ldots, c_n$. Here f_m and a_{mk} are considered known since we are given $b_m(x,y)$, $f_m(x,y)$ and $a_k(x,y)$.

So the solution to the 2-D Fredholm integral equation of the second kind given by eq.(3.2) with degenerate kernel given by eq.(3.1) reduces to solving for c_m from the system of the n linear equations given by eq.(3.9), since c_m will then be used in the series given in eq.(3.5) to obtain the solution $u(x,y)$ of eq.(3.2).

If we use matrix notation, the system of n linear equations given by eq.(3.9) can be written in the form

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \lambda \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = F + \lambda A C \quad (3.10)$$

or as

$$(I - \lambda A) C = F \quad (3.11)$$

From the theory of linear systems of equations, we known that eq.(3.11) has a unique solution if $|I - \lambda A| \neq 0$ and has either infinite or no solution when $|I - \lambda A| = 0$.

To illustrate this approach, see the following examples.

Example (3.1):

Consider the nonhomogeneous 2-D Fredholm integral equation of the second kind

$$u(x,y) = xy - \frac{2}{3}x - \frac{4}{3}y + \int_0^2\int_0^1 (xz + ym)u(z,m)dzdm \qquad (3.12)$$

This Fredholm integral equation has a degenerate kernel of the form given by eq.(3.1), since

$$k(x,y,z,m) = xz + ym$$

$$= \sum_{k=1}^{2} a_k(x,y)b_k(z,m) \qquad (3.13)$$

where $a_1(x,y) = x$, $a_2(x,y) = y$, $b_1(z,m) = z$ and $b_2(z,m) = m$.

To solve for c_m in eq.(3.9) and hence $u(x,y)$ of eq.(3.12) we must prepare f_1, f_2 from eq.(3.7) and $a_{11}, a_{12}, a_{21}, a_{22}$ from eq.(3.8). From eq.(3.12) we have

$$f(x,y) = xy - \frac{2}{3}x - \frac{4}{3}y$$

hence according to eq.(3.7),

$$f_1 = \int_0^2\int_0^1 b_1(x,y)f(x,y)dxdy = \int_0^2\int_0^1 x\left(xy - \frac{2}{3}x - \frac{4}{3}y\right)dxdy = -\frac{10}{9}.$$

$$f_2 = \int_0^2\int_0^1 b_2(x,y)f(x,y)dxdy = \int_0^2\int_0^1 y(xy - \frac{2}{3}x - \frac{4}{3}y)dxdy = -\frac{26}{9}.$$

and the column matrix F of eq.(3.10) becomes:

$$F=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} -\frac{10}{9} \\ -\frac{26}{9} \end{pmatrix}$$

To prepare the matrix A in eq.(3.10), we use eq.(3.8) to evaluate the elements a_{mk} with $a_k(x,y)$ and $b_k(x,y)$ as in eq.(3.13) for k, $m = 1, 2$,

$$a_{11}=\int_0^2\int_0^1 b_1(x,y)a_1(x,y)dxdy=\int_0^2\int_0^1 x^2dxdy=\frac{2}{3}$$

$$a_{12}=\int_0^2\int_0^1 b_1(x,y)a_2(x,y)dxdy=\int_0^2\int_0^1 xydxdy=1$$

$$a_{21}=\int_0^2\int_0^1 b_2(x,y)a_1(x,y)dxdy=\int_0^2\int_0^1 yxdxdy=1$$

$$a_{22}=\int_0^2\int_0^1 b_2(x,y)a_2(x,y)dxdy=\int_0^2\int_0^1 y^2dxdy=\frac{8}{3}$$

Hence $C=F+\lambda AC$ of eq.(3.10) becomes:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} -\frac{10}{9} \\ -\frac{26}{9} \end{pmatrix}+\begin{pmatrix} \frac{2}{3} & 1 \\ 1 & \frac{8}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and if we transform the matrix product to the left side, we obtain $C-AC=F$, that is:

$$\begin{pmatrix} -\frac{1}{3} & -1 \\ -1 & -\frac{5}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} -\frac{10}{9} \\ -\frac{26}{9} \end{pmatrix}$$

The solution of the above system is $c_1 = \frac{2}{3}$ and $c_2 = \frac{4}{3}$. Thus $u(x,y) = xy$ is the solution of eq.(3.12).

Example (3.2):

Consider the nonhomogeneous 2-D Fredholm integral equation of the second kind

$$u(x,y) = x^2 + y^2 + \lambda \int_0^1 \int_0^1 (x + y + z + m) u(z,m) dz dm \qquad (3.14)$$

This Fredholm integral equation has a degenerate kernel of the form given by eq.(3.1), since:

$$k(x,y,z,m) = x + y + z + m = \sum_{k=1}^{2} a_k(x,y) b_k(z,m) \qquad (3.15)$$

where $a_1(x,y) = x + y$, $a_2(x,y) = 1$, $b_1(z,m) = 1$ and $b_2(z,m) = z + m$.

To solve for c_m in eq.(3.9) and hence $u(x, y)$ of eq.(3.14), we must prepare f_1, f_2 from eq.(3.7) and $a_{11}, a_{12}, a_{21}, a_{22}$ from eq.(3.8). From eq.(3.14) we have $f(x,y) = x^2 + y^2$, hence according to eq.(3.7),

$$f_1 = \int_0^1 \int_0^1 (x^2 + y^2) dx dy = \frac{2}{3}.$$

$$f_2 = \int_0^1 \int_0^1 (x^2 + y^2)(x + y) dx dy = \frac{5}{6}.$$

and the column matrix F of eq.(3.10) becomes:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}$$

To prepare the matrix A in eq.(3.10), we use eq.(3.8) to evaluate the elements a_{mk} with $a_k(x,y)$ and $b_k(z,m)$ as in eq.(3.15) for $k, m = 1, 2,$

$$a_{11} = \int_0^1\int_0^1 (x+y)\,dx\,dy = 1$$

$$a_{12} = \int_0^1\int_0^1 dx\,dy = 1$$

$$a_{21} = \int_0^1\int_0^1 (x+y)^2\,dx\,dy = \frac{7}{6}$$

$$a_{22} = \int_0^1\int_0^1 (x+y)\,dx\,dy = 1$$

Hence $C = F + \lambda AC$ of eq.(3.10) becomes

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix} + \lambda \begin{pmatrix} 1 & 1 \\ \frac{7}{6} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and if we transfer the matrix product to the left side, we obtain

$$\begin{pmatrix} 1-\lambda & -\lambda \\ -\frac{7}{6}\lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}$$

In general, before solving the above system, we must evaluate the determinant of the matrix $I - \lambda A$,

$$|I - \lambda A| = \begin{vmatrix} 1-\lambda & -\lambda \\ -\frac{7}{6}\lambda & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \frac{7}{6}\lambda^2 = 1 - 2\lambda - \frac{1}{6}\lambda^2$$

If $1-2\lambda-\frac{1}{6}\lambda^2 \neq 0$, the above linear system has a unique solution for c_1 and c_2 which we can evaluate by finding the inverse $(I-\lambda A)^{-1}$

$$(I-\lambda A)^{-1}=\begin{pmatrix} \frac{6(-1+\lambda)}{-6+12\lambda+\lambda^2} & \frac{-6\lambda}{-6+12\lambda+\lambda^2} \\ \frac{-7\lambda}{-6+12\lambda+\lambda^2} & \frac{6(-1+\lambda)}{-6+12\lambda+\lambda^2} \end{pmatrix}$$

Therefore

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} \frac{6(-1+\lambda)}{-6+12\lambda+\lambda^2} & \frac{-6\lambda}{-6+12\lambda+\lambda^2} \\ \frac{-7\lambda}{-6+12\lambda+\lambda^2} & \frac{6(-1+\lambda)}{-6+12\lambda+\lambda^2} \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}$$

Thus

$$c_1=\frac{4(-1+\lambda)-5\lambda}{-6+12\lambda+\lambda^2} \quad \text{and} \quad c_2=-\frac{14\lambda}{3(-6+12\lambda+\lambda^2)}+\frac{5(-1+\lambda)}{-6+12\lambda+\lambda^2}$$

and hence the solution of eq.(3.12) is given by:

$$u(x,y)=x^2+y^2+\lambda\left[\frac{4(-1+\lambda)-5\lambda}{-6+12\lambda+\lambda^2}(x+y)-\frac{14\lambda}{3(-6+12\lambda+\lambda^2)}+\frac{5(-1+\lambda)}{-6+12\lambda+\lambda^2}\right]$$

Remark (3.1):

The degenerate kernel method can be also used to solve the multi-dimensional Fredholm linear integral equation of the second kind

$$u(x_1,x_2,\dots,x_m)=f(x_1,x_2,\dots,x_m)+\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\cdots\int_{\alpha_1}^{\beta_1} k(x_1,\dots,x_m,z_1,\dots,z_m)u(z_1,\dots,z_m)dz_1\dots dz_m \quad (3.16)$$

in case $k(x_1,x_2,...,x_m,z_1,z_2,...,z_m)$ takes the degenerate form:

$$k(x_1,...,x_m,z_1,...,z_m)=\sum_{k=1}^{n} a_k(x_1,...,x_m)b_k(z_1,...,z_m) \qquad (3.17)$$

In this case, the solution of the above integral equation takes the form:

$$u(x_1,x_2,..., x_m)=f(x_1,x_2,..., x_m)+\lambda\sum_{k=1}^{n} a_k(x_1,x_2,...,x_m)c_k$$

where:

$$c_k=f_k+\lambda\sum_{i=1}^{n} a_{ki}c_i \ , \quad k=1,2,...,n$$

$$f_k=\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1} b_k(x_1,x_2,..., x_m)f(x_1,x_2,..., x_m)dx_1dx_2... dx_m$$

$$a_{ki}=\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1} b_k(x_1,x_2,..., x_m)a_i(x_1,x_2,..., x_m)dx_1dx_2... dx_m$$

and $k, i=1,2,...,n$.

Second, consider the homogeneous 2-D Fredholm integral equation

$$u(x,y)=\lambda\int_c^d\int_a^b k(x,y,z,m)u(z,m)dzdm \qquad (3.18)$$

where $k(x,y,z,m)$ is a degenerate kernel given by eq.(3.1) then

$$u(x,y)=\lambda\sum_{k=1}^{n} a_k(x,y)\int_c^d\int_a^b b_k(z,m)u(z,m)dzdm \qquad (3.19)$$

we will follow the same steps as those we used for the nonhomogeneous equation to reduce eq.(3.18) to

$$u(x,y)=\lambda\sum_{k=1}^{n}c_k a_k(x,y) \tag{3.20}$$

and then to a system of n homogeneous equation in c_m

$$c_m = \lambda\sum_{k=1}^{n} a_{mk}c_k\ ,\ \ m=1,2,...,n \tag{3.21}$$

Or in matrix notation

$$(I-\lambda A)C=0 \tag{3.22}$$

Instead of the nonhomogeneous system of linear equations given in eq.(3.9) and eq.(3.10). Here A and C are defined as in eq.(3.10).

From the theory of systems of linear equations we can conclude that if $|I-\lambda A|\neq 0$ then the only solution to the homogeneous equation given by eq.(3.22) is the trivial solution $c\equiv 0$. By using eq.(3.20), the solution to the homogeneous 2-D Fredholm integral equation given by eq.(3.18) is the trivial solution $u(x,y)\equiv 0$ when $|I-\lambda A|\neq 0$. On the other hand, when $|I-\lambda A|=0$, then eq.(3.22) and hence eq.(3.18) may have either no solution or infinitely many solutions. To determine which of those possibilities, we need to discuss next the eigenvalue and eigenfunctions of the homogeneous problem.

For the homogeneous 2-D Fredholm integral equation given by eq.(3.18), the parameter $\lambda\neq 0$ for which eq.(3.18) does not a trivial solution is called the eigenvalue or characteristic value of eq.(3.18). The nontrivial solution $u(x,y)\neq 0$ corresponding to the eigenvalue is called the eigenfunction or characteristic function of eq.(3.18).

In this sense, the eigenvalues of eq.(3.18) are the solutions of $|I-\lambda A|=0$, since if λ is not the solution of this equation, then $|I-\lambda A|\neq 0$, and hence eq.(3.21) and in turn eq.(3.18) have the trivial solution. There may exist more than one eigenfunction $\psi_j(x,y)$ corresponding to a specific eigenvalue λ_j. Then number p of such eigenfunctions $\psi_{j+1}, \psi_{j+2}, \ldots, \psi_{j+p}$ is called the multiplicity or degeneracy of λ_j, and λ_j is called a simple eigenvalue when $p=1$.

To illustrate this approach, consider the following example.

Example (3.3):

Consider the homogeneous 2-D Fredholm integral equation:

$$u(x,y) = \lambda \int_0^{\pi}\int_0^{\pi} \left(my\cos 2x\cos 2z + z^2 x\cos y\cos m\right) u(z,m)\,dz\,dm \qquad (3.23)$$

This is a homogeneous 2-D Fredholm integral equation with degenerate kernel

$$k(x,y,z,m) = my\cos 2x\cos 2z + z^2 x\cos y\cos m$$

$$= \sum_{k=1}^{n} a_k(x,y)\, b_k(z,m)$$

hence $a_1(x,y) = y\cos 2x$, $a_2(x,y) = x\cos y$, $b_1(z,m) = m\cos 2z$, $b_2(z,m) = z^2\cos m$.

We follow the previous method to find c_1 and c_2 from eq.(3.21) or eq.(3.22). The solution $u(x,y)$ of eq.(3.23) is:

$$u(x,y) = \lambda \sum_{k=1}^{n} c_k a_k(x,y)$$

$$= \lambda c_1 y\cos 2x + \lambda c_2 x\cos y$$

To evaluate c_1 and c_2 from eq.(3.20) we must evaluate a_{11}, a_{12}, a_{21} and a_{22}, the elements of matrix A:

$$a_{11} = \int_0^\pi\int_0^\pi b_1(z,m)a_1(x,y)dzdm = \int_0^\pi\int_0^\pi m^2\cos 2z\,\cos^2 2z\;dzdm = \frac{\pi^4}{6}\cdot$$

$$a_{12} = \int_0^\pi\int_0^\pi b_1(z,m)a_2(x,y)dzdm = \int_0^\pi\int_0^\pi mz\,\cos m\cos 2z\;dzdm = 0\,.$$

$$a_{21} = \int_0^\pi\int_0^\pi b_2(z,m)a_1(x,y)dzdm = \int_0^\pi\int_0^\pi z^2 m\cos m\cos 2z\;dzdm = -\pi\,.$$

$$a_{22} = \int_0^\pi\int_0^\pi b_2(z,m)a_2(x,y)dzdm = \int_0^\pi\int_0^\pi z^3\cos^2 m\;dzdm = \frac{\pi^5}{8}\cdot$$

Hence eq.(3.22) becomes:

$$\begin{pmatrix} 1-\lambda\frac{\pi^4}{6} & 0 \\ \lambda\pi & 1-\lambda\frac{\pi^5}{8} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.24)$$

For $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ not to be the trivial solution, we must have a zero determinant for $I-\lambda A$ in eq.(3.24),

$$\begin{vmatrix} 1-\lambda\frac{\pi^4}{6} & 0 \\ \lambda\pi & 1-\lambda\frac{\pi^5}{8} \end{vmatrix} = (1-\lambda\frac{\pi^4}{6})(1-\lambda\frac{\pi^5}{8}) = 0$$

which a quadratic equation in λ whose solution are $\lambda_1 = \frac{6}{\pi^4}$ and $\lambda_2 = \frac{8}{\pi^5}$ which in turn are the eigenvalues of eq.(3.23). As a solution to eq.(3.23), we have two eigenfunctions $u_1(x,y)$ and $u_2(x,y)$, corresponding to the two eigenvalues $\lambda_1 = \frac{6}{\pi^4}$ and $\lambda_2 = \frac{8}{\pi^5}$ respectively.

Next we consider the two eigenvalues separately and find their corresponding eigenfunctions.

(a) $\lambda_1 = \frac{6}{\pi^4}$

If we substitute $\lambda_1 = \frac{6}{\pi^4}$ in eq.(3.24) to solve for c_1 and c_2, we have:

$$c_1 = \frac{\pi^3}{6}\left(\frac{6}{8}\lambda - 1\right)c_2$$

From eq.(3.20) the eigenfunction $u_1(x,y)$ corresponding to $\lambda_1 = \frac{6}{\pi^4}$ is:

$$u_1(x,y) = \frac{6}{\pi^4}\left[c_1 y \cos(2x) + c_2 x \cos y\right]$$

$$= \frac{1}{\pi}\left(\frac{6}{8}\pi - 1\right)c_2 y \cos(2x) + \frac{6}{\pi^4}c_2 x \cos y$$

This means that the eigenfunction is known except for c_2, which determines it's amplitude, we may arbitrarily let $c_2 = \lambda$ to have

$$u_1(x,y) = \left(\frac{6}{8}\pi - 1\right)y \cos(2x) + \frac{6}{\pi^3}x \cos y$$

Now we consider the case of the second eigenvalue:

(b) $\lambda_2 = \frac{8}{\pi^5}$

We again substitute $\lambda_2 = \frac{8}{\pi^5}$ in eq.(3.24) to obtain $c_1 = 0$.

From eq.(3.20), the eigenfunction $u_2(x,y)$ corresponding to $\lambda_2=\dfrac{8}{\pi^5}$ is

$$u_2(x,y)=\frac{8}{\pi^5}c_2 x\cos y$$

Now we may let $c_2=\dfrac{\pi^5}{8}$ to have $u_2(x,y)=x\cos y$.

Example (3.4):

Consider the homogeneous 2-D Fredholm integral equations

$$u(x,y)=\lambda\int_{-1}^{1}\int_{-1}^{1}(x^2z+y^2m)u(x,y)dzdm \tag{3.25}$$

This integral equation has degenerate kernel

$$k(x,y,z,m)=x^2z+y^2m$$
$$=\sum_{k=1}^{2}a_k(x,y)b_k(z,m)$$

where $a_1(x,y)=x^2$, $a_2(x,y)=y^2$, $b_1(z,m)=z$ and $b_2(z,m)=m$. The solution $u(x,y)$ of eq.(3.25) is

$$u(x,y)=\lambda\sum_{k=1}^{2}c_k a_k(x,y)=\lambda c_1x^2+\lambda c_2y^2$$

To evaluate c_1 and c_2 from eq.(3.21) we must evaluate a_{11}, a_{12}, a_{21} and a_{22}, the elements of the matrix A:

$$a_{11}=\int_{-1}^{1}\int_{-1}^{1}x^3dxdy=0.$$

$$a_{12}=\int_{-1}^{1}\int_{-1}^{1}xy^2dxdy=0.$$

$$a_{21} = \int_{-1}^{1}\int_{-1}^{1} yx^2 dxdy = 0.$$

$$a_{22} = \int_{-1}^{1}\int_{-1}^{1} y^3 dxdy = 0.$$

Hence eq.(3.22) becomes:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.26}$$

Therefore $|I - \lambda A| = 1 \neq 0$ then the only solution to eq.(3.26) is the trivial solution $C = 0.$ By using eq.(3.20), the solution of eq.(3.25) is $u(x, y) = 0$.

Remark (3.2):

The degenerate kernel method can be also used to solve the multi-dimensional Fredholm linear integral equation:

$$u(x_1,...,x_m) = \lambda \int_{\alpha_m}^{x_m}\int_{\alpha_{m-1}}^{x_{m-1}} ... \int_{\alpha_1}^{x_1} k(x_1,...,x_m, z_1,...,z_m) u(z_1,...,z_m) dz_1 ... dz_m \tag{3.27}$$

where $k(x_1, x_2,..., x_m, z_1, z_2,..., z_m)$ takes the degenerate form given by eq.(3.17). In this case, the solution of eq.(3.27) takes the form:

$$u(x_1, x_2,...,x_m) = \lambda \sum_{k=1}^{n} a_k(x_1, x_2,...,x_m) c_k$$

where

$$c_k = \lambda \sum_{i=1}^{n} a_{ki} c_i \ , \quad k = 1,2,...,n$$

$$a_{ki} = \int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}} ... \int_{\alpha_1}^{\beta_1} b_k(x_1, x_2,..., x_m) a_i(x_1, x_2,..., x_m) dx_1 dx_2 ... dx_m$$

and $k, i = 1,2,...,n.$

Now, in many cases a nondegenerate kernel $k(x,y,z,m)s$ may be approximated by a degenerate kernel as a partial sum of the Taylor (or other) series expansion of $k(x,y,z,m)$.

Let us consider the nonhomogeneous 2-D Fredholm integral equation given by eq.(3.2) and its associated equation

$$v(x,y) = f(x,y) + \lambda \int_c^d \int_a^b k_1(x,y,z,m) v(z,m)\,dz\,dm \qquad (3.28)$$

with kernel k_1 as the degenerate kernel approximation to $k(x,y,z,m)$. In principle, we may use the previous method to solve eq.(3.28) for $v(x,y)$, which is considered as an approximate to the solution $u(x,y)$ of eq.(3.2). Of course, there will be an error involved in such an approximation, which is defined as $\varepsilon = \left| u(x,y) - v(x,y) \right|$, and we may attempt to estimate this error to give us a measure of how good this approximation is.

To illustrate this fact, consider the following example.

Example (3.5):

Consider the nonhomogenuous 2-D Fredholm integral equation

$$u(x,y) = \sin x - y + 1 + \int_0^1 \int_0^1 [y - x\cos(xz)]u(z,m)\,dz\,dm \qquad (3.29)$$

We note here that the kernel

$$k(x,y,z,m) = y - x\cos(xz)$$

is not degenerate, but a finite number of terms of its Maclaurion series

$$y - x\left[1 - \frac{x^2 z^2}{2!} + \frac{x^4 z^4}{4!} - \cdots\right] = y - x + \frac{x^3 z^2}{2!} - \frac{x^5 z^4}{4!} + \cdots \qquad (3.30)$$

is degenerate, or in other words, separable in (x,y) and (z,m).

So, if we consider only two terms of the series in eq.(3.30), we have a degenerate kernel:

$$k_1(x,y,z,m) = y - x + \frac{x^3 z^2}{2!}$$

as an approximation to $k(x,y,z,m) = y - x\cos(xz)$ of eq.(3.29). The associated equation in $u_1(x,y)$

$$u_1(x,y) = \sin x - y + 1 + \int_0^1\int_0^1 \left[y - x + \frac{x^3 z^2}{2!} \right] u_1(z,m)\,dz\,dm \qquad (3.31)$$

has a degenerate kernel and can be solved by the previous method.

Hence from eq.(3.1), we have:

$$k_1(x,y,z,m) = y - x + \frac{x^3 z^2}{2!} = \sum_{k=1}^{2} a_k(x,y) b_k(z,m)$$

where $a_1(x,y) = y - x$, $a_2(x,y) = \frac{x^3}{2!}$, $b_1(z,m) = 1$ and $b_2(z,m) = z^2$.

Also, from eq.(3.7), one can get

$$f_1 = \int_0^1\int_0^1 (\sin x - y + 1)\,dx\,dy = -\cos(1) + \frac{3}{2}\cdot$$

$$f_2 = \int_0^1\int_0^1 x^2(\sin x - y + 1)\,dx\,dy = \cos(1) + 2\sin(1) - \frac{11}{6}\cdot$$

Moreover, the elements of the matrix A are

$$a_{11} = \int_0^1\int_0^1 (y - x)\,dx\,dy = 0.$$

$$a_{12} = \int_0^1\int_0^1 \frac{x^3}{2!}\,dx\,dy = \frac{1}{8}\cdot$$

$$a_{21} = \int_0^1\int_0^1 x^2(y-x)dxdy = -\frac{1}{12}.$$

$$a_{22} = \int_0^1\int_0^1 \frac{x^5}{2!}dxdy = \frac{1}{12}.$$

Therefore eq.(3.11) becomes

$$\begin{pmatrix} 1 & -\frac{1}{8} \\ \frac{1}{12} & \frac{11}{12} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\cos(1)+\frac{3}{2} \\ \cos(1)+2\sin(1)-\frac{11}{6} \end{pmatrix}$$

which has the solution:

$$c_1 = -0.854\cos(1)+1.236+0.27\sin(1)$$

and

$$c_2 = 1.169\cos(1)-2.112+2.157\sin(1)$$

Thus, by using eq.(3.5), the solution of eq.(3.31) is

$$u_1(x,y) = \sin x - y + 1 + \left[-0.854\cos(1)+1.236+0.27\sin(1)\right](y-x) + \left[1.169\cos(1)-2.112+2.157\sin(1)\right]\frac{x^3}{2}$$

which is approximation solution to eq.(3.29). In this special case, the exact solution of eq.(3.29) is $u(x,y)=1$.

The following table shows that the approximated solution $u_1(x,y)$ and the exact solution $u(x,y)=1$ of eq.(3.29) at some specific points in the region $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$.

Table (3.1): The approximated solution and the exact solution of example(3.5) at some specific points.

(x, y)	*The Exact Solution u(x, y)*	*The Approximated Solution $u_1(x, y)$*
(0,0)	**1**	**1**
(0.1,0)	**1**	**1**
(0.1,0.1)	**1**	**1**
(0.9,0.0002)	**1**	**1.004**
(0.01,0.5)	**1**	**1.01**
(0.5,0.5)	**1**	**1**
(0.9,0.07)	**1**	**1.004**
(0.02,0.3)	**1**	**1**
(0.5,1)	**1**	**1.001**
(1,0)	**1**	**1.007**
(0,1)	**1**	**1.001**
(0.8,1)	**1**	**1.003**
(1,0.65)	**1**	**1.008**
(1,1)	**1**	**1.009**

On the other hand, if we consider three terms of the series in eq.(3.30), we have also a degenerate kernel:

$$k_2(x,y,z,m) = y - x + \frac{x^3 z^2}{2!} - \frac{x^5 z^4}{4!}$$

as an approximation to $k(x,y,z,m) = y - x\cos(xz)$ of eq.(3.29). The associated equation in $u_2(x,y)$,

$$u_2(x,y) = \sin x - y + 1 + \int_0^1\int_0^1 \left[y - x + \frac{x^3 z^2}{2!} - \frac{x^5 z^4}{4!} \right] u_2(z,m)\,dz\,dm \quad (3.32)$$

has a degenerate kernel:

$$k_2(x,y,z,m)=\sum_{k=1}^{3} a_k(x,y)b_k(z,m)$$

where:

$$a_1(x,y)=y-x,\ a_2(x,y)=\frac{x^3}{2!},\ a_3(x,y)=\frac{-x^5}{4!},\ b_1(z,m)=1,$$

$$b_2(z,m)=z^2 \text{ and } b_3(z,m)=z^4\cdot$$

Also from eq.(3.7), one can get

$$f_1=\int_0^1\int_0^1(\sin x-y+1)dxdy=-\cos(1)+\frac{3}{2}\cdot$$

$$f_2=\int_0^1\int_0^1(\sin x-y+1)x^2dxdy=\cos(1)+2\sin(1)-\frac{11}{6}\cdot$$

$$f_3=\int_0^1\int_0^1(\sin x-y+1)x^4dxdy=-13\cos(1)-20\sin(1)+\frac{241}{10}\cdot$$

Moreover, the elements of the matrix A are

$$a_{11}=\int_0^1\int_0^1(y-x)dxdy=0.$$

$$a_{12}=\int_0^1\int_0^1\frac{x^3}{2!}dxdy=\frac{1}{8}\cdot$$

$$a_{13}=\int_0^1\int_0^1-\frac{x^5}{4!}dxdy=-\frac{1}{144}\cdot$$

$$a_{21}=\int_0^1\int_0^1 x^2(y-x)dxdy=-\frac{1}{12}\cdot$$

$$a_{22}=\int_0^1\int_0^1\frac{x^5}{2!}dxdy=\frac{1}{12}\cdot$$

$$a_{23} = \int_0^1\int_0^1 -\frac{x^7}{4!}dxdy = -\frac{1}{192}\cdot$$

$$a_{31} = \int_0^1\int_0^1 x^4(y-x)dxdy = -\frac{1}{15}\cdot$$

$$a_{32} = \int_0^1\int_0^1 \frac{x^7}{2!}dxdy = \frac{1}{16}\cdot$$

$$a_{33} = \int_0^1\int_0^1 -\frac{x^9}{4!}dxdy = -\frac{1}{240}\cdot$$

Therefore eq.(3.11) becomes

$$\begin{pmatrix} 1 & \frac{-1}{8} & \frac{1}{144} \\ \frac{1}{12} & \frac{11}{12} & \frac{1}{192} \\ \frac{1}{15} & \frac{-1}{16} & \frac{241}{240} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -\cos(1)+\frac{3}{2} \\ \cos(1)+2\sin(1)-\frac{11}{6} \\ -13\cos(1)-20\sin(1)+\frac{241}{10} \end{pmatrix}$$

which has the solution

$$c_1 = -0.757\cos(1)+1.056+0.42\sin(1),$$

$$c_2 = -1.233\cos(1)+23.791-19.804\sin(1)$$

and

$$c_3 = -12.819\cos(1)+23.791-19.804\sin(1)$$

Thus by using eq.(3.5), the solution of eq.(3.32) is

$$u_2(x,y) = \sin x - y + 1 + \left[-0.757\cos(1)+1.056+0.42\sin(1)\right](y-x) + \left[-1.233\cos(1)+23.791-19.804\sin(1)\right]\frac{x^3}{2!} - \left[-12.819\cos(1)+23.791-19.804\sin(1)\right]\frac{x^5}{4!}$$

which is the approximation solution to eq.(3.29).

The following table shows that the approximated solution $u_2(x,y)$ and the exact solution $u(x,y)=1$ of eq.(3.29) at some specific points in the region *D*.

Table (3.2): The approximated solution and the exact solution of example (3.5) at some specific points.

(x, y)	*The Exact Solution* *u(x, y)*	*The Approximated Solution* *u_2(x, y)*
(0,0)	**1**	**1**
(0.97,0.008)	**1**	**1.089**
(0.01,0.001)	**1**	**1**
(0,0.5)	**1**	**1**
(0.7,0.1)	**1**	**1**
(0.9,0)	**1**	**1**
(0.8,0.07)	**1**	**1**
(0.1,0)	**1**	**1**
(0.9,0.8)	**1**	**1**
(1,0)	**1**	**1**
(0,1)	**1**	**1**
(0.007,0.85)	**1**	**1**
(0.9,0.9)	**1**	**1**
(0.99,1)	**1**	**1**
(1,1)	**1**	**1**

Now, if we consider four terms of the series in eq.(3.30), we have also a degenerate kernel

$$k_3(x,y,z,m)=y-x+\frac{x^3z^2}{2!}-\frac{x^5z^4}{4!}+\frac{x^7z^6}{6!}$$

as an approximate to $k(x,y,z,m)=y-x\cos(xz)$ of eq.(3.29). The associated equation in $u_3(x,y)$,

$$u_3(x,y)=\sin x-y+1+\int_0^1\int_0^1\left[y-x+\frac{x^3z^2}{2!}-\frac{x^5z^4}{4!}+\frac{x^7z^6}{6!}\right]u_3(z,m)dzdm \tag{3.33}$$

has a degenerate kernel:

$$k_3(x,y,z,m)=\sum_{k=1}^{3}a_k(x,y)b_k(z,m)$$

where:

$$a_1(x,y)=y-x,\ a_2(x,y)=\frac{x^3}{2!},\ a_3(x,y)=\frac{-x^5}{4!},\ a_4(x,y)=\frac{x^7}{6!},$$

$$b_1(z,m)=1,\ b_2(z,m)=z^2,\ b_3(z,m)=z^4 \text{ and } b_4(z,m)=z^6.$$

Also from eq.(3.7), one can get:

$$f_1=\int_0^1\int_0^1(\sin x-y+1)dxdy=-\cos(1)+\frac{3}{2}\cdot$$

$$f_2=\int_0^1\int_0^1(\sin x-y+1)x^2dxdy=\cos(1)+2\sin(1)-\frac{11}{6}\cdot$$

$$f_3=\int_0^1\int_0^1(\sin x-y+1)x^4dxdy=-13\cos(1)-20\sin(1)+\frac{241}{10}\cdot$$

$$f_4=\int_0^1\int_0^1(\sin x-y+1)x^6dxdy=389\cos(1)-\frac{10079}{14}+606\sin(1).$$

Moreover, the elements of the matrix A are

$$a_{11}=\int_0^1\int_0^1(y-x)dxdy=0.$$

$$a_{12}=\int_0^1\int_0^1\frac{x^3}{2!}dxdy=\frac{1}{8}\cdot$$

$$a_{13} = \int_0^1\int_0^1 -\frac{x^5}{4!}dxdy = -\frac{1}{144}\cdot$$

$$a_{14} = \int_0^1\int_0^1 \frac{x^7}{6!}dxdy = \frac{1}{5760}\cdot$$

$$a_{21} = \int_0^1\int_0^1 x^2(y-x)dxdy = -\frac{1}{12}\cdot$$

$$a_{22} = \int_0^1\int_0^1 \frac{x^5}{2!}dxdy = \frac{1}{12}\cdot$$

$$a_{23} = \int_0^1\int_0^1 -\frac{x^7}{4!}dxdy = -\frac{1}{192}\cdot$$

$$a_{24} = \int_0^1\int_0^1 \frac{x^9}{6!}dxdy = \frac{1}{7200}\cdot$$

$$a_{31} = \int_0^1\int_0^1 x^4(y-x)dxdy = -\frac{1}{15}.$$

$$a_{32} = \int_0^1\int_0^1 \frac{x^7}{2!}dxdy = \frac{1}{16}\cdot$$

$$a_{33} = \int_0^1\int_0^1 -\frac{x^9}{4!}dxdy = -\frac{1}{240}\cdot$$

$$a_{34} = \int_0^1\int_0^1 \frac{x^{11}}{6!}dxdy = \frac{1}{8640}\cdot$$

$$a_{41} = \int_0^1\int_0^1 x^6(y-x)dxdy = \frac{-3}{56}\cdot$$

$$a_{42} = \int_0^1\int_0^1 \frac{x^9}{2!}dxdy = \frac{1}{20}\cdot$$

$$a_{43} = \int_0^1\int_0^1 \frac{-x^{11}}{4!} dxdy = \frac{-1}{288} \cdot$$

$$a_{44} = \int_0^1\int_0^1 \frac{x^{13}}{6!} dxdy = \frac{1}{10080} \cdot$$

Therefore eq.(3.11) becomes

$$\begin{pmatrix} 1 & \frac{-1}{8} & \frac{1}{144} & \frac{1}{5760} \\ \frac{1}{12} & \frac{11}{12} & \frac{1}{192} & \frac{1}{7200} \\ \frac{1}{15} & \frac{-1}{16} & \frac{241}{240} & \frac{-1}{8640} \\ \frac{3}{56} & \frac{-1}{20} & \frac{1}{288} & \frac{10079}{10080} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -\cos(1) + \frac{3}{2} \\ \cos(1) + 2\sin(1) - \frac{11}{6} \\ -13\cos(1) - 20\sin(1) + \frac{241}{10} \\ 389\cos(1) - \frac{10079}{14} + 606\sin(1) \end{pmatrix}$$

which has the solution:

$$c_1 = -0.831\cos(1) + 1.194 + 0.304\sin(1),$$

$$c_2 = 1.18\cos(1) - 2.134 + 2.174\sin(1),$$

$$c_3 = -12.773\cos(1) + 23.705 - 19.732\sin(1)$$

and

$$c_4 = 389.187\cos(1) - 720.253 + 606.221\sin(1).$$

Thus by using eq.(3.5), the solution of eq.(3.33) is:

$$u_3(x,y) = \sin x - y + 1 + [-0.831\cos(1) + 1.194 + 0.304\sin(1)](y - x)$$
$$+ [1.18\cos(1) - 2.134 + 2.174\sin(1)\frac{x^3}{2!} -$$
$$[-12.773\cos(1) + 23.705 - 19.732\sin(1)]\frac{x^5}{4!} +$$
$$[389.187\cos(1) - 720.253 + 606.221\sin(1)]\frac{x^7}{6!}$$

which is the approximation solution to eq.(3.29).

The following table shows that the approximated solution $u_3(x,y)$ and the exact solution $u(x,y)=1$ of eq.(3.29) at some specific points in the region *D*.

Table (3.3): The approximation solution and the exact solution of the example (3.5) at some specific points.

(x, y)	*The Exact Solution u(x, y)*	*The Approximated Solution $u_3(x, y)$*
(0,0)	**1**	**1**
(0.97,0.008)	**1**	**1**
(0.01,0.001)	**1**	**1**
No (0,0.5)	**1**	**1**
(0.7,0.1)	**1**	**1**
(0.9,0)	**1**	**1**
(0.8,0.07)	**1**	**1**
(0.1,0)	**1**	**1**
(0.9,0.8)	**1**	**1**
(1,0)	**1**	**1**
(0,1)	**1**	**1**
(0.007,0.85)	**1**	**1**
(0.9,0.9)	**1**	**1**
(0.99,1)	**1**	**1**
(1,1)	**1**	**1**

3.2.2 The Method of Iterated Kernels:

It is known, the method of iterated kernels is one of the important methods that can be used to solve the 1-D to Fredholm linear integral equation of the second kind, [Jerri J., 1985].

In this section we generalize this method to solve the 2-D and hence the m-D Fredholm linear integral equations.

To do this, consider the 2-D Fredholm linear integral equation of the second kind

$$u(x,y)=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,z,m)u(z,m)dzdm \tag{3.34}$$

We use the method of iterated kernels to find the solution of eq.(3.34). This method starts by the zeroth approximation $u_0(x,y)=f(x,y)$ for the solution $u(x,y)$ in the integrals of eq.(3.34) to obtain the first approximation $u_1(x,y)$,

$$u_1(x,y)=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,z,m)f(z,m)dzdm$$

$$=f(x,y)+\lambda\varphi_1(x,y)s \tag{3.35}$$

where:

$$\varphi_1(x,y)=\int_c^d\int_a^b k(x,y,z,m)f(z,m)dzdm \tag{3.36}$$

The function $u_1(x,y)$ defined in eq.(3.35) is substituted again in the integrals of eq.(3.34) to obtain the second approximation $u_2(x,y)$.

$$u_2(x,y)=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,z,m)u_1(x,y)dzdm$$

$$=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,z,m)\Big[f(z,m)+$$

$$\lambda\int_c^d\int_a^b k(z,m,s,t)f(s,t)dsdt\Big]dzdm$$

$$= f(x,y) + \lambda \int_c^d \int_a^b k(x,y,z,m) f(z,m) dzdm +$$

$$\lambda^2 \int_c^d \int_a^b \left[\int_c^d \int_a^b k(x,y,z,m) k(z,m,s,t) dzdm \right] f(s,t) dsdt$$

$$= f(x,y) + \lambda \varphi_1(x,y) + \lambda^2 \int_c^d \int_a^b k_2(x,y,s,t) f(s,t) dsdt \quad (3.37)$$

with

$$k_2(x,y,s,t) = \int_c^d \int_a^b k(x,y,z,m) k_1(z,m,s,t) dzdm \quad (3.38)$$

and

$$k_1(z,m,s,t) = k(z,m,s,t).$$

If we define

$$\varphi_2(x,y) = \int_c^d \int_a^b k_2(x,y,s,t) f(s,t) dsdt$$

Then $u_2(x,y)$ in eq.(3.37) becomes

$$u_2(x) = f(x) + \lambda \varphi_1(x) + \lambda^2 \varphi_2(x) ...$$

This second approximation is then substituted in eq.(3.34) and following the same steps as those used above to obtain $u_3(x,y)$

$$u_3(x,y) = f(x,y) + \lambda \varphi_1(x,y) + \lambda^2 \varphi_2(x,y) + \lambda^3 \int_c^d \int_a^b k_3(x,y,s,t) f(s,t) dsdt$$

$$= f(x,y) + \lambda \varphi_1(x,y) + \lambda^2 \varphi_2(x,y) + \lambda^3 \varphi_3(x,y) \quad (3.39)$$

where

$$k_3(x,y,s,t) = \int_c^d \int_a^b k(x,y,z,m) k_2(z,m,s,t) dzdm$$

$$\varphi_3(x,y)=\int_c^d\int_a^b k_3(x,y,s,t)f(s,t)dsdt$$

and $k_2(z,m,s,t)$ is given by eq.(3.38). If this process is continued n times, we obtain $u_n(x,y)$, the nth approximation for the solution of eq.(3.34) as

$$u_n(x,y)=f(x,y)+\lambda\varphi_1(x,y)+\lambda^2\varphi_2(x,y)+\cdots+\lambda^n\varphi_n(x,y)$$

$$=f(x,y)+\sum_{i=1}^{n}\lambda^i\varphi_i(x,y) \tag{3.40}$$

where

$$\varphi_i(x,y)=\int_c^d\int_a^b k_i(x,y,s,t)f(s,t)dsdt \tag{3.41}$$

and

$$k_i(x,y,s,t)=\int_c^d\int_a^b k(x,y,z,m)k_{i-1}(z,m,s,t)dsdt,\ i=2,3,\ldots,n \tag{3.42}$$

$k_i(x,y)$ is called the i-th iterate kernel. It remains to find under what condition the series give by eq.(3.40) converges to $u(x,y)$, the solution of eq.(3.34). It turns out that the series give by eq.(3.40) converges for $|\lambda B|<1$, where :

$$B=\left(\int_c^d\int_a^b\int_c^d\int_a^b k^2(x,y,z,m)dz\,dm\,dx\,dy\right)^{\frac{1}{2}}$$

The convergent series

$$u(x,y)=f(x,y)+\sum_{i=1}^{\infty}\lambda^i\varphi_i(x,y) \tag{3.43}$$

is called the Neumann series.

To illustrate this approach, consider the following example.

Example (3.6):

Consider the 2-D Fredholm linear integral equation of the second kind

$$u(x,y)=1+\frac{1}{2}\int_0^1\int_0^1 xye^{z+m}u(z,m)dzdm \qquad (3.44)$$

We start by the zeroth approximation $u(x,y)=f(x,y)=1$, then the first approximation $u_1(x,y)$ is given by

$$u_1(x,y)=1+\frac{1}{2}\int_0^1\int_0^1 xye^{z+m}dzdm$$

$$=1+\frac{1}{2}xy\,(e^1-1)^2$$

Here $\varphi_1(x,y)=(e^1-1)^2xy$.

The second approximation $u_2(x,y)$ is obtained form:-

$$u_2(x,y)=1+\lambda^{\frac{1}{2}}(e^1-1)^2xy+\frac{1}{4}\int_0^1\int_0^1\int_0^1\int_0^1 xye^{z+m}zme^{s+t}dzdmdsdt$$

$$=1+\frac{1}{2}(e^1-1)^2xy+\frac{1}{4}xy\,(e^1-1)^2$$

here $\varphi_2(x,y)=xy\,(e^1-1)^2$.

It this process is continued n times, we obtain $u_n(x,y)$, the n-th approximation for the solution of eq.(3.41)

$$u_n(x,y)=1+\sum_{i=1}^{n}\left(\tfrac{1}{2}\right)^i\varphi_i(x,y)$$

$$=1+(e^1-1)^2xy\,.$$

To arrive at the Neumann series solution given by eq.(3.43) for eq.(3.44) we must prepare $k_i(x,y,z,m)$, the i-th iterated of the kernel

$k(x,y,z,m) = xye^{z+m}$. Here we have $k_1(x,y,z,m) = k(x,y,z,m) = xye^{z+m}$. For $i = 2$ we obtain the second iterate $k_2(x,y,z,m)$,

$$k_2(x,y,s,t) = \int_0^1\int_0^1 k(x,y,z,m)k(z,m,s,t)dzdm$$

$$= \int_0^1\int_0^1 xye^{z+m}zme^{s+t}dzdm$$

$$= xye^{s+t}\int_0^1\int_0^1 zme^{z+m}dzdt = xye^{s+t} \qquad (3.45)$$

Now, we use this result again in eq.(3.41) for $i = 3$ to obtain

$$k_3(x,y,s,t) = \int_0^1\int_0^1 k(x,y,z,m)k_2(z,m,s,t)dzdm$$

$$= \int_0^1\int_0^1 xye^{z+m}zme^{s+t}dsdt = xye^{s+t} \qquad (3.46)$$

and it obvious from eq.(3.45)-(3.46) and eq.(3.41) that if these calculations are repeated, we obtain the general expression for the i-th iterate of the kernel as

$$k_i(x,y,s,t) = xye^{s+t}.$$

By substituting $k_i(x,y,s,t)$ in eq. (3.41) are can obtain

$$\varphi_i(x,y) = \int_0^1\int_0^1 xye^{s+t}dsdt = xy(e^1-1)^2.$$

this is now substituted in the Neumann series given by eq.(3.43) to obtain the final solution of eq.(3.44):

$$u(x,y) = 1 + \sum_{i=1}^{\infty}\lambda^i\varphi_i(x,y)$$

$$= 1 + \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i xy(e^1-1)^2 = 1 + xy(e^1-1)^2.$$

Not that

$$B = \int_0^1\int_0^1\int_0^1\int_0^1 x^2 y^2 e^{2z+2m} dz dm dx dy$$

$$= \frac{1}{36}(e^2 - 1)^2 .$$

and $\lambda = \frac{1}{2}$, so $|\lambda B| = \frac{1}{72}(e^2 - 1)^2 = 0.567 < 1.$

Remark (3.3):

The method of iterated kernels can be also used to solve the multi-dimensional Fredholm linear integral equation of the second kind given by eq.(3.16). In this case the solution of eq.(3.16) takes the form

$$u(x_1, x_2, ..., x_m) = f(x_1, x_2, ..., x_m) + \sum_{i=1}^{\infty} \lambda^i \varphi_i(x_1, x_2, ..., x_m)$$

where

$$\varphi_i(x_1, ..., x_m) = \int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}} ... \int_{\alpha_1}^{\beta_1} k_i(x_1, ..., x_m, z_1, ..., z_m) f(z_1, ..., z_m) dz_1 ... dz_m$$

$$k_i(x_1, x_2, ..., x_m, z_1, z_2, ..., z_m) =$$

$$\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}} ... \int_{\alpha_1}^{\beta_1} k_i(x_1, ..., x_m, s_1, ..., s_m) k_{i-1}(s_1, ..., s_m, z_1, ..., z_m) ds_1 ... ds_m$$

and

$$k_i(x_1, x_2, ..., x_m, z_1, z_2, ..., z_m) = k(x_1, x_2, ..., x_m, z_1, z_2, ..., z_m) .$$

3.2.3 The Method of Fredholm Resolvent Kernel:

It is Known that, the method of Fredholm resolvent kernel can be used to solve the 1-D Fredholm linear integral equation of the second kind, [Jerri A., 1985]. In this section, we modify it to be a method for solving the 2-D and hence the m-D Fredholm linear integral equation of the second kind. To do this, consider the 2-D Fredholm linear integral equation given by eq.(3.34). The solution of the integral equation may often appear as an integral

$$u(x,y) = f(x,y) + \lambda \int_c^d \int_a^b \Gamma(x,y,z,m;\lambda) f(z,m)\,dz\,dm$$

where

$$\Gamma(x,y,z,m;\lambda) = \frac{D(x,y,z,m;\lambda)}{D(\lambda)}, D(\lambda) \neq 0$$

where $\Gamma(x,y,z,m;\lambda), D(x,y,z,m;\lambda)$ and $D(\lambda)$ are called the Fredholm resolvent kernel of eq.(3.34), the Fredholm minor and the Fredholm determinant, respectively. The function $D(x,y,z,m;\lambda)$ is defined as:

$$D(x,y,z,m;\lambda) = k(x,y,z,m) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} B_n(x,y,z,m)$$

where

$$B_n(x,y,z,m) = C_n k(x,y,z,m) - n \int_c^d \int_a^b k(x,y,s,t) B_{n-1}(s,t,z,m)\,ds\,dt,$$

$$B_o(x,y,z,m) = k(x,y,z,m)$$

where

$$C_n = \int_c^d \int_a^b B_{n-1}(z,m,z,m)\,dz\,dm\,\lambda,\ n = 1,2,\ldots,\ C_o = 1$$

and $D(\lambda)$ is defined as

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} C_n,\ C_o = 1.$$

To illustrate this approach consider the following example.

Example (3.7):

Consider example (3.6) according to eq.(3.34). We use the method of Fredholm resolvent kernel to solve this example. Therefore the solution of this example takes the form

$$u(x,y)=1+\frac{1}{2}\int_0^1\int_0^1 \Gamma(x,y,z,m;\tfrac{1}{2})dzdm$$

To evaluate the resolvent kernel $\Gamma(x,y,z,m;\frac{1}{2})$ we must find the functions $D(x,y,z,m;\lambda)$ and $D(\lambda)$. For this purpose, we must find the functions $D(x,y,z,m;\lambda)$ and $D(\lambda)$. For this purpose, we must find $B_n(x,y,z,m)$ and C_n. Here:

$$B_o(x,y,z,m)=k(x,y,z,m)=xye^{z+m},\ C_o=1$$

$$C_1=\int_0^1\int_0^1 B_0(z,m,z,m)dzdm$$

$$=\int_0^1\int_0^1 xzme^{z+m}dzdm=1.$$

For C_2 we need $B_1(z,m,z,m)$ which can evaluated below:

$$B_1(x,y,z,m)=C_1k(x,y,z,m)-\int_0^1\int_0^1 k(x,y,s,t)B_0(s,t,z,m)dsdt$$

$$=xye^{z+m}-\int_0^1\int_0^1 xye^{s+t}ste^{z+m}dsdt$$

$$=xye^{z+m}-xye^{z+y}=0$$

Therefore

$$C_2 = \int_0^1\int_0^1 B_1(z,m,z,m)\,dz\,dm = 0.$$

and

$$B_2(x,y,z,m) = C_2 k(x,y,z,m) - \int_0^1\int_0^1 k(x,y,s,t)B_1(s,t,z,m)\,ds\,dt$$

Thus

$$C_n = 0,\ n = 2,3,...$$

and

$$B_n = 0,\ n = 1,2,...$$

Hence:

$$D(\lambda) = D\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{n!}C_n$$

$$= \left(\frac{-1}{2}\right)^0 C_o + \left(\frac{-1}{2}\right)C_1$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \neq 0.$$

and

$$D(x,y,z,m;\tfrac{1}{2}) = xye^{z+m}$$

Therefore

$$\Gamma(x,y,z,m;\tfrac{1}{2}) = \frac{D(x,y,z,m;\frac{1}{2})}{D\left(\frac{1}{2}\right)} = 2xye^{z+m}$$

and the solution of this example is:

$$u(x,y)=1+\frac{1}{2}\int_0^1\int_0^1 \Gamma(x,y,z,m;\lambda)dzdm$$

$$=1+\int_0^1\int_0^1 xye^{z+m}dzdm$$

$$=1+xy(e^1-1)^2.$$

Remark (3.4):

The method of Fredholm resolvent kernel can be also used to solve the m-D Fredholm linear integral equation of the second kind given by eq.(3.16).

In this case, the solution of eq.(3.16) takes the form

$$u(x_1,x_2,...x_m)=f(x_1,x_2,...x_m)+$$

$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1}\Gamma(x_1,...,x_m,z_1,...,z_m;\lambda)f(z_1,...,z_m)dz_1..dz_m$$

where

$$\Gamma(x_1,...,x_m,z_1,...,z_m;\lambda)=\frac{D(x_1,...,x_m,z_1,...,z_m;\lambda)}{D(\lambda)},\ D(\lambda)\neq 0$$

$$D(x_1,x_2,...,x_m,z_1,z_2,...,z_m;\lambda)=k(x_1,x_2,...,x_m,z_1,z_2,...,z_m)+$$

$$\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}B_n(x_1,x_2,...,x_m,z_1,z_2,...,z_m),$$

$$B_n(x_1,x_2,...,x_m,z_1,z_2,...,z_m)=c_nk(x_1,x_2,...,x_m,z_1,z_2,...,z_m)-$$

$$n\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}...\int_{\alpha_1}^{\beta_1}k(x_1,...,x_m,s_1,...,s_m)B_{n-1}(s_1,...,s_m,z_1,...,z_m)ds_1..ds_m,$$

$$C_n = \int_{\alpha_m}^{\beta_m} \int_{\alpha_{m-1}}^{\beta_{m-1}} \cdots \int_{\alpha_1}^{\beta_1} B_{n-1}(z_1,\ldots,z_m,z_1,\ldots,z_m)dz_1\ldots dz_m\,,\; n=1,\ldots,n,\; C_o=1$$

and $D(\lambda)$ is defined as:

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)}{n!} C_n\,,\; C_0 = 1\,,\; C_0 = 1\,.$$

3.3 Some Methods for Solving The Multi-Dimensional Volterra Linear Integral Equations:

Recall that these are many methods for solving the 1-D volterra linear integral equations, like the Laplace transform method, the method of successive approximations and the resolvent kernel method, [Jerri A., 1985], the expansion methods, [Delves L. and Mohamed J., 1985], the quadrature methods, [Chambars L., 1976] and the variationl method, [Zaboon A., 1993] and etc.

In this section, we generalize some of these methods like the resolvent kernel method and the method of successive approximations to solve the m-D volterra linear integral equation.

3.3.1 The Resolvent Kernel Method : Neumann Series:

It is known the resolvent kernel method is a method that can be used to solve the 1-D linear Volterra integral equation of the first kind.

Here, we use the same method to solve the 2-D linear Volterra integral equation of the second kind

$$u(x,y) = f(x,y) + \lambda \int_c^y \int_a^x k(x,y,z,m)u(z,m)dzdm \qquad (3.47)$$

The solution of the 2-D linear Volterra linear integral equation of the second kind may often appear as an integral

$$u(x,y) = f(x,y) + \lambda \int_c^y \int_a^x \Gamma(x,y,z,m;\lambda) f(z,m)\, dz\, dm \qquad (3.48)$$

in terms of the given function $f(x,y)$ where $\Gamma(x,y,z,m;\lambda)$ is called the resolvent kernel of the integral equation given by eq.(3.47).

When $k(x,y,z,m)$ and $f(x,y)$ in eq.(3.47) are both continuous, it is easy to construct the resolvent kernel $\Gamma(x,y,z,m;\lambda)$ for eq.(3.47) in terms of the following Neumann series :

$$\Gamma(x,y,z,m;\lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,y,z,m) \qquad (3.49)$$

where $k_{n+1}(x,y,z,m)$, the iterated kernel, is evaluated as follows:

$$k_{n+1}(x,y,z,m) = \int_m^y \int_z^x k(x,y,s,t) k_n(s,t,z,m)\, ds\, dt \qquad (3.50)$$

and

$$k_1(x,y,s,t) = k(x,y,s,t)$$

This is easily shown by assuming the following series form for the solution $u(x,y)$:

$$u(x,y) = u_0(x,y) + \lambda u_1(x,y) + \lambda^2 u_2(x,y) + \ldots \qquad (3.51)$$

and substituting it in eq.(3.47) to obtain

$$u_0(x,y) + \lambda u_1(x,y) + \lambda^2 u_2(x,y) + \ldots = f(x,y) +$$
$$\lambda \int_c^y \int_a^x k(x,y,z,m) u_0(z,m)\, dz\, dm + \lambda \int_c^y \int_a^x k(x,y,z,m) u_1(z,m)\, dz\, dm$$
$$+ \lambda^2 \int_c^y \int_a^x k(x,y,z,m) u_2(z,m)\, dz\, dm + \ldots \qquad (3.52)$$

Now, we equate the coefficients of each λ of the same power on both sides of eq.(3.52) to obtain

$$u_0(x,y)=f(x,y) \tag{3.53}$$

$$u_1(x,y)=\int_c^y\int_a^x k(x,y,z,m)u_0(z,m)dzdm \tag{3.54}$$

$$u_2(x,y)=\int_c^y\int_a^x k(x,y,z,m)u_1(z,m)dzdm \tag{3.55}$$

$$\vdots$$

$$u_n(x,y)=\int_c^y\int_a^x k(x,y,z,m)u_{n-1}(z,m)dzdm$$

So if we substitute $u_0(x,y)=f(x,y)$ from eq.(3.53) in eq.(3.54), we get:

$$u_1(x,y)=\int_c^y\int_a^x k(x,y,z,m)f(z,m)dzdm \tag{3.56}$$

then use this resulting value of $u_1(x,y)$ in eq.(3.0) to get:

$$u_2(x,y)=\int_c^y\int_a^x k(x,y,z,m)\left[\int_c^y\int_a^x k(z,m,s,t)f(s,t)dsdt\right]dzdm \tag{3.57}$$

and interchange the integral in eq.(3.57) to obtain

$$u_2(x,y)=\int_c^y\int_a^x f(x,y)+\left[\int_t^y\int_s^x k(x,y,z,m)k(z,m,s,t)dzdm\right]dsdt \tag{3.58}$$

That is:

$$u_2(x,y)=\int_c^y\int_a^x f(s,t)k_2(x,y,s,t)dsdt \tag{3.59}$$

Just as the function $k(x,y,z,m)$ in eq.(3.54) is taken as $k_1(x,y,z,m)$ to give $u_1(x,y)$, the inside integrals in eq.(3.58) defines $k_2(x,y,s,t)$, the iterated kernel, to give $u_2(x,y)$,

$$k_2(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k(z,m,s,t)dzdm$$

$$=\int_t^y\int_s^x k(x,y,z,m)k_1(z,m,s,t)dzdm \qquad (3.60)$$

since $k_1(z,m,s,t)=k(z,m,s,t)$.

In general, following the same steps, we can derive the general term for the iterated kernel,

$$k_{n+1}(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_{n-1}(z,m,s,t)dzdm \qquad (3.61)$$

The final solution given by eq.(3.48) is then obtained from $u(x,y)$ given by eq.(3.51) with $u_0(x,y)=f(x,y)$ as in eq.(3.53), $u_1(x,y)$ as in eq.(3.54), $u_2(x,y)$ as in eq.(3.55), and so on.

To illustrate this method, consider the following example.

<u>Example (3.8):</u>

Consider the 2-D volterra linear integral equation of the second kind

$$u(x,y)=f(x,y)+\lambda\int_0^y\int_0^x e^{x+y-z-m}u(z,m)dzdm \qquad (3.62)$$

Here we have

$$k_1(x,y,s,t)=k(x,y,s,t)=e^{x+y-s-t} \qquad (3.63)$$

So, if we use $k(x,y,z,m)=e^{x+y-z-m}$ and $k_1(z,m,s,t)=k(z,m,s,t)=e^{z+m-s-t}$ in eq.(3.60), we obtain:

$$k_2(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_1(z,m,s,t)dzdm$$

$$=\int_t^y\int_s^x e^{x+y-z-m}e^{z+m-s-t}dzdm$$

$$=\int_t^y\int_s^x e^{x+y-s-t}dzdm=(x-s)(y-t)e^{x+y-s-t} \quad (3.64)$$

From eq.(3.61) with $n = 2$, we have:

$$k_3(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_2(z,m,s,t)dzdm$$

$$=\int_t^y\int_s^x e^{x+x-z-m}(z-s)(m-t)e^{z+m-s-t}dzdm$$

$$=e^{x+y-s-t}\int_t^y\int_s^x (z-s)(m-t)dzdm$$

$$=e^{x+y-s-t}\left(\frac{z^2}{2}-sz\right)\Bigg|_s^x\left(\frac{m^2}{2}-tm\right)\Bigg|_t^y$$

$$=\frac{(x-s)^2}{2}\frac{(y-t)^2}{2}e^{x+y-s-t} \quad (3.65)$$

Similarly, from eq.(3.61) with $n = 3$, we have

$$k_4(x,y,s,t)=\int_t^y\int_s^x k(x,y,z,m)k_3(z,m,s,t)dzdm$$

$$=\int_t^y\int_s^x e^{x+y-z-m}\frac{(z-s)^2}{2}\frac{(m-t)^2}{2}e^{z+m-s-t}dzdm$$

$$k_4(x,y,s,t) = e^{x+y-s-t} \left.\frac{(z-s)^3}{3}\right|_s^x \left.\frac{(m-t)^3}{3}\right|_t^y$$

$$= e^{x+y-s-t} \frac{(x-s)^3}{6} \frac{(y-t)^3}{6} \qquad (3.66)$$

These calculations can be continued to find that

$$k_{n+1}(x,y,s,t) = \frac{(x-s)^n (y-t)^n}{(n!)^2} e^{x+y-s-t} \qquad (3.67)$$

Hence from eq.(3.49) and (3.67), the resolvent kernel for eq.(3.62) is

$$\Gamma(x,y,s,t;\lambda) = k_1(x,y,s,t) + \lambda k_2(x,y,s,t) + \lambda^2 k_3(x,y,s,t) + \ldots + \lambda^n k_{n+1}(x,y,s,t)$$

$$= e^{x+y-s-t} + \lambda(x-s)(y-t)e^{x+y-s-t} + \lambda^2 \frac{(x-s)^2(y-t)^2}{2} e^{x+y-s-t} + \lambda^3 \frac{(x-s)^3(y-t)^3}{3} e^{x+y-s-t} + \ldots + \lambda^n \frac{(x-s)^n(y-t)^n}{(n!)^2} e^{x+y-s-t} + \ldots$$

$$= e^{x+y-s-t} \sum_{n=1}^{\infty} \frac{(x-s)^n (y-t)^n}{(n!)^2} \qquad (3.68)$$

So from eq.(3.48) and the resolvent kernel given by eq.(3.68), the solution of the integral equation given by eq.(3.62) is

$$u(x,y) = f(x,y) + \int_0^y \int_0^x e^{x+y-s-t} \sum_{n=1}^{\infty} \frac{(x-s)^n (y-t)^n}{(n!)^2} f(s,t)\,ds\,dt \qquad (3.69)$$

Remark (3.5):

It is not often that the series representation of $\Gamma(x,y,z,m;\lambda)$ will converge to an expression in closed form (see example (3.7)). In this case, we may have to evaluate numerically a finite number of terms of the Neumann series gives by eq.(3.49) which gives only an approximation of the resolvent kernel $\Gamma(x,y,z,m;\lambda)$.

Remark (3.6):

The resolvent kernels method can be also used to solve the m-D volterra linear integral equation of the second kind given by eq.(1.4).

In this case, the solution of eq.(1.4) takes the form

$$u(x_1,x_2,\dots,x_m) = f(x_1,x_2,\dots,x_m) +$$

$$\lambda \int_{\alpha_m}^{x_m} \int_{\alpha_{m-1}}^{x_{m-1}} \dots \int_{\alpha_1}^{x_1} \Gamma(x_1,\dots,x_m,z_1,\dots,z_m;\lambda) f(z_1,\dots,z_m) dz_1 \dots dz_m$$

where

$$\Gamma(x_1,\dots,x_m,z_1,\dots,z_m) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x_1,\dots,x_m,z_1,\dots,z_m)$$

$$k_{n+1}(x_1,x_2,\dots,x_m,z_1,z_2,\dots,z_m) =$$

$$\int_{z_m}^{x_m} \int_{z_{m-1}}^{x_{m-1}} \dots \int_{z_1}^{x_1} k(x_1,\dots,x_m,s_1,\dots,s_m) k_n(s_1,\dots,s_m,z_1,\dots,z_m) ds_1 \dots ds_m$$

and

$$k_1(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m) = k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m).$$

3.3.2 The Method of Successive Approximations:

This method can be used to solve the 1-D Volterra linear integral equations of the second kind, [Jerri J., 1985].

Here, we use this method to solve the 2-D Volterra linear integral equation of the second kind given by eq.(3.47).

This method starts with substituting a zeroth approximation $u_o(x,y)$ in the integral (of eq.(3.47) with λ = 1) to obtain a first approximation $u_1(x,y)$,

$$u_1(x,y) = f(x,y) + \lambda \int_c^y \int_a^x k(x,y,z,m) u_0(z,m) dz dm \qquad (3.70)$$

Then $u_1(x,y)$ is substituted again in the integral of eq.(3.47) to obtain a second approximation $u_2(x,y)$,

$$u_2(x,y) = f(x,y) + \lambda \int_c^y \int_a^x k(x,y,z,m) u_1(z,m) dz dm$$

This process can be continued to obtain the nth approximation,

$$u_n(x,y) = f(x,y) + \lambda \int_c^y \int_a^x k(x,y,z,m) u_{n-1}(z,m) dz dm \qquad (3.71)$$

Notice that if $f(x,y)$ is continuous for $a \le x \le b$, $c \le y \le d$ and if $k(x,y,z,m)$ is also continuous for $a \le x \le b$, $c \le y \le d$ and $a \le z \le b$, $c \le m \le d$, then it can be proved that the sequence $u_n(x,y)$ will converge to the solution $u(x,y)$ of eq.(3.47).

To illustrate this method, consider the following example.

Example (3.9):

Consider the 2-D Volterra integral equation of the second kind

$$u(x,y) = x\,y + \int_0^y \int_0^x (x\,m)\,u(z,m)dzdm \tag{3.72}$$

we may first remark here that is always an advantage in making a reasonable zeroth approximation, a matter that becomes clearer after solving a number of problems.

We start with $u_0(x,t) = 0$ in the above integral equation to obtain $u_1(x,y)$ according to eq.(3.70)

$$u_1(x,y) = x\,y \tag{3.73}$$

so if we let $u(x,y) = u_1(x,y)$ in eq.(3.70) we obtain

$$u_2(x,y) = x\,y - \int_0^y \int_0^x (x\,m)\,u_1(z,m)dzdm$$

$$= x\,y - \int_0^y \int_0^x (x\,m)(z\,m)dzdm$$

$$= x\,y - \frac{1}{6}x^3 y^3. \tag{3.74}$$

Now,

$$u_3(x,y) = (x\,y) - \int_0^y \int_0^x (x\,m)u_2(z,m)dzdx$$

$$u_3(x,y) = x\,y - \int_0^y \int_0^x (x\,m)\left(z\,m - \frac{1}{6}z^3 m^3\right)dzdm$$

$$= x\,y - \frac{1}{6}x^3y^3 + \frac{1}{120}x^5y^5$$

$$= x\,y - \frac{1}{3!}x^3y^3 + \frac{1}{5!}x^5y\,. \qquad (3.75)$$

From eq.(3.73), eq.(3.74) and eq.(3.75) it looks clear now that if we continue this process, we obtain the nth approximation $u_n(x,y)$ as

$$u_n(x,y) = x\,y - \frac{1}{3!}x^3y^3 + \frac{1}{5!}x^5y^5 - \cdots + (-1)^n\frac{(x\,y)^{2n+1}}{(2n+1)!} \qquad (7.76)$$

which is obviously the nth partial sum of Maclaurian series of $\sin(xy)$. Hence the solution to eq.(3.72) is

$$u(x,y) = \lim_{n\to\infty} u_n(x,y) = \sin(xy). \qquad (3.77)$$

<u>Remark (3.7):</u>

Recall that the 2-D Volterra integral equation of the first kind takes the form

$$f(x,y) = \lambda\int_c^y\int_a^x k(x,y,z,m)u(z,m)dzdm \qquad (3.78)$$

which can be reduced to the 2-D Volterra integral equation of the second kind when $k(x,y,x,y) \neq 0$ since if we differentiate both side of eq. (3.78) with respect to y and with respect to x we can obtain:

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x} = \lambda\int_c^y\int_a^x \frac{\partial^2 k(x,y,z,m)}{\partial y\,\partial x}u(z,m)dzdm + \lambda k(x,y,x,y)u(x,y)$$

This can easily be rewritten

$$u(x,y) = \frac{1}{\lambda k(x,y,x,y)}\frac{\partial^2 f(x,y)}{\partial y\partial x} - \int_c^y\int_a^x \frac{1}{\lambda k(x,y,x,y)}\frac{\partial^2 k(x,y,z,m)}{\partial y\partial x}u(x,y)dzdm$$

As a 2-D Volterra integral equation of the second kind. So when $k(x,y,x,y) \neq 0$ in eq.(3.78) we can reduce it to a 2-D Volterra integral equation of the second kind which can be solved by one of the methods that discussed previously.

Next, the following example is very useful to understand the above study

<u>Example (3.10):</u>

Consider the 2-D Volterra linear integral equation of the first kind

$$(e^{2(x+1)} - e^{x+1})(e^{-(y+1)} - e^{-2(y+1)}) = \int_1^y \int_1^x (x+1)(y+1)e^{xz-ym} u(z,m)\,dz\,dm$$

here $k(x,y,z,m) = (x+1)(y+1)e^{xz-ym}$, thus:

$$k(x,y,x,y) = (x+1)(y+1)e^{x^2+y^2} \neq 0 \;, \;\; \forall (x,y) \in D$$

where $D = \{(x,y) | 1 \leq x,y \leq 2\}$. Also

$$f(x,y) = (e^{2(x+1)} - e^{x+1})(e^{-(y+1)} - e^{-2(y+1)})$$

Hence:

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x} = (2e^{2(x+1)} - e^{x+1})(e^{-(y+1)} - 2e^{-(y+1)}).$$

Moreover:

$$\frac{\partial^2 k(x,y,z,m)}{\partial y\,\partial x} = (1 - m(y+1)(1 - z(x+1)) + z(x+1))e^{xz-ym}$$

Therefore, eq.(3.78) reduces to the following 2-D Volterra integral equation of the second kind

$$u(x,y)=\frac{1}{(x+1)(y+1)e^{xz-ym}}(2e^{2(x+1)}-e^{x+1})(e^{-2(y+1)}-e^{-(y+1)})-$$

$$\int_{1}^{y}\int_{1}^{x}\frac{1}{(x+1)(y+1)e^{xz-ym}}(1-m(y+1)(x+2)+(x+1)z)e^{xz+ym}u(z,m)dzdm$$

then which can be solved by any one of the previous methods.

Conclusion and Recommendations

From the present study, we can conclude the following:

1. The classification of the 1-D integral and integro-differential equations can be extended to include the m-D integral and integro- differential equations.
2. The multi-dimensional integro-differential equations can be also regarded as partial integro-differential equations.
3. In most cases, the multi-dimensional integral and integro-differential equations and the partial integro- differential equations, are so difficult to be solved analytically.
4. Some of the existence and uniqueness theorems of the 1-D integral equations can be extended to include the m-D integral equations.
5. Some of the methods, say the degenerate Kernel method, the method of iterated kernels and the method of Fredholm resolvent kernel are extended to solve special types of the m-D linear integral equations.

For future work, the following problems could be recommended:

1. Modify the quadrature methods to solve the multi-dimensional integral equations.
2. Generalize the Laplace transform method to solve the multi-dimensional Volterra linear integral equations.
3. Solve real life applications in which its mathematical modeling can be represented as m-D integral equations.
4. The study of the m-D non-linear integral equations to include the m-D non-linear integro-differential equations.
5. Devote the study of the system of the non-linear m-D integral equations.

Contents

Introduction

The integral equations arise quite frequently as mathematical models in diverse disciplines. The origins of the study of the integral equations may be traced to the work of Abel, Lotka, Fredholm, Malthas, Verhulst and Volterra on problems in mechanics, mathematical biology and economics, [Lakshmikantham V. and Rama M., 1995].

The one-dimensional integral equations are one of the important classes of the integral equations that has many real life applications, say in population dynamics, the surge in birth rates, the mortality of equipment and the rate of replacement, biological species living together, the torsion of a wire or rod, the control of a rotating shaft, the propagation of a nerve impulse, the smoke filtration in a cigarette, the chance of crossing dense traffic, the shape of a hanging chain, the deflection of a rotating rod, and the shape of a wire that allows a bead to descend on it in a predetermined time, [Jerri A., 1985].

Many researchers and authors studies the one-dimensional integral equations say [Hochstadt H., 1973], [Delves L. and Walsh J., 1974], [Chambers L., 1976], [Delves L. and Mohamed J., 1985], [Jerri J., 1985], [Corduneanu C., 1991] [Atkinson K., 1997]. Moreover, [Najieb, S., 2002] studied the one-dimensional fuzzy integral equations, [Mustafa M., 2004] devoted the numerical solutions for system of the one-dimensional integral equations, [Al-Shakry A., 2001] descried the one-dimensional delay integral equations with their solutions, [Al-Shather, A., 2003] introduced some approximate solutions for solving the one-dimensional fractional integral equations with or without delay, [Abdul-Jabbar, R., 2005] presented the inverse problem for the one-dimensional fractional integral equations, [Al-Shather, A.,1999] studied the one-dimensional singular integral equations.

The one-dimensional integral equations play an important role in the mathematical modeling of many physical problems, biological phenomena and engineering scions, [Belotscrkovsky S. and Lifanov I, 1985] and [Anfinogenov A., 2000].

Many researches studied the two-dimensional integral equations, say [David K., 1999] gave some analytically methods for solving the multi-dimensional integral equations, [Hasson H., 2005] devoted the variational technique for solving the two-dimensional linear integral equations and [Al-Bayati, B., 2005] used the expansion methods for solving the two-dimensional delay integral equations.

The main purpose of this work is to extend the study of the one-dimensional integral equations to include the multi-dimensional integral equations. This study include, the classification of the multi-dimensional integral equations, the relation between the partial differential equations and the multi-dimensional integral equations, the existence and uniqueness theorems for the solutions of the multi-dimensional integral equations and some methods for solving them.

This thesis consists of three chapters.

In chapter one, we extend the classification of the one-dimensional integral and integro-differential equations to be valid for the multi-dimensional integral and integro-differential equations. Also, the relation among some special types of the partial differential equations with the multi-dimensional integral equations is discussed. Moreover, some basic concepts for the partial integro-differential equations is introduced. In chapter two, we generalize some existence and uniqueness theorems for the solution of special types of the multi-dimensional linear and non-linear integral equations (to the best of our knowledge, all theorems appeared in this chapter seems to be new).

In chapter three, we modify some methods that used to solve the one-dimensional linear integral equations to solve the multi-dimensional linear integral equations. These methods are the degenerate kernel method, the method of iterated kernels, the method of Fredholm resolvent kernel for solving the multi-dimensional Fredholm linear integral equations and the resolvent kernel method: Neumann series and the method of successive approximation for solving the multi-dimensional Volterra linear integral equations.

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Chapter Two

Existence and Uniqueness Theorems for the Multi-Dimensional Volterra and Fredholm Integral Equations

Chapter Three

Methods for Solving the Multi-Dimensional Volterra and Fredholm Integral Equations

Chapter One

The Multi-Dimensional Integral Equations

المستخلص

الهدف الرئيسي من هذا العمل هو تعميم دراسة المعادلات التكاملية ذات البعد الواحد الى دراسه المعادلات التكاملية ذات الابعاد المتعددة.

هذه الدراسة شملت تصنيف المعادلات التكاملية والمعادلات التكاملية التفاضلية ذات الابعاد المتعددة.

كما قمنا بأعطاء بعض النظريات الموسعة لوجود و وحدانية الحلول للمعادلات التكاملية ذات الابعاد المتعددة.

بالاضافة الى ذلك قمنا بتعميم بعض الطرق لحل المعادلات التكاملية ذات الابعاد المتعددة.

بسم الله الرحمن الرحيم

﴿اللّهُ لاَ إِلَهَ إِلاَّ هُوَ الْحَيُّ الْقَيُّومُ لاَ تَأْخُذُهُ سِنَةٌ وَلاَ نَوْمٌ لَّهُ مَا فِي السَّمَاوَاتِ وَمَا فِي الأَرْضِ مَن ذَا الَّذِي يَشْفَعُ عِنْدَهُ إِلاَّ بِإِذْنِهِ يَعْلَمُ مَا بَيْنَ أَيْدِيهِمْ وَمَا خَلْفَهُمْ وَلاَ يُحِيطُونَ بِشَيْءٍ مِّنْ عِلْمِهِ إِلاَّ بِمَا شَاء وَسِعَ كُرْسِيُّهُ السَّمَاوَاتِ وَالأَرْضَ وَلاَ يَؤُودُهُ حِفْظُهُمَا وَهُوَ الْعَلِيُّ الْعَظِيمُ﴾

صدق الله العظيم

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المفاتيح الاستدلالية :- Integral equation . Multi-dimensional. Degenerate method. Successive method

المستخلص

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Abstract

The main aim of this work is to generalize the study of the one-dimensional integral equations to include the multi-dimensional integral equations.

This study includes the classification of the multi-dimensional integral and integro-differential equations.

Also, some extended theorems for the existence and uniqueness of solution for the multi-dimensional integral equations are given.

Moreover, some generalized methods are used to solve the multi-dimensional integral equations, with some illustrative examples.

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