Abstract

The main purpose of this work is to study the numerical solutions for special types of the fractional differential equations via the finite difference methods with their stability. This study includes the following aspects:

1. Give some definitions of the fractional order ordinary derivatives with their generalization for the partial ones.
2. Study the existence of the solutions and extermal solutions for special types of the fractional order ordinary differential equations.
3. Use the explicit and the implicit finite difference methods with a study of their stability to solve special types of the one-sided and two-sided fractional order ordinary and partial differential equations.

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Laylan, 2007.

Appendix
Computer Programs

Program (2.1):

$$y^{(0.5)}(x) := xy^2(x) + \frac{2.667}{\Gamma(0.5)} x^{1.5} - x^5,$$

together with initial condition

$y(0) = 0$

We use the explicit finite difference method to solve this example.
First, we construct the exact solution of this example

$$\frac{1}{\Gamma(1-0.5)} \cdot \frac{d}{dx}\int_0^x \frac{y^2}{(x-y)^{0.5}}\,dy \to 1.5045055561273500986 \cdot x^{\frac{3}{2}}$$

$$\frac{2.667}{\Gamma(1-0.5)} = 1.505$$

$$L(x) := \frac{2.667}{\Gamma(1-0.5)} \cdot x^{\frac{3}{2}}$$

$c(x) := x$

$y(x) := x^2$

$$L(x) - c(x) \cdot (y(x))^2 \to 1.5046936193218660173 \cdot x^{\frac{3}{2}} - x^5$$

$$s(x) := \frac{2.667}{\Gamma(1-0.5)} \cdot x^{\frac{3}{2}} - x^5$$

$n := 2 \quad L := 0 \quad R := 1$

$$h := \frac{(R-L)}{n}$$

$i := 0..n$

$x_i := L + i \cdot h$

$\alpha := 0.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..2$

$g_i := g(i)$

$i := 0$

$u_0 := 0$

$i := 0..n-1$

$$u_{i+1} := -1 \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1} \right) + h^{\alpha} \cdot \left[c(x_i) \cdot (u_i)^2 + s(x_i) \right]$$

$$u = \begin{pmatrix} 0 \\ 0 \\ 0.354 \end{pmatrix}$$

$i := 0..2$

$ue_i := (x_i)^2$

$$ue = \begin{pmatrix} 0 \\ 0.25 \\ 1 \end{pmatrix}$$

$n := 10 \quad L := 0 \quad R := 1$

$$h := \frac{(R - L)}{n}$$

$i := 0..n$

$x_i := L + i \cdot h$

$\alpha := 0.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..10$

$g_i := g(i)$

$i := 0$

$u_0 := 0$

$i := 0 .. n-1$

$$u_{i+1} := -1 \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1} \right) + h^{\alpha} \cdot \left[c(x_i) \cdot (u_i)^2 + s(x_i) \right]$$

$i := 0 .. 10$

$ue_i := (x_i)^2$

u =

	0
0	0
1	0
2	0.015
3	0.05
4	0.105
5	0.178
6	0.269
7	0.376
8	0.495
9	0.621
10	0.744

ue =

	0
0	0
1	0.01
2	0.04
3	0.09
4	0.16
5	0.25
6	0.36
7	0.49
8	0.64
9	0.81
10	1

$n := 100 \quad L := 0$

$i := 0 .. 100$

$R := 1$

$$h := \frac{(R - L)}{n}$$

$i := 0 .. n$

$x_i := L + i \cdot h$

$\alpha := 0.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 100$

$g_i := g(i)$

$i := 0$

$u_0 := 0$

$i := 0 .. n-1$

$$u_{i+1} := -1 \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1} \right) + h^{\alpha} \cdot \left[c(x_i) \cdot (u_i)^2 + s(x_i) \right]$$

$i := 0 .. 100$

$ue_i := (x_i)^2$

ue =	u =
0	0
1.00E-04	0
4.00E-04	1.51E-04
9.00E-04	5.01E-04
1.60E-03	1.05E-03
2.50E-03	1.80E-03
3.60E-03	2.75E-03
4.90E-03	3.90E-03
6.40E-03	5.25E-03
8.10E-03	6.80E-03
0.01	8.55E-03
0.0121	0.011
0.0144	0.013
0.0169	0.015
0.0196	0.018
0.0225	0.02
0.0256	0.023
0.0289	0.026
0.0324	0.03
0.0361	0.033
0.04	0.037
0.0441	0.041
0.0484	0.045
0.0529	0.049
0.0576	0.054
0.0625	0.059
0.0676	0.064
0.0729	0.069
0.0784	0.074
0.0841	0.08
0.09	0.086
0.0961	0.091
0.1024	0.098
0.1089	0.104
0.1156	0.11
0.1225	0.117
0.1296	0.124
0.1369	0.131
0.1444	0.139
0.1521	0.146
0.16	0.154
0.1681	0.162
0.1764	0.17
0.1849	0.178

0.1936
0.2025
0.2116
0.2209
0.2304
0.2401
0.25
0.2601
0.2704
0.2809
0.2916
0.3025
0.3136
0.3249
0.3364
0.3481
0.36
0.3721
0.3844
0.3969
0.4096
0.4225
0.4356
0.4489
0.4624
0.4761
0.49
0.5041
0.5184
0.5329
0.5476
0.5625
0.5776
0.5929
0.6084
0.6241
0.64
0.6561
0.6724
0.6889
0.7056
0.7225
0.7396
0.7569
0.7744
0.7921
0.81

0.187
0.195
0.204
0.213
0.223
0.232
0.242
0.252
0.262
0.272
0.283
0.293
0.304
0.315
0.327
0.338
0.35
0.362
0.374
0.386
0.398
0.411
0.424
0.436
0.45
0.463
0.477
0.49
0.504
0.518
0.532
0.547
0.561
0.576
0.591
0.606
0.622
0.637
0.653
0.669
0.684
0.701
0.717
0.733
0.75
0.766
0.783

0.8281	0.8
0.8464	0.817
0.8649	0.834
0.8836	0.852
0.9025	0.869
0.9216	0.886
0.9409	0.904
0.9604	0.921
0.9801	0.939
1	0.957

$n := 170 \quad R := 1 \quad h := \frac{(R - L)}{n}$

$i := 0 .. n$

$x_i := L + i \cdot h$

$\alpha := 0.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 170$

$g_i := g(i)$

$i := 0$

$u_0 := 0$

$i := 0 .. n - 1$

$$u_{i+1} := -1 \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1} \right) + h^{\alpha} \cdot \left[c(x_i) \cdot (u_i)^2 + s(x_i) \right]$$

$i := 0 .. 170$

$ue_i := (x_i)^2$

ue =	u=
0	0
3.46E-05	0
1.38E-04	5.21E-05
3.11E-04	1.73E-04
5.54E-04	3.64E-04
8.65E-04	6.23E-04
1.25E-03	9.52E-04
1.70E-03	1.35E-03
2.21E-03	1.82E-03
2.80E-03	2.35E-03
3.46E-03	2.96E-03
4.19E-03	3.63E-03

4.98E-03
5.85E-03
6.78E-03
7.79E-03
8.86E-03
0.01
0.011211
0.012491
0.013841
0.01526
0.016747
0.018305
0.019931
0.021626
0.023391
0.025225
0.027128
0.0291
0.031142
0.033253
0.035433
0.037682
0.04
0.042388
0.044844
0.04737
0.049965
0.05263
0.055363
0.058166
0.061038
0.063979
0.06699
0.070069
0.073218
0.076436
0.079723
0.08308
0.086505
0.09
0.093564
0.097197
0.1009
0.104671
0.108512
0.112422
0.116401

4.38E-03
5.19E-03
6.07E-03
7.03E-03
8.05E-03
9.14E-03
0.01
0.012
0.013
0.014
0.016
0.017
0.019
0.02
0.022
0.024
0.026
0.028
0.03
0.032
0.034
0.036
0.038
0.041
0.043
0.045
0.048
0.051
0.053
0.056
0.059
0.062
0.065
0.068
0.071
0.074
0.077
0.081
0.084
0.087
0.091
0.094
0.098
0.102
0.106
0.109
0.113

0.12045
0.124567
0.128754
0.13301
0.137336
0.14173
0.146194
0.150727
0.155329
0.16
0.16474
0.16955
0.174429
0.179377
0.184394
0.189481
0.194637
0.199862
0.205156
0.210519
0.215952
0.221453
0.227024
0.232664
0.238374
0.244152
0.25
0.255917
0.261903
0.267958
0.274083
0.280277
0.28654
0.292872
0.299273
0.305744
0.312284
0.318893
0.325571
0.332318
0.339135
0.346021
0.352976
0.36
0.367093
0.374256
0.381488

0.117
0.121
0.126
0.13
0.134
0.138
0.143
0.147
0.152
0.156
0.161
0.166
0.171
0.176
0.18
0.185
0.191
0.196
0.201
0.206
0.212
0.217
0.223
0.228
0.234
0.24
0.245
0.251
0.257
0.263
0.269
0.275
0.281
0.288
0.294
0.3
0.307
0.313
0.32
0.327
0.333
0.34
0.347
0.354
0.361
0.368
0.375

0.388789	0.382
0.396159	0.39
0.403599	0.397
0.411107	0.404
0.418685	0.412
0.426332	0.419
0.434048	0.427
0.441834	0.435
0.449689	0.442
0.457612	0.45
0.465606	0.458
0.473668	0.466
0.481799	0.474
0.49	0.482
0.49827	0.49
0.506609	0.498
0.515017	0.507
0.523495	0.515
0.532042	0.523
0.540657	0.532
0.549343	0.54
0.558097	0.549
0.56692	0.558
0.575813	0.566
0.584775	0.575
0.593806	0.584
0.602907	0.593
0.612076	0.602
0.621315	0.611
0.630623	0.62
0.64	0.629
0.649446	0.638
0.658962	0.648
0.668547	0.657
0.678201	0.666
0.687924	0.676
0.697716	0.685
0.707578	0.695
0.717509	0.705
0.727509	0.714
0.737578	0.724
0.747716	0.734
0.757924	0.744
0.768201	0.754
0.778547	0.764
0.788962	0.774
0.799446	0.784

0.81	0.794
0.820623	0.804
0.831315	0.814
0.842076	0.825
0.852907	0.835
0.863806	0.845
0.874775	0.856
0.885813	0.866
0.89692	0.877
0.908097	0.888
0.919343	0.898
0.930657	0.909
0.942042	0.919
0.953495	0.93
0.965017	0.941
0.976609	0.952
0.98827	0.963
1	0.973

$n := 200 \quad R := 1 \quad h := \frac{(R - L)}{n}$

$i := 0 .. n$

$x_i := L + i \cdot h$

$\alpha := 0.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 170$

$g_i := g(i)$

$i := 0$

$u_0 := 0$

$i := 0 .. 169$

$$u_{i+1} := -1 \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1} \right) + h^{\alpha} \cdot \left[c(x_i) \cdot (u_i)^2 + s(x_i) \right]$$

$i := 0 .. 170$

$ue_i := (x_i)^2$

Program (3.1.2)

$$\frac{\partial u}{\partial t} = e^{\pi t}\frac{\partial^2 u}{\partial x^2}$$

$$-\pi x e^{-\pi t}\sin(\pi x) + \pi^2 x\sin(\pi x) - 2\pi\cos(\pi x)$$

$$u(x,0) = x\sin(\pi x) \ \text{ for } 0 \le x \le 1$$

$$u(0,t) = 0 \ \text{ for } 0 \le t \le 1$$

$$u(1,t) = 0 \ \text{ for } 0 \le t \le 1$$

This example has the exact solution

$$u(x,t) := e^{-\pi \cdot t} \cdot \sin(\pi \cdot x) \cdot x$$

$$c(x,t) := e^{\pi t}$$

$$s(x,t) := -\pi \cdot \exp(-\pi \cdot t) \cdot \sin(\pi \cdot x) \cdot x - \exp(\pi \cdot t) \cdot \big(-\exp(-\pi \cdot t) \cdot$$

$$\sin(\pi \cdot x) \cdot \pi^2 \cdot x + 2 \cdot \exp(-\pi \cdot t) \cdot \cos(\pi \cdot x) \cdot \pi\big)$$

We use the explicit finite difference method to solve this example.

$n := 2$

$m := 2$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$$\beta := \frac{k}{h^2}$$

$$\alpha := \frac{1}{2 \cdot e^{\pi}}$$

$\beta = 2$

$\alpha = 0.022$

$i := 0 .. n$

$x_i := L + i \cdot h$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$i := 0 .. 2$

$u_{i,0} := x_i \cdot \sin(\pi \cdot x_i)$

$i := 0 .. 2$

$u_{0,i} := 0$
$j := 0$
$i := 1 .. n-1$
$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0.5 & 0.182 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 1$
$i := 1 .. n-1$
$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0.5 & 0.182 & -1.016 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 0 .. 2$
$i := 0 .. 2$
$ue_{i,j} := e^{-(\pi \cdot t_j)} \cdot \sin(\pi \cdot x_i) \cdot x_i$

$$ue = \begin{pmatrix} 0 & 0 & 0 \\ 0.5 & 0.10393979 & 0.02160696 \\ 0 & 0 & 0 \end{pmatrix}$$

$n := 10$
$m := 10$
$L := 0$
$R := 1$
$T := 1$
$h := \frac{(R-L)}{n}$
$k := \frac{T}{m}$
$\beta := \frac{k}{h^2}$
$\alpha := \frac{1}{2 \cdot e^{\pi}}$
$\beta = 10$

$\alpha = 0.022$
$i := 0 .. n$
$x_i := L + i \cdot h$
$j := 0 .. m$
$t_j := 0 + j \cdot k$
$i := 0 .. 10$
$u_{i,0} := x_i \cdot \sin(\pi \cdot x_i)$
$i := 0 .. 10$
$u_{0,i} := 0$
$j := 0$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 1$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 2$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 3$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 4$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 5$
$i := 1 .. n - 1$
$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$
$j := 6$
$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$

$j := 7$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$

$j := 8$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$

$j := 9$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$$

$j := 0 .. 10$

$i := 0 .. 10$

$$ue_{i,j} := e^{-(\pi \cdot t_j)} \cdot \sin(\pi \cdot x_i) \cdot x_i$$

ue=

0	0	0	0	0	0	0	0	0	0	0
0.031	0.023	0.016	0.012	8.80E-03	6.42E-03	4.69E-03	3.43E-03	2.50E-03	1.83E-03	1.34E-03
0.118	0.086	0.063	0.046	0.033	0.024	0.018	0.013	9.52E-03	6.96E-03	5.08E-03
0.243	0.177	0.129	0.095	0.069	0.05	0.037	0.027	0.02	0.014	0.01
0.38	0.278	0.203	0.148	0.108	0.079	0.058	0.042	0.031	0.023	0.016
0.5	0.365	0.267	0.195	0.142	0.104	0.076	0.055	0.041	0.03	0.022
0.571	0.417	0.304	0.222	0.162	0.119	0.087	0.063	0.046	0.034	0.025
0.566	0.414	0.302	0.221	0.161	0.118	0.086	0.063	0.046	0.034	0.024
0.47	0.343	0.251	0.183	0.134	0.098	0.071	0.052	0.038	0.028	0.02
0.278	0.203	0.148	0.108	0.079	0.058	0.042	0.031	0.023	0.016	0.012
0	0	0	0	0	0	0	0	0	0	0

u=

0	0	0	0	0	0	0	0	0	0	0
0.031	0.012	0.121	-4.414	290.59	-2.88E+04	4.17E+06	-8.65E+08	2.54E+11	-1.05E+14	6.01E+16
0.118	0.073	0.031	2.69	-246.567	2.97E+04	-4.85E+06	1.09E+09	-3.41E+11	1.47E+14	-8.77E+16
0.243	0.162	0.104	0.097	67.786	-1.34E+04	2.81E+06	-7.45E+08	2.59E+11	-1.21E+14	7.60E+16
0.38	0.261	0.187	0.148	0.099	2.38E+03	-8.69E+05	2.99E+08	-1.21E+11	6.17E+13	-4.12E+16
0.5	0.347	0.261	0.192	0.142	0.103	0.08	0.032	0.309	-37.208	2.07E+04
0.571	0.399	0.309	0.217	0.171	-2.38E+03	8.69E+05	-2.99E+08	1.21E+11	-6.17E+13	4.12E+16
0.566	0.399	0.317	0.213	-67.556	1.34E+04	-2.81E+06	7.45E+08	-2.59E+11	1.21E+14	-7.60E+16
0.47	0.335	0.276	-2.465	246.734	-2.97E+04	4.85E+06	-1.09E+09	3.41E+11	-1.47E+14	8.77E+16
0.278	0.203	0.04	4.532	-290.503	2.88E+04	-4.17E+06	8.65E+08	-2.54E+11	1.05E+14	-6.01E+16
0	0	0	0	0	0	0	0	0	0	0

$n := 10 \quad m := 10000$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R - L)}{n}$

$k := \frac{T}{m}$

$\beta := \frac{k}{h^2}$

$\alpha := \frac{1}{2 \cdot e^{\pi}}$

$\beta = 0.01$

$\alpha = 0.022$

$i := 0 .. n$

$x_i := L + i \cdot h$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$i := 0 .. 10$

$u_{i,0} := x_i \cdot \sin(\pi \cdot x_i)$

$i := 0 .. 10000$

$u_{0,i} := 0$

$j := 0$

$i := 1 .. n - 1$

$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$

$j := 1$

$i := 1 .. n - 1$

$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$

$j := 2$

$i := 1 .. n - 1$

$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i, t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i, t_j) + u_{i,j}$

$j := 3$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 4$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 5$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 6$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 7$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 8$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 9$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^2} \cdot c(x_i,t_j) \cdot \left[u_{i+1,j} - 2 \cdot (u_{i,j}) + u_{i-1,j}\right] + k \cdot s(x_i,t_j) + u_{i,j}$$

$j := 0 .. 10$

$i := 0 .. 10$

$$ue_{i,j} := e^{-(\pi \cdot t_j)} \cdot \sin(\pi \cdot x_i) \cdot x_j$$

ue=

0	0	0	0	0	0	0	0	0	0	0
0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031
0.118	0.118	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117
0.243	0.243	0.243	0.242	0.242	0.242	0.242	0.242	0.242	0.242	0.242
0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.379	0.379	0.379
0.5	0.5	0.5	0.5	0.499	0.499	0.499	0.499	0.499	0.499	0.498
0.571	0.57	0.57	0.57	0.57	0.57	0.57	0.569	0.569	0.569	0.569
0.566	0.566	0.566	0.566	0.566	0.565	0.565	0.565	0.565	0.565	0.565
0.47	0.47	0.47	0.47	0.47	0.469	0.469	0.469	0.469	0.469	0.469
0.278	0.278	0.278	0.278	0.278	0.278	0.278	0.278	0.277	0.277	0.277
0	0	0	0	0	0	0	0	0	0	0

u=

0	0	0	0	0	0	0	0	0	0	0
0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031
0.118	0.118	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117
0.243	0.243	0.243	0.242	0.242	0.242	0.242	0.242	0.242	0.242	0.242
0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.379	0.379	0.379
0.5	0.5	0.5	0.5	0.499	0.499	0.499	0.499	0.499	0.499	0.498
0.571	0.57	0.57	0.57	0.57	0.57	0.57	0.569	0.569	0.569	0.569
0.566	0.566	0.566	0.566	0.566	0.565	0.565	0.565	0.565	0.565	0.565
0.47	0.47	0.47	0.47	0.47	0.47	0.469	0.469	0.469	0.469	0.469
0.278	0.278	0.278	0.278	0.278	0.278	0.278	0.278	0.278	0.277	0.277
0	0	0	0	0	0	0	0	0	0	0

Program (3.1.3):

$$\frac{\partial u(x,t)}{\partial t} = x^{\frac{4}{5}} \frac{\partial^{1.8} u(x,t)}{\partial x^{1/8}} + x(x-1) - \frac{t}{\Gamma(0.2)}(10x-1), 0 \le x \le 1, 0 \le t \le 1$$

$u(x,0) = 0$ for $0 \le x \le 1$.

$u(0,t) = 0$ for $0 \le t \le 1$.

$u(1,t) = 0$ for $0 \le t \le 1$.

We use the explicit finite difference method to solve this example.

$s(x,t) := x\cdot(x-1) - .21782488421166726157\cdot t\cdot(10\cdot x - 1)$

$n := 2$

$m := 2$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R - L)}{n}$

$k := \frac{T}{m}$

$i := 0..2$

$x_i := L + i\cdot h$

$j := 0..10$

$t_j := 0 + j\cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p\cdot(\alpha - i) \end{array} \right.$$

$i := 1..2$

$g_i := g(i)$

$i := 0..2$

$u_{i,0} := 0$

$i := 0..2$

$u_{0,i} := 0$

$s(x,t) := x\cdot(x-1) - .21782488421166726157\cdot t\cdot(10\cdot x - 1)$

$j := 0$

$i := 1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.125 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 1$
$i := 1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.125 & -0.242825 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 0..2$
$i := 0..2$
$ue_{i,j} := x_i\cdot(x_i - 1)\cdot t_j$

$$ue = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.125 & -0.25 \\ 0 & 0 & 0 \end{pmatrix}$$

$n := 3$
$m := 4$
$L := 0$
$R := 1$
$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$i := 0..3$
$x_i := L + i\cdot h$
$j := 0..4$
$t_j := 0 + j\cdot k$
$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..3$

$g_i := g(i)$

$i := 0..3$

$u_{i,0} := 0$

$i := 0..4$

$u_{0,i} := 0$

$s(x,t) := x \cdot (x-1) - .2178248842116672 6157 \cdot t \cdot (10 \cdot x - 1)$

$j := 0$

$i := 1..2$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.056 & -0.243 & 0 & 0 \\ 0 & -0.056 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 1$

$i := 1..2$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.056 & -0.11 & 0 & 0 \\ 0 & -0.056 & -0.11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 2$

$i := 1..2$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.056 & -0.11 & -0.163 & 0 \\ 0 & -0.056 & -0.11 & -0.164 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 3$

$i := 1..2$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.056 & -0.11 & -0.163 & -0.217 \\ 0 & -0.056 & -0.11 & -0.164 & -0.218 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 0..4$

$i := 0..3$

$ue_{i,j} := x_i \cdot (x_i - 1) \cdot t_j$

$$ue = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.056 & -0.111 & -0.167 & -0.222 \\ 0 & -0.056 & -0.111 & -0.167 & -0.222 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$n := 10$

$m := 10$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$i := 0..10$

$x_i := L + i \cdot h$

$j := 0..10$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..10$

$g_i := g(i)$

$i := 0..10$

$u_{i,0} := 0$

$i := 0..10$

$u_{0,i} := 0$

$s(x,t) := x\cdot(x-1) - .21782488421166726157\cdot t\cdot(10\cdot x - 1)$

$j := 0$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.10954	-0.16318	-0.21705	0	0	0	0	0	0
0	-0.016	-0.10991	-0.16441	-0.21838	0	0	0	0	0	0
0	-0.021	0	0	0	0	0	0	0	0	0
0	-0.024	0	0	0	0	0	0	0	0	0
0	-0.025	0	0	0	0	0	0	0	0	0
0	-0.024	0	0	0	0	0	0	0	0	0
0	-0.021	0	0	0	0	0	0	0	0	0
0	-0.016	0	0	0	0	0	0	0	0	0
0	-9.00E-03	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 1$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.16318	-0.21705	0	0	0	0	0	0
0	-0.016	-0.03188	-0.16441	-0.21838	0	0	0	0	0	0
0	-0.021	-0.04191	0	0	0	0	0	0	0	0
0	-0.024	-0.04792	0	0	0	0	0	0	0	0
0	-0.025	-0.04993	0	0	0	0	0	0	0	0
0	-0.024	-0.04793	0	0	0	0	0	0	0	0
0	-0.021	-0.04194	0	0	0	0	0	0	0	0
0	-0.016	-0.03194	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.01794	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 2$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.018	-0.027	-0.217	0	0	0	0	0	0
0	-0.016	-0.032	-0.048	-0.218	0	0	0	0	0	0
0	-0.021	-0.042	-0.063	0	0	0	0	0	0	0
0	-0.024	-0.048	-0.072	0	0	0	0	0	0	0
0	-0.025	-0.05	-0.075	0	0	0	0	0	0	0
0	-0.024	-0.048	-0.072	0	0	0	0	0	0	0
0	-0.021	-0.042	-0.063	0	0	0	0	0	0	0
0	-0.016	-0.032	-0.048	0	0	0	0	0	0	0
0	-9.00E-03	-0.018	-0.027	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 3$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	0	0	0	0	0	0
0	-0.016	-0.03188	-0.0476	-0.06351	0	0	0	0	0	0
0	-0.021	-0.04191	-0.0627	-0.08338	0	0	0	0	0	0
0	-0.024	-0.04792	-0.07175	-0.09547	0	0	0	0	0	0
0	-0.025	-0.04993	-0.07477	-0.09953	0	0	0	0	0	0
0	-0.024	-0.04793	-0.07179	-0.09557	0	0	0	0	0	0
0	-0.021	-0.04194	-0.0628	-0.0836	0	0	0	0	0	0
0	-0.016	-0.03194	-0.04781	-0.0654	0	0	0	0	0	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 4$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	0	0	0	0	0
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	0	0	0	0	0
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	0	0	0	0	0
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	0	0	0	0	0
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	0	0	0	0	0
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	0	0	0	0	0
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0	0	0	0	0
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	0	0	0	0	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 5$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	0	0	0	0
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	0	0	0	0
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	0	0	0	0
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	0	0	0	0
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	0	0	0	0
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0	0	0	0
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	0	0	0	0
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	0	0	0	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 6$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	0	0	0
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	0	0	0
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	0	0	0
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	0	0	0
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	0	0	0
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	0	0	0
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	0	0	0
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	0	0	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 7$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	-0.05951	0	0
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	-0.16934	0	0
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	-0.09031	0	0
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	-0.6463	0	0
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	4.271436	0	0
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	-22.1344	0	0
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	59.86746	0	0
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	-95.2276	0	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	75.98654	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 8$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	-0.05951	-0.130733	0
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	-0.16934	0.096096	0
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	-0.09031	-1.611601	0
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	-0.6463	15.528214	0
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	4.271436	-105.6109	0
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	-22.1344	408.58627	0
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	59.86746	-977.7611	0
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	-95.2276	1.43E+03	0
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	75.98654	-1.10E+03	0
0	0	0	0	0	0	0	0	0	0	0

$j := 9$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	-0.05951	-0.1307328	0.191682
0	-0.016	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	-0.16934	0.0960958	-3.210517
0	-0.021	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	-0.09031	-1.6116011	42.861105
0	-0.024	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	-0.6463	15.528214	-392.947
0	-0.025	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	4.271436	-105.61085	2.10E+03
0	-0.024	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	-22.1344	408.58627	-7.09E+03
0	-0.021	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	59.86746	-977.76106	1.55E+04
0	-0.016	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	-95.2276	1.43E+03	-2.16E+04
0	-9.00E-03	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	75.98654	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 0 .. 10$

$i := 0 .. 10$

$ue_{i,j} := x_i \cdot (x_i - 1) \cdot t_j$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-03	-0.018	-0.027	-0.036	-0.045	-0.054	-0.063	-0.072	-0.081	-0.09
0	-0.016	-0.032	-0.048	-0.064	-0.08	-0.096	-0.112	-0.128	-0.144	-0.16
0	-0.021	-0.042	-0.063	-0.084	-0.105	-0.126	-0.147	-0.168	-0.189	-0.21
0	-0.024	-0.048	-0.072	-0.096	-0.12	-0.144	-0.168	-0.192	-0.216	-0.24
0	-0.025	-0.05	-0.075	-0.1	-0.125	-0.15	-0.175	-0.2	-0.225	-0.25
0	-0.024	-0.048	-0.072	-0.096	-0.12	-0.144	-0.168	-0.192	-0.216	-0.24
0	-0.021	-0.042	-0.063	-0.084	-0.105	-0.126	-0.147	-0.168	-0.189	-0.21
0	-0.016	-0.032	-0.048	-0.064	-0.08	-0.096	-0.112	-0.128	-0.144	-0.16
0	-9.00E-03	-0.018	-0.027	-0.036	-0.045	-0.054	-0.063	-0.072	-0.081	-0.09
0	0	0	0	0	0	0	0	0	0	0

$n := 10$

$m := 1000$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$i := 0 .. 10$

$x_i := L + i \cdot h$

$j := 0 .. 1000$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 10$

$g_i := g(i)$

$i := 0 .. 10$

$u_{i,0} := 0$

$i := 0 .. 1000$

$u_{0,i} := 0$

$s(x,t) := x \cdot (x-1) - .2178248842116672615 7 \cdot t \cdot (10 \cdot x - 1)$

$j := 0$

$i := 1 .. 9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-0.0178	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	-0.05951	-0.130733	0.191682
0	-1.60E-04	-0.03188	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	-0.16934	0.096096	-3.210517
0	-2.10E-04	-0.04191	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	-0.09031	-1.611601	42.861105
0	-2.40E-04	-0.04792	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	-0.6463	15.528214	-392.947
0	-2.50E-04	-0.04993	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	4.271436	-105.61085	2.10E+03
0	-2.40E-04	-0.04793	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	-22.1344	408.58627	-7.09E+03
0	-2.10E-04	-0.04194	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	59.86746	-977.76106	1.55E+04
0	-1.60E-04	-0.03194	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	-95.2276	1.43E+03	-2.16E+04
0	-9.00E-05	-0.01794	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	75.98654	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 1$

$i := 1 .. 9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-0.02664	-0.03529	-0.04428	-0.05216	-0.06393	-0.05951	-0.130733	0.191682
0	-1.60E-04	-3.20E-04	-0.0476	-0.06351	-0.07858	-0.09666	-0.10165	-0.16934	0.096096	-3.210517
0	-2.10E-04	-4.20E-04	-0.0627	-0.08338	-0.1045	-0.1226	-0.15787	-0.09031	-1.611601	42.861105
0	-2.40E-04	-4.80E-04	-0.07175	-0.09547	-0.11915	-0.14366	-0.15847	-0.6463	15.528214	-392.947
0	-2.50E-04	-5.00E-04	-0.07477	-0.09953	-0.12421	-0.14896	-0.3037	4.271436	-105.61085	2.10E+03
0	-2.40E-04	-4.80E-04	-0.07179	-0.09557	-0.11927	-0.17846	0.863106	-22.1344	408.58627	-7.09E+03
0	-2.10E-04	-4.20E-04	-0.0628	-0.0836	-0.11279	0.065191	-3.67788	59.86746	-977.76106	1.55E+04
0	-1.60E-04	-3.20E-04	-0.04781	-0.0654	-0.05272	-0.51191	6.197361	-95.2276	1.43E+03	-2.16E+04
0	-9.00E-05	-1.80E-04	-0.02716	-0.03346	-0.07438	0.335459	-5.42508	75.98654	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 2$

$i := 1 .. 9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-0.035	-0.044	-0.052	-0.064	-0.06	-0.131	0.192
0	-1.60E-04	-3.20E-04	-4.80E-04	-0.064	-0.079	-0.097	-0.102	-0.169	0.096	-3.211
0	-2.10E-04	-4.20E-04	-6.30E-04	-0.083	-0.105	-0.123	-0.158	-0.09	-1.612	42.861
0	-2.40E-04	-4.80E-04	-7.20E-04	-0.095	-0.119	-0.144	-0.158	-0.646	15.528	-392.947
0	-2.50E-04	-5.00E-04	-7.50E-04	-0.1	-0.124	-0.149	-0.304	4.271	-105.611	2.10E+03
0	-2.40E-04	-4.80E-04	-7.20E-04	-0.096	-0.119	-0.178	0.863	-22.134	408.586	-7.09E+03
0	-2.10E-04	-4.20E-04	-6.30E-04	-0.084	-0.113	0.065	-3.678	59.867	-977.761	1.55E+04
0	-1.60E-04	-3.20E-04	-4.80E-04	-0.065	-0.053	-0.512	6.197	-95.228	1.43E+03	-2.16E+04
0	-9.00E-05	-1.80E-04	-2.70E-04	-0.033	-0.074	0.335	-5.425	75.987	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 3$

$i := 1 .. 9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-0.04428	-0.05216	-0.06393	-0.059506	-0.130733	0.191682
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-0.07858	-0.09666	-0.10165	-0.169338	0.096096	-3.210517
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-0.1045	-0.1226	-0.15787	-0.090314	-1.611601	42.861105
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-0.11915	-0.14366	-0.15847	-0.646298	15.528214	-392.94702
0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-0.12421	-0.14896	-0.3037	4.271436	-105.6109	2.10E+03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-0.11927	-0.17846	0.863106	-22.134366	408.58627	-7.09E+03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-0.11279	0.065191	-3.67788	59.867458	-977.7611	1.55E+04
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-0.05272	-0.51191	6.197361	-95.227578	1.43E+03	-2.16E+04
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-0.07438	0.335459	-5.42508	75.986544	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 4$

$i := 1 .. 9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot (x_i)^{\frac{4}{5}} \cdot \left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-0.05216	-0.06393	-0.05951	-0.130733	0.191682
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-0.09666	-0.10165	-0.16934	0.096096	-3.210517
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-0.1226	-0.15787	-0.09031	-1.611601	42.861105
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-0.14366	-0.15847	-0.6463	15.528214	-392.947
u=	0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-0.14896	-0.3037	4.271436	-105.61085	2.10E+03
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-0.17846	0.863106	-22.1344	408.58627	-7.09E+03
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	0.065191	-3.67788	59.86746	-977.76106	1.55E+04
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-0.51191	6.197361	-95.2276	1.43E+03	-2.16E+04
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	0.335459	-5.42508	75.98654	-1.10E+03	1.61E+04
	0	0	0	0	0	0	0	0	0	0	0

$j := 5$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-0.06393	-0.059506	-0.130733	0.191682
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-0.10165	-0.169338	0.096096	-3.210517
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-0.15787	-0.090314	-1.611601	42.861105
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-0.15847	-0.646298	15.528214	-392.94702
u=	0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-0.3037	4.271436	-105.6109	2.10E+03
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	0.863106	-22.134366	408.58627	-7.09E+03
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-3.67788	59.867458	-977.7611	1.55E+04
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	6.197361	-95.227578	1.43E+03	-2.16E+04
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-5.42508	75.986544	-1.10E+03	1.61E+04
	0	0	0	0	0	0	0	0	0	0	0

$j := 6$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot(x_i)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-0.059506	-0.130733	0.191682
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-0.169338	0.096096	-3.210517
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-0.090314	-1.611601	42.861105
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-0.646298	15.528214	-392.94702
u=	0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-1.75E-03	4.271436	-105.6109	2.10E+03
	0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-22.134366	408.58627	-7.09E+03
	0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	59.867458	-977.7611	1.55E+04
	0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-95.227578	1.43E+03	-2.16E+04
	0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	75.986544	-1.10E+03	1.61E+04
	0	0	0	0	0	0	0	0	0	0	0

$j := 7$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot\left(x_i\right)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s\left(x_i,t_j\right) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.19E-04	-0.130733	0.191682
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	0.096096	-3.210517
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.611601	42.861105
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	15.528214	-392.94702
0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-1.75E-03	-2.00E-03	-105.61085	2.10E+03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	408.586266	-7.09E+03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-977.761061	1.55E+04
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	1.43E+03	-2.16E+04
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.20E-04	-1.10E+03	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 8$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot\left(x_i\right)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s\left(x_i,t_j\right) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.19E-04	-8.09E-04	0.191682
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-3.210517
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	42.861105
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-392.94702
0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-1.75E-03	-2.00E-03	-2.25E-03	2.10E+03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-7.09E+03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	1.55E+04
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-2.16E+04
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.20E-04	-8.10E-04	1.61E+04
0	0	0	0	0	0	0	0	0	0	0

$j := 9$

$i := 1..9$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot\left(x_i\right)^{\frac{4}{5}}\cdot\left(u_{i+1,j} + \sum_{k=1}^{i+1} g_k u_{i-k+1,j}\right) + k\cdot s\left(x_i,t_j\right) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.19E-04	-8.09E-04	-8.99E-04
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-1.60E-03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	-2.10E-03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-2.40E-03
0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-1.75E-03	-2.00E-03	-2.25E-03	-2.50E-03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-2.40E-03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	-2.10E-03
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-1.60E-03
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.20E-04	-8.10E-04	-9.00E-04
0	0	0	0	0	0	0	0	0	0	0

$j := 0..10$
$i := 0..10$
$ue_{i,j} := x_i \cdot (x_i - 1) \cdot t_j$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.20E-04	-8.10E-04	-9.00E-04
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-1.60E-03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	-2.10E-03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-2.40E-03
0	-2.50E-04	-5.00E-04	-7.50E-04	-1.00E-03	-1.25E-03	-1.50E-03	-1.75E-03	-2.00E-03	-2.25E-03	-2.50E-03
0	-2.40E-04	-4.80E-04	-7.20E-04	-9.60E-04	-1.20E-03	-1.44E-03	-1.68E-03	-1.92E-03	-2.16E-03	-2.40E-03
0	-2.10E-04	-4.20E-04	-6.30E-04	-8.40E-04	-1.05E-03	-1.26E-03	-1.47E-03	-1.68E-03	-1.89E-03	-2.10E-03
0	-1.60E-04	-3.20E-04	-4.80E-04	-6.40E-04	-8.00E-04	-9.60E-04	-1.12E-03	-1.28E-03	-1.44E-03	-1.60E-03
0	-9.00E-05	-1.80E-04	-2.70E-04	-3.60E-04	-4.50E-04	-5.40E-04	-6.30E-04	-7.20E-04	-8.10E-04	-9.00E-04
0	0	0	0	0	0	0	0	0	0	0

Program (3.1.4):

$$\frac{\partial u}{\partial t} = (1-x)^{1.8}\frac{\partial^{1.8}u}{\partial_+ x^{1/8}} + tx^{0.8}\frac{\partial^{1.8}u}{\partial_- x^{1/8}} + x(1-x) -$$
$$\frac{1}{\Gamma(0.2)}\left[t^2(-10x+1) - t(5.4 - 9.8x + 4.4x^2)\right], 0 \le x \le 1, 0 \le t \le 1$$

$u(x,0) = 0$ for $0 \le x \le 1$.

$u(0,t) = 0$ for $0 \le t \le 1$.

$u(1,t) = 0$ for $0 \le t \le 1$.

We use the explicit finite difference method to solve this example.

$$L(x,t) := \frac{t}{\Gamma(2-1.8)} \cdot \frac{d^2}{dx^2}\left[\int_0^x y \cdot \frac{(1-y)}{(x-y)^{0.8}}\, dy\right]$$

$$c(x,t) := t \cdot x^{\frac{4}{5}}$$

$$F1(x,t) := .21782488421166726157 \cdot t^2 \cdot (-10 \cdot x + 1)$$

$$R(x,t) := \frac{t}{\Gamma(2-1.8)} \cdot \frac{d^2}{dx^2}\left[\int_x^1 y \cdot \frac{(1-y)}{(y-x)^{0.8}}\, dy\right]$$

$$d(x,t) := (1-x)^{\frac{9}{5}}$$

$$F2(x,t) := .21782488421166726157 \cdot t\left[7.2 \cdot (1-x)^2 - 1.8 \cdot (1-x) - .8 \cdot x + 3.6(1-x) \cdot x + .8 \cdot x^2\right]$$

$$s(x,t) := x\cdot(1-x) - \frac{1}{\Gamma(0.2)}\cdot\left[t^2\cdot(-10x+1) - t\cdot\left(5.4 - 9.8 + 4.4x^2\right)\right]$$

$n := 2$

$m := 2$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R-L)}{n}$$

$$k := \frac{T}{m}$$

$i := 0 .. n$

$x_i := L + i\cdot h$

$j := 0 .. m$

$t_j := 0 + j\cdot k$

$\alpha := 1.8$

$$g(k) := \left|\begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!}\right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p\cdot(\alpha - i) \end{array}\right.$$

$i := 1 .. 2$

$g_i := g(i)$

$i := 0 .. 2$

$u_{i,0} := 0$

$i := 0 .. 2$

$u_{0,i} := 0$

$j := 0$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^\alpha}\cdot c\left(x_i,t_j\right)\cdot\left(\sum_{l=1}^{i+1} g_l\cdot u_{i-l+1,j} + u_{i+1,j}\right) + \frac{k}{h^\alpha}\cdot d\left(x_i,t_j\right)\cdot\left(\sum_{l=1}^{2-i+1} g_l\cdot u_{i+l-1,j} + u_{i-1,j}\right) + k\cdot s\left(x_i,t_j\right) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 1$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{2-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.125 & -0.046 \\ 0 & 0 & 0 \end{pmatrix}$$

$j := 0..2$

$i := 0..2$

$ue_{i,j} := x_i \cdot (1 - x_i) \cdot t_j$

$$ue = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.125 & 0.25 \\ 0 & 0 & 0 \end{pmatrix}$$

$n := 3$

$m := 4$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$i := 0..n$

$x_i := L + i \cdot h$

$j := 0..m$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..3$

$g_i := g(i)$

$i := 0..3$

$u_{i,0} := 0$

$i := 0..4$

$u_{0,i} := 0$

$j := 0$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{3-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.056 & -0.046 & 0 & 0 \\ 0 & 0.056 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 1$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{3-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.056 & 5.24 \times 10^{-3} & 0 & 0 \\ 0 & 0.056 & 0.066 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 2$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{3-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.056 & 5.24 \times 10^{-3} & 0.041 & 0 \\ 0 & 0.056 & 0.066 & 0.028 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 3$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{3-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.056 & 5.24 \times 10^{-3} & 0.041 & -0.063 \\ 0 & 0.056 & 0.066 & 0.028 & 0.134 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$j := 0 .. 4$

$i := 0 .. 3$

$ue_{i,j} := x_i \cdot (1 - x_i) \cdot t_j$

$$ue = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05555556 & 0.11111111 & 0.16666667 & 0.22222222 \\ 0 & 0.05555556 & 0.11111111 & 0.16666667 & 0.22222222 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$n := 10$

$m := 10$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R - L)}{n}$

$k := \frac{T}{m}$

$i := 0 .. n$

$x_i := L + i \cdot h$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 10$

$g_i := g(i)$

$i := 0 .. 10$

$u_{i,0} := 0$

$i := 0 .. 10$

$u_{0,i} := 0$

$j := 0$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	5.24E-03	0.041	-0.063	0	0	0	0	0	0
0	0.016	0.066	0.028	0.134	0	0	0	0	0	0
0	0.021	0	0	0	0	0	0	0	0	0
0	0.024	0	0	0	0	0	0	0	0	0
0	0.025	0	0	0	0	0	0	0	0	0
0	0.024	0	0	0	0	0	0	0	0	0
0	0.021	0	0	0	0	0	0	0	0	0
0	0.016	0	0	0	0	0	0	0	0	0
0	9.00E-03	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 1$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.041	-0.063	0	0	0	0	0	0
0	0.016	0.011	0.028	0.134	0	0	0	0	0	0
0	0.021	0.024	0	0	0	0	0	0	0	0
0	0.024	0.033	0	0	0	0	0	0	0	0
0	0.025	0.038	0	0	0	0	0	0	0	0
0	0.024	0.039	0	0	0	0	0	0	0	0
0	0.021	0.036	0	0	0	0	0	0	0	0
0	0.016	0.028	0	0	0	0	0	0	0	0
0	9.00E-03	0.016	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 2$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-0.063	0	0	0	0	0	0
0	0.016	0.011	-0.018	0.134	0	0	0	0	0	0
0	0.021	0.024	8.11E-03	0	0	0	0	0	0	0
0	0.024	0.033	0.027	0	0	0	0	0	0	0
0	0.025	0.038	0.04	0	0	0	0	0	0	0
0	0.024	0.039	0.045	0	0	0	0	0	0	0
0	0.021	0.036	0.044	0	0	0	0	0	0	0
0	0.016	0.028	0.036	0	0	0	0	0	0	0
0	9.00E-03	0.016	0.021	0	0	0	0	0	0	0

$j := 3$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	0	0	0	0	0	0
0	0.016	0.011	-0.018	0.651	0	0	0	0	0	0
0	0.021	0.024	8.11E-03	-0.021	0	0	0	0	0	0
0	0.024	0.033	0.027	7.00E-03	0	0	0	0	0	0
0	0.025	0.038	0.04	0.029	0	0	0	0	0	0
0	0.024	0.039	0.045	0.042	0	0	0	0	0	0
0	0.021	0.036	0.044	0.045	0	0	0	0	0	0
0	0.016	0.028	0.036	0.039	0	0	0	0	0	0
0	9.00E-03	0.016	0.021	0.024	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 4$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	0	0	0	0	0
0	0.016	0.011	-0.018	0.651	-10.085	0	0	0	0	0
0	0.021	0.024	8.11E-03	-0.021	2.732	0	0	0	0	0
0	0.024	0.033	0.027	7.00E-03	4.64E-03	0	0	0	0	0
0	0.025	0.038	0.04	0.029	0.013	0	0	0	0	0
0	0.024	0.039	0.045	0.042	0.031	0	0	0	0	0
0	0.021	0.036	0.044	0.045	0.04	0	0	0	0	0
0	0.016	0.028	0.036	0.039	0.039	0	0	0	0	0
0	9.00E-03	0.016	0.021	0.024	0.024	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 5$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	-154.744	0	0	0	0
0	0.016	0.011	-0.018	0.651	-10.085	151.694	0	0	0	0
0	0.021	0.024	8.11E-03	-0.021	2.732	-61.035	0	0	0	0
0	0.024	0.033	0.027	7.00E-03	4.64E-03	9.403	0	0	0	0
0	0.025	0.038	0.04	0.029	0.013	0.169	0	0	0	0
0	0.024	0.039	0.045	0.042	0.031	0.06	0	0	0	0
0	0.021	0.036	0.044	0.045	0.04	0.053	0	0	0	0
0	0.016	0.028	0.036	0.039	0.039	0.046	0	0	0	0
0	9.00E-03	0.016	0.021	0.024	0.024	0.037	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 6$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	-154.744	2.11E+03	0	0	0
0	0.016	0.011	-0.018	0.651	-10.085	151.694	-2.30E+03	0	0	0
0	0.021	0.024	8.11E-03	-0.021	2.732	-61.035	1.15E+03	0	0	0
0	0.024	0.033	0.027	7.00E-03	4.64E-03	9.403	-287.626	0	0	0
0	0.025	0.038	0.04	0.029	0.013	0.169	27.189	0	0	0
0	0.024	0.039	0.045	0.042	0.031	0.06	0.406	0	0	0
0	0.021	0.036	0.044	0.045	0.04	0.053	-0.055	0	0	0
0	0.016	0.028	0.036	0.039	0.039	0.046	-0.047	0	0	0
0	9.00E-03	0.016	0.021	0.024	0.024	0.037	-0.118	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 7$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	0	0
0	0.016	0.011	-0.018	0.651	-10.085	151.694	-2.30E+03	3.59E+04	0	0
0	0.021	0.024	8.11E-03	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	0	0
0	0.024	0.033	0.027	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	0	0
0	0.025	0.038	0.04	0.029	0.013	0.169	27.189	-1.14E+03	0	0
0	0.024	0.039	0.045	0.042	0.031	0.06	0.406	73.785	0	0
0	0.021	0.036	0.044	0.045	0.04	0.053	-0.055	4.613	0	0
0	0.016	0.028	0.036	0.039	0.039	0.046	-0.047	1.897	0	0
0	9.00E-03	0.016	0.021	0.024	0.024	0.037	-0.118	2.391	0	0
0	0	0	0	0	0	0	0	0	0	0

$j := 8$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	0
0	0.016	0.011	-0.018	0.651	-10.085	151.694	-2.30E+03	3.59E+04	-5.76E+05	0
0	0.021	0.024	8.11E-03	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	0
0	0.024	0.033	0.027	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	0
0	0.025	0.038	0.04	0.029	0.013	0.169	27.189	-1.14E+03	3.30E+04	0
0	0.024	0.039	0.045	0.042	0.031	0.06	0.406	73.785	-4.15E+03	0
0	0.021	0.036	0.044	0.045	0.04	0.053	-0.055	4.613	118.147	0
0	0.016	0.028	0.036	0.039	0.039	0.046	-0.047	1.897	-19.231	0
0	9.00E-03	0.016	0.021	0.024	0.024	0.037	-0.118	2.391	-36.911	0
0	0	0	0	0	0	0	0	0	0	0

$j := 9$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c\left(x_i, t_j\right) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d\left(x_i, t_j\right) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s\left(x_i, t_j\right) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	-7.25E-03	0.106	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
0	0.016	0.011	-0.018	0.651	-10.085	151.694	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
0	0.021	0.024	8.11E-03	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
0	0.024	0.033	0.027	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
0	0.025	0.038	0.04	0.029	0.013	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
0	0.024	0.039	0.045	0.042	0.031	0.06	0.406	73.785	-4.15E+03	1.50E+05
0	0.021	0.036	0.044	0.045	0.04	0.053	-0.055	4.613	118.147	-1.32E+04
0	0.016	0.028	0.036	0.039	0.039	0.046	-0.047	1.897	-19.231	684.795
0	9.00E-03	0.016	0.021	0.024	0.024	0.037	-0.118	2.391	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 0..10$

$i := 0..10$

$ue_{i,j} := x_i \cdot \left(1 - x_i\right) \cdot t_j$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-03	0.018	0.027	0.036	0.045	0.054	0.063	0.072	0.081	0.09
0	0.016	0.032	0.048	0.064	0.08	0.096	0.112	0.128	0.144	0.16
0	0.021	0.042	0.063	0.084	0.105	0.126	0.147	0.168	0.189	0.21
0	0.024	0.048	0.072	0.096	0.12	0.144	0.168	0.192	0.216	0.24
0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
0	0.024	0.048	0.072	0.096	0.12	0.144	0.168	0.192	0.216	0.24
0	0.021	0.042	0.063	0.084	0.105	0.126	0.147	0.168	0.189	0.21
0	0.016	0.032	0.048	0.064	0.08	0.096	0.112	0.128	0.144	0.16
0	9.00E-03	0.018	0.027	0.036	0.045	0.054	0.063	0.072	0.081	0.09
0	0	0	0	0	0	0	0	0	0	0

$n := 10$

$m := 1000$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R - L)}{n}$

$k := \frac{T}{m}$

$i := 0..n$

$x_i := L + i \cdot h$

$j := 0..m$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^{k} \dfrac{1}{k!} \right] \\ \text{for } i \in 0 .. (k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1 .. 10$

$g_i := g(i)$

$i := 0 .. 10$

$u_{i,0} := 0$

$i := 0 .. 1000$

$u_{0,i} := 0$

$j := 0$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	-7.25E-03	0.106	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
0	1.60E-04	0.011	-0.018	0.651	-10.085	151.694	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
0	2.10E-04	0.024	8.11E-03	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
0	2.40E-04	0.033	0.027	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
0	2.50E-04	0.038	0.04	0.029	0.013	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
0	2.40E-04	0.039	0.045	0.042	0.031	0.06	0.406	73.785	-4.15E+03	1.50E+05
0	2.10E-04	0.036	0.044	0.045	0.04	0.053	-0.055	4.613	118.147	-1.32E+04
0	1.60E-04	0.028	0.036	0.039	0.039	0.046	-0.047	1.897	-19.231	684.795
0	9.00E-05	0.016	0.021	0.024	0.024	0.037	-0.118	2.391	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 1$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i, t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i, t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i, t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	0.106	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
0	1.60E-04	3.18E-04	-0.018	0.651	-10.085	151.694	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
0	2.10E-04	4.18E-04	8.11E-03	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
0	2.40E-04	4.79E-04	0.027	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
0	2.50E-04	4.99E-04	0.04	0.029	0.013	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
0	2.40E-04	4.79E-04	0.045	0.042	0.031	0.06	0.406	73.785	-4.15E+03	1.50E+05
0	2.10E-04	4.19E-04	0.044	0.045	0.04	0.053	-0.055	4.613	118.147	-1.32E+04
0	1.60E-04	3.20E-04	0.036	0.039	0.039	0.046	-0.047	1.897	-19.231	684.795
0	9.00E-05	1.80E-04	0.021	0.024	0.024	0.037	-0.118	2.391	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 2$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot c(x_i,t_j)\cdot\left(\sum_{l=1}^{i+1} g_l\cdot u_{i-l+1,j} + u_{i+1,j}\right) + \frac{k}{h^{\alpha}}\cdot d(x_i,t_j)\cdot\left(\sum_{l=1}^{10-i+1} g_l\cdot u_{i+l-1,j} + u_{i-1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	9.00E-05	1.78E-04	2.63E-04	-1.034	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
	0	1.60E-04	3.18E-04	4.74E-04	0.651	-10.085	151.694	-2304	3.59E+04	-5.76E+05	9.55E+06
	0	2.10E-04	4.18E-04	6.25E-04	-0.021	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
	0	2.40E-04	4.79E-04	7.16E-04	7.00E-03	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
u=	0	2.50E-04	4.99E-04	7.47E-04	0.029	0.013	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
	0	2.40E-04	4.79E-04	7.17E-04	0.042	0.031	0.06	0.406	73.785	-4.15E+03	1.50E+05
	0	2.10E-04	4.19E-04	6.28E-04	0.045	0.04	0.053	-0.055	4.613	118.147	-1.32E+04
	0	1.60E-04	3.20E-04	4.79E-04	0.039	0.039	0.046	-0.047	1.897	-19.231	684.795
	0	9.00E-05	1.80E-04	2.69E-04	0.024	0.024	0.037	-0.118	2.391	-36.911	652.528
	0	0	0	0	0	0	0	0	0	0	0

$j := 3$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot c(x_i,t_j)\cdot\left(\sum_{l=1}^{i+1} g_l\cdot u_{i-l+1,j} + u_{i+1,j}\right) + \frac{k}{h^{\alpha}}\cdot d(x_i,t_j)\cdot\left(\sum_{l=1}^{10-i+1} g_l\cdot u_{i+l-1,j} + u_{i-1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	12.097	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
	0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	-10.085	151.694	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
	0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	2.732	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
	0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	4.64E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
u=	0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	0.013	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
	0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	0.031	0.06	0.406	73.785	-4.15E+03	1.50E+05
	0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	0.04	0.053	-0.055	4.613	118.147	-1.32E+04
	0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	0.039	0.046	-0.047	1.897	-19.231	684.795
	0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	0.024	0.037	-0.118	2.391	-36.911	652.528
	0	0	0	0	0	0	0	0	0	0	0

$j := 4$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}}\cdot c(x_i,t_j)\cdot\left(\sum_{l=1}^{i+1} g_l\cdot u_{i-l+1,j} + u_{i+1,j}\right) + \frac{k}{h^{\alpha}}\cdot d(x_i,t_j)\cdot\left(\sum_{l=1}^{10-i+1} g_l\cdot u_{i+l-1,j} + u_{i-1,j}\right) + k\cdot s(x_i,t_j) + u_{i,j}$$

	0	0	0	0	0	0	0	0	0	0	0
	0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	-154.744	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
	0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	151.694	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
	0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	-61.035	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
	0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	9.403	-287.626	6.70E+03	-1.43E+05	2.95E+06
u=	0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	0.169	27.189	-1.14E+03	3.30E+04	-8.41E+05
	0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	0.06	0.406	73.785	-4.15E+03	1.50E+05
	0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	0.053	-0.055	4.613	118.147	-1.32E+04
	0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	0.046	-0.047	1.897	-19.231	684.795
	0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	0.037	-0.118	2.391	-36.911	652.528
	0	0	0	0	0	0	0	0	0	0	0

$j := 5$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	5.05E-04	2.11E+03	-3.04E+04	4.58E+05	-7.18E+06
0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	9.28E-04	-2.30E+03	3.59E+04	-5.76E+05	9.55E+06
0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	1.23E-03	1.15E+03	-2.07E+04	3.69E+05	-6.69E+06
0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	1.42E-03	-287.626	6.70E+03	-1.43E+05	2.95E+06
0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	1.48E-03	27.189	-1.14E+03	3.30E+04	-8.41E+05
0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	1.43E-03	0.406	73.785	-4.15E+03	1.50E+05
0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	1.25E-03	-0.055	4.613	118.147	-1.32E+04
0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	9.54E-04	-0.047	1.897	-19.231	684.795
0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	5.37E-04	-0.118	2.391	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 6$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	5.05E-04	5.82E-04	-3.04E+04	4.58E+05	-7.18E+06
0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	9.28E-04	1.08E-03	3.59E+04	-5.76E+05	9.55E+06
0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	1.23E-03	1.43E-03	-2.07E+04	3.69E+05	-6.69E+06
0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	1.42E-03	1.65E-03	6.70E+03	-1.43E+05	2.95E+06
0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	1.48E-03	1.73E-03	-1.14E+03	3.30E+04	-8.41E+05
0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	1.43E-03	1.66E-03	73.785	-4.15E+03	1.50E+05
0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	1.25E-03	1.46E-03	4.613	118.147	-1.32E+04
0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	9.54E-04	1.11E-03	1.897	-19.231	684.795
0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	5.37E-04	6.26E-04	2.391	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 7$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	5.05E-04	5.82E-04	6.57E-04	4.58E+05	-7.18E+06
0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	9.28E-04	1.08E-03	1.22E-03	-5.76E+05	9.55E+06
0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	1.23E-03	1.43E-03	1.63E-03	3.69E+05	-6.69E+06
0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	1.42E-03	1.65E-03	1.88E-03	-1.43E+05	2.95E+06
0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	1.48E-03	1.73E-03	1.97E-03	3.30E+04	-8.41E+05
0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	1.43E-03	1.66E-03	1.90E-03	-4.15E+03	1.50E+05
0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	1.25E-03	1.46E-03	1.66E-03	118.147	-1.32E+04
0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	9.54E-04	1.11E-03	1.27E-03	-19.231	684.795
0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	5.37E-04	6.26E-04	7.15E-04	-36.911	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 8$

$i := 1 .. n-1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	5.05E-04	5.82E-04	6.57E-04	7.30E-04	-7.18E+06
0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	9.28E-04	1.08E-03	1.22E-03	1.36E-03	9.55E+06
0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	1.23E-03	1.43E-03	1.63E-03	1.83E-03	-6.69E+06
0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	1.42E-03	1.65E-03	1.88E-03	2.11E-03	2.95E+06
0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	1.48E-03	1.73E-03	1.97E-03	2.21E-03	-8.41E+05
0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	1.43E-03	1.66E-03	1.90E-03	2.13E-03	1.50E+05
0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	1.25E-03	1.46E-03	1.66E-03	1.87E-03	-1.32E+04
0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	9.54E-04	1.11E-03	1.27E-03	1.43E-03	684.795
0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	5.37E-04	6.26E-04	7.15E-04	8.03E-04	652.528
0	0	0	0	0	0	0	0	0	0	0

$j := 9$

$i := 1 .. n - 1$

$$u_{i,j+1} := \frac{k}{h^{\alpha}} \cdot c(x_i,t_j) \cdot \left(\sum_{l=1}^{i+1} g_l \cdot u_{i-l+1,j} + u_{i+1,j} \right) + \frac{k}{h^{\alpha}} \cdot d(x_i,t_j) \cdot \left(\sum_{l=1}^{10-i+1} g_l \cdot u_{i+l-1,j} + u_{i-1,j} \right) + k \cdot s(x_i,t_j) + u_{i,j}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	9.00E-05	1.78E-04	2.63E-04	3.46E-04	4.26E-04	5.05E-04	5.82E-04	6.57E-04	7.30E-04	8.02E-04
0	1.60E-04	3.18E-04	4.74E-04	6.27E-04	7.79E-04	9.28E-04	1.08E-03	1.22E-03	1.36E-03	1.50E-03
0	2.10E-04	4.18E-04	6.25E-04	8.29E-04	1.03E-03	1.23E-03	1.43E-03	1.63E-03	1.83E-03	2.02E-03
0	2.40E-04	4.79E-04	7.16E-04	9.51E-04	1.19E-03	1.42E-03	1.65E-03	1.88E-03	2.11E-03	2.33E-03
0	2.50E-04	4.99E-04	7.47E-04	9.93E-04	1.24E-03	1.48E-03	1.73E-03	1.97E-03	2.21E-03	2.45E-03
0	2.40E-04	4.79E-04	7.17E-04	9.55E-04	1.19E-03	1.43E-03	1.66E-03	1.90E-03	2.13E-03	2.36E-03
0	2.10E-04	4.19E-04	6.28E-04	8.36E-04	1.04E-03	1.25E-03	1.46E-03	1.66E-03	1.87E-03	2.07E-03
0	1.60E-04	3.20E-04	4.79E-04	6.38E-04	7.96E-04	9.54E-04	1.11E-03	1.27E-03	1.43E-03	1.58E-03
0	9.00E-05	1.80E-04	2.69E-04	3.59E-04	4.48E-04	5.37E-04	6.26E-04	7.15E-04	8.03E-04	8.92E-04
0	0	0	0	0	0	0	0	0	0	0

$j := 0 .. 10$

$i := 0 .. 10$

$ue_{i,j} := x_i \cdot (1 - x_i) \cdot t_j$

Program (3.1.5):

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\Gamma(0.5)} x^{\frac{1}{2}} \frac{\partial^{1.5} u}{\partial x^{1.5}} - 4x^2 + 2x^3 - 2.546x^2t^2 + 2.546xt^2, 0 \le x \le 2, 0 \le t \le 1$$

We use the explicit finite difference method to solve this example

$\alpha := 1.5$

$$g(k) := \left| \begin{array}{l} p \leftarrow \left[(-1)^k \frac{1}{k!} \right] \\ \text{for } i \in 0..(k-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 1..10$

$g_i := g(i)$

$$c(x,t) := \frac{1}{\Gamma(0.5)} \cdot x^{\frac{1}{2}}$$

$$d(x,t) := (2-x)^{\frac{1}{2}}$$

$ue(x,t) := x^2 \cdot (x-2) \cdot t^2$

$$s(x,t) := 2 \cdot x^2 \cdot (-2+x) - 2.5464790894703253723 \cdot x \cdot t^2 \cdot (x-1.) - \left(-4.5135166683820502956 \cdot x^2 + 9.0270333367641005912 \cdot x - 2.2567583341910251478\right) \cdot t^2$$

$n := 2$

$m := 2$

$L := 0$

$R := 2$

$T := 1$

$$h := \frac{(R-L)}{n}$$

$$k := \frac{T}{m}$$

$i :=$ "0..n"

$x(i) := L + i \cdot h$

$i :=$ "0..m"

$t(j) := 0 + j \cdot k$

$i := 0..2$

$u_{i,0} := 0$

$i := 0..2$

$u_{0,i} := 0$

$$u_{1,1} := \left[\frac{k^2}{\left(2h^{\alpha}\right)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(1))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{2} g_k\cdot u_{(2-k),0}\right] + \left[\frac{k^2}{\left(2h^{\alpha}\right)}\cdot\left[(2-x(1))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{2} g_k\cdot u_{(k),0}\right] + u_{(1,0)} + \frac{k^2}{2}\cdot s(x(1),t(0))$$

$$u_{1,2} := \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(1))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{2} g_k\cdot u_{(2-k),1}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[(2-x(1))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{2} g_k\cdot u_{(k),1}\right] + (k)^2\cdot s(x(1),t(1)) + 2\cdot u_{1,1} - u_{1,0}$$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.25 & -0.994 \\ 0 & 0 & 0 \end{pmatrix}$$

$i := 0..2$

$j := 0..2$

$ue_{i,j} := ue(x(i),t(j))$

$$ue = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.25 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$ue(x,t) := x^2\cdot(x-2)\cdot t^2$

$n := 3$

$m := 4$

$L := 0$

$R := 2$

$T := 1$

$$h := \frac{(R-L)}{n}$$

$$k := \frac{T}{m}$$

$i := "0..n"$
$x(i) := L + i \cdot h$
$i := "0..m"$
$t(j) := 0 + j \cdot k$
$i := 0..3$
$u_{i,0} := 0$
$i := 0..4$
$u_{0,i} := 0$
$i := 1..n-1$

$$u_{i,1} := \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),0}\right] + \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{3-i+1} g_k \cdot u_{(i+k-1),0}\right] + u_{(i,0)} + \frac{k^2}{2} \cdot s(x(i), t(0))$$

$j := 1..m-1$
$i := 1..n-1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{3-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.037 & -0.146 & -0.307 & -0.499 \\ 0 & -0.074 & -0.29 & -0.63 & -1.069 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$i := 0..3$
$j := 0..4$
$ue_{i,j} := ue(x(i), t(j))$

$$ue = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.037 & -0.148 & -0.333 & -0.593 \\ 0 & -0.074 & -0.296 & -0.667 & -1.185 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$ue(x,t) := x^2 \cdot (x-2) \cdot t^2$
$n := 10$
$m := 1000$
$L := 0$
$R := 2$
$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$$i := "0..n"$$

$$x(i) := L + i \cdot h$$

$$i := "0..m"$$

$$t(j) := 0 + j \cdot k$$

$$i := 0 .. 10$$

$$u_{i,0} := 0$$

$$i := 0 .. 10$$

$$u_{0,i} := 0$$

$$i := 1 .. 9$$

$$u_{i,1} := \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),0}\right] + \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),0}\right] + u_{(i,0)} + \frac{k^2}{2} \cdot s(x(i), t(0))$$

$$j := 1$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$j := 2$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$j := 3$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$j := 4$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$j := 5$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$$j := 6$$

$$i := 1 .. n - 1$$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 7$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

$j := 8$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

$j := 9$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-7.20E-08	-2.88E-07	-6.48E-07	-1.15E-06	-1.80E-06	-2.59E-06	-3.53E-06	-4.61E-06	-5.83E-06	-7.20E-06
0	-2.56E-07	-1.02E-06	-2.30E-06	-4.10E-06	-6.40E-06	-9.22E-06	-1.25E-05	-1.64E-05	-2.07E-05	-2.56E-05
0	-5.04E-07	-2.02E-06	-4.54E-06	-8.06E-06	-1.26E-05	-1.81E-05	-2.47E-05	-3.23E-05	-4.08E-05	-5.04E-05
0	-7.68E-07	-3.07E-06	-6.91E-06	-1.23E-05	-1.92E-05	-2.77E-05	-3.76E-05	-4.92E-05	-6.22E-05	-7.68E-05
0	-1.00E-06	-4.00E-06	-9.00E-06	-1.60E-05	-2.50E-05	-3.60E-05	-4.90E-05	-6.40E-05	-8.10E-05	-1.00E-04
0	-1.15E-06	-4.61E-06	-1.04E-05	-1.84E-05	-2.88E-05	-4.15E-05	-5.64E-05	-7.37E-05	-9.33E-05	-1.15E-04
0	-1.18E-06	-4.70E-06	-1.06E-05	-1.88E-05	-2.94E-05	-4.23E-05	-5.76E-05	-7.53E-05	-9.53E-05	-1.18E-04
0	-1.02E-06	-4.10E-06	-9.22E-06	-1.64E-05	-2.56E-05	-3.69E-05	-5.02E-05	-6.55E-05	-8.29E-05	-1.02E-04
0	-6.48E-07	-2.59E-06	-5.83E-06	-1.04E-05	-1.62E-05	-2.33E-05	-3.18E-05	-4.15E-05	-5.25E-05	-6.48E-05
0	0	0	0	0	0	0	0	0	0	0

$i := 0 .. 10$

$j := 0 .. 10$

$ue_{i,j} := ue(x(i),t(j))$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-7.20E-08	-2.88E-07	-6.48E-07	-1.15E-06	-1.80E-06	-2.59E-06	-3.53E-06	-4.61E-06	-5.83E-06	-7.20E-06
0	-2.56E-07	-1.02E-06	-2.30E-06	-4.10E-06	-6.40E-06	-9.22E-06	-1.25E-05	-1.64E-05	-2.07E-05	-2.56E-05
0	-5.04E-07	-2.02E-06	-4.54E-06	-8.06E-06	-1.26E-05	-1.81E-05	-2.47E-05	-3.23E-05	-4.08E-05	-5.04E-05
0	-7.68E-07	-3.07E-06	-6.91E-06	-1.23E-05	-1.92E-05	-2.77E-05	-3.76E-05	-4.92E-05	-6.22E-05	-7.68E-05
0	-1.00E-06	-4.00E-06	-9.00E-06	-1.60E-05	-2.50E-05	-3.60E-05	-4.90E-05	-6.40E-05	-8.10E-05	-1.00E-04
0	-1.15E-06	-4.61E-06	-1.04E-05	-1.84E-05	-2.88E-05	-4.15E-05	-5.65E-05	-7.37E-05	-9.33E-05	-1.15E-04
0	-1.18E-06	-4.70E-06	-1.06E-05	-1.88E-05	-2.94E-05	-4.23E-05	-5.76E-05	-7.53E-05	-9.53E-05	-1.18E-04
0	-1.02E-06	-4.10E-06	-9.22E-06	-1.64E-05	-2.56E-05	-3.69E-05	-5.02E-05	-6.55E-05	-8.29E-05	-1.02E-04
0	-6.48E-07	-2.59E-06	-5.83E-06	-1.04E-05	-1.62E-05	-2.33E-05	-3.18E-05	-4.15E-05	-5.25E-05	-6.48E-05
0	0	0	0	0	0	0	0	0	0	0

$n := 10$

$m := 10$

$L := 0$

$R := 2$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$i :=$ "0..n"

$x(i) := L + i \cdot h$

$i :=$ "0..m"

$t(j) := 0 + j \cdot k$

$i := 0 .. 10$

$u_{i,0} := 0$

$i := 0 .. 10$

$u_{0,i} := 0$

$i := 1 .. 9$

$$u_{i,1} := \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),0}\right] + \left[\frac{k^2}{\left(2h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),0}\right] + u_{(i,0)} + \frac{k^2}{2} \cdot s(x(i), t(0))$$

$j := 1$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 2$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 3$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 4$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 5$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[\frac{1}{\Gamma(0.5)} \cdot (x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{i+1} g_k \cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{\left(h^{\alpha}\right)} \cdot \left[(2 - x(i))^{\frac{1}{2}}\right] \cdot \sum_{k=0}^{10-i+1} g_k \cdot u_{(i+k-1),j}\right] + k^2 \cdot s(x(i), t(j)) + 2 \cdot u_{i,j} - u_{i,j-1}$$

$j := 6$

$i := 1 .. n - 1$

$$u_{i,j+1} := \left[\frac{k^2}{(h^\alpha)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{(h^\alpha)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

$j := 7$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{(h^\alpha)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{(h^\alpha)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

$j := 8$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{(h^\alpha)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{(h^\alpha)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

$j := 9$

$i := 1 .. n-1$

$$u_{i,j+1} := \left[\frac{k^2}{(h^\alpha)}\cdot\left[\frac{1}{\Gamma(0.5)}\cdot(x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{i+1} g_k\cdot u_{(i+1-k),j}\right] + \left[\frac{k^2}{(h^\alpha)}\cdot\left[(2-x(i))^{\frac{1}{2}}\right]\cdot\sum_{k=0}^{10-i+1} g_k\cdot u_{(i+k-1),j}\right] + k^2\cdot s(x(i),t(j)) + 2\cdot u_{i,j} - u_{i,j-1}$$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-7.20E-04	-2.85E-03	-6.25E-03	-0.011	-0.016	-0.02	-0.024	-0.024	-0.022	-0.014
0	-2.56E-03	-9.97E-03	-0.021	-0.036	-0.051	-0.065	-0.077	-0.084	-0.086	-0.082
0	-5.04E-03	-0.019	-0.041	-0.068	-0.095	-0.119	-0.138	-0.149	-0.152	-0.148
0	-7.68E-03	-0.03	-0.062	-0.101	-0.14	-0.174	-0.199	-0.213	-0.215	-0.209
0	-0.01	-0.038	-0.081	-0.13	-0.18	-0.223	-0.254	-0.27	-0.272	-0.264
0	-0.012	-0.044	-0.093	-0.15	-0.208	-0.257	-0.293	-0.312	-0.316	-0.308
0	-0.012	-0.045	-0.095	-0.155	-0.215	-0.268	-0.309	-0.333	-0.341	-0.338
0	-0.01	-0.04	-0.084	-0.138	-0.194	-0.246	-0.289	-0.32	-0.337	-0.344
0	-6.48E-03	-0.025	-0.055	-0.093	-0.135	-0.179	-0.221	-0.257	-0.288	-0.311
0	0	0	0	0	0	0	0	0	0	0

$ue(x,t) := x^2\cdot(x-2)\cdot t^2$

$i := 0 .. 10$

$j := 0 .. 10$

$ue_{i,j} := ue(x(i),t(j))$

u=

0	0	0	0	0	0	0	0	0	0	0
0	-7.20E-04	-2.88E-03	-6.48E-03	-0.012	-0.018	-0.026	-0.035	-0.046	-0.058	-0.072
0	-2.56E-03	-0.01	-0.023	-0.041	-0.064	-0.092	-0.125	-0.164	-0.207	-0.256
0	-5.04E-03	-0.02	-0.045	-0.081	-0.126	-0.181	-0.247	-0.323	-0.408	-0.504
0	-7.68E-03	-0.031	-0.069	-0.123	-0.192	-0.276	-0.376	-0.492	-0.622	-0.768
0	-0.01	-0.04	-0.09	-0.16	-0.25	-0.36	-0.49	-0.64	-0.81	-1
0	-0.012	-0.046	-0.104	-0.184	-0.288	-0.415	-0.564	-0.737	-0.933	-1.152
0	-0.012	-0.047	-0.106	-0.188	-0.294	-0.423	-0.576	-0.753	-0.953	-1.176
0	-0.01	-0.041	-0.092	-0.164	-0.256	-0.369	-0.502	-0.655	-0.829	-1.024
0	-6.48E-03	-0.026	-0.058	-0.104	-0.162	-0.233	-0.318	-0.415	-0.525	-0.648
0	0	0	0	0	0	0	0	0	0	0

Program (3.2.1):

$$\frac{\partial u(x,t)}{\partial t} = x^{\frac{4}{5}} \frac{\partial^{1.8} u(x,t)}{\partial x^{1/8}} + x(x-1) - \frac{t}{\Gamma(0.2)}(10x-1), 0 \le x \le 1, 0 \le t \le 1$$

$u(x,0) = 0$ for $0 \le x \le 1$.

$u(0,t) = 0$ for $0 \le t \le 1$.

$u(1,t) = 0$ for $0 \le t \le 1$.

We use the implicit finite difference method to solve this example.

$s(x,t) := x\cdot(x-1) - .2178248842116672615 7\cdot t\cdot(10\cdot x-1)$

$n := 2$

$m := 2$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R-L)}{n}$

$k := \frac{T}{m}$

$j := 0 .. m$

$t_j := 0 + j\cdot k$

$\alpha := 1.8$

$$g(z) := \left| \begin{array}{l} p \leftarrow \left[(-1)^z \frac{1}{z!} \right] \\ \text{for } i \in 0 .. (z-1) \\ \quad p \leftarrow p\cdot(\alpha - i) \end{array} \right.$$

$i := 0 .. n$

$u_{i,0} := 0$

$i := 0 .. m$

$u_{0,i} := 0$

$h := \frac{1}{2}$

$j := 0 .. 2$

$x_j := \frac{j}{2}$

$x = \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}$

$z := 0..\, n - 2$

$m_{z,z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{h}{(k)^{\alpha}} \cdot g(1)$

$m = (\, 2.8 \,)$

$i := 0..\, n - 2$

$b_i := s\left(x_{i+1}, t_1\right) \cdot k$

$b = (\, -0.343 \,)$

$m^{-1} = (\, 0.357 \,)$

$u := m^{-1} \cdot b$

$u = (\, -0.122 \,)$

$i := 0..\, n - 2$

$ue_i := x_{i+1}\left(x_{i+1} - 1\right) \cdot \frac{1}{2}$

$ue = (\, -0.125 \,)$

$z := 0..\, n - 2$

$m_{z,z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{h}{(k)^{\alpha}} \cdot g(1)$

$m = (\, 2.8 \,)$

$i := 0$

$d_i := s\left(x_{i+1}, t_2\right) \cdot k + u$

$m^{-1} = (\, 0.357 \,)$

$m^{-1} \cdot d_0 = (\, -0.244 \,)$

$\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) = -0.25$

$n := 3$

$m := 4$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R - L)}{n}$

$k := \frac{T}{m}$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(z) := \left| \begin{array}{l} p \leftarrow \left[(-1)^z \frac{1}{z!} \right] \\ \text{for } i \in 0 .. (z - 1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 0 .. n$

$u_{i, 0} := 0$

$i := 0 .. m$

$u_{0, i} := 0$

$h := \frac{1}{3}$

$j := 0 .. 3$

$x_j := \frac{j}{3}$

$$x = \begin{pmatrix} 0 \\ 0.333 \\ 0.667 \\ 1 \end{pmatrix}$$

$z := 0 .. n - 2$

$$m_{z, z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{\frac{1}{4}}{\left(\frac{1}{3}\right)^{\alpha}} \cdot g(1)$$

$$m = \begin{pmatrix} 2.35 & 0 \\ 0 & 3.35 \end{pmatrix}$$

$i := 0$

$$m_{i, i+1} := -\left(x_{i+1}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ 0 & 3.35 \end{pmatrix}$$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{1,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$i := 0 .. n - 2$

$b_i := s\left(x_{i+1}, t_1\right) \cdot k$

$$b = \begin{pmatrix} -0.087 \\ -0.133 \end{pmatrix}$$

$$m^{-1} = \begin{pmatrix} 0.467 & 0.105 \\ 0.131 & 0.328 \end{pmatrix}$$

$u := m^{-1} \cdot b$

$$u = \begin{pmatrix} -0.055 \\ -0.055 \end{pmatrix}$$

$i := 0 .. n - 2$

$$ue_i := x_{i+1}\left(x_{i+1} - 1\right) \cdot \frac{1}{4}$$

$$ue = \begin{pmatrix} -0.056 \\ -0.056 \end{pmatrix}$$

$z := 0 .. n - 2$

$$m_{z,z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{\frac{1}{4}}{\left(\frac{1}{3}\right)^{\alpha}} \cdot g(1)$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$i := 0$

$$m_{i,i+1} := -\left(x_{i+1}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$$

$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$m_{1,i} := -(x_{i+2})^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$

$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$

$i := 0 .. n - 2$

$b_i := s(x_{i+1}, t_2) \cdot k + -0.055$

$b = \begin{pmatrix} -0.174 \\ -0.265 \end{pmatrix}$

$m^{-1} = \begin{pmatrix} 0.467 & 0.105 \\ 0.131 & 0.328 \end{pmatrix}$

$u := m^{-1} \cdot b$

$u = \begin{pmatrix} -0.109 \\ -0.11 \end{pmatrix}$

$i := 0 .. n - 2$

$ue_i := x_{i+1}(x_{i+1} - 1) \cdot \frac{2}{4}$

$ue = \begin{pmatrix} -0.111 \\ -0.111 \end{pmatrix}$

$z := 0 .. n - 2$

$m_{z,z} := 1 - (x_{z+1})^{\frac{4}{5}} \cdot \frac{\frac{1}{4}}{\left(\frac{1}{3}\right)^{\alpha}} \cdot g(1)$

$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$

$i := 0$

$m_{i,i+1} := -(x_{i+1})^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{1,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$i := 0 .. n - 2$

$b_i := s\left(x_{i+1}, t_3\right) \cdot k + u_j$

$$b = \begin{pmatrix} -0.26 \\ -0.397 \end{pmatrix}$$

$$m^{-1} = \begin{pmatrix} 0.467 & 0.105 \\ 0.131 & 0.328 \end{pmatrix}$$

$u := m^{-1} \cdot b$

$$u = \begin{pmatrix} -0.163 \\ -0.164 \end{pmatrix}$$

$i := 0 .. n - 2$

$$ue_i := x_{i+1}\left(x_{i+1} - 1\right) \cdot \frac{3}{4}$$

$$ue = \begin{pmatrix} -0.167 \\ -0.167 \end{pmatrix}$$

$z := 0 .. n - 2$

$$m_{z,z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{\frac{1}{4}}{\left(\frac{1}{3}\right)^{\alpha}} \cdot g(1)$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$i := 0$

$$m_{i,i+1} := -\left(x_{i+1}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{1,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$$m = \begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}$$

$i := 0 .. n - 2$

$b_i := s\left(x_{i+1}, t_4\right) \cdot k + u_i$

$$b = \begin{pmatrix} -0.346 \\ -0.528 \end{pmatrix}$$

$$m^{-1} = \begin{pmatrix} 0.467 & 0.105 \\ 0.131 & 0.328 \end{pmatrix}$$

$u := m^{-1} \cdot b$

$$u = \begin{pmatrix} -0.217 \\ -0.219 \end{pmatrix}$$

$i := 0 .. n - 2$

$ue_i := x_{i+1}\left(x_{i+1} - 1\right)$

$$ue = \begin{pmatrix} -0.222 \\ -0.222 \end{pmatrix}$$

$n := 10$

$m := 10$

$L := 0$

$R := 1$

$T := 1$

$$h := \frac{(R - L)}{n}$$

$$k := \frac{T}{m}$$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$\alpha := 1.8$

$$g(z) := \left| \begin{array}{l} p \leftarrow \left[(-1)^{z} \frac{1}{z!} \right] \\ \text{for } i \in 0 .. (z-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 0 .. n$

$u_{i,0} := 0$

$i := 0 .. m$

$u_{0,i} := 0$

$h := \frac{1}{10}$

$j := 0 .. 10$

$x_j := \frac{j}{10}$

$z := 0 .. n - 2$

$$m_{z,z} := 1 - (x_{z+1})^{\frac{4}{5}} \cdot \frac{\frac{1}{10}}{\left(\frac{1}{10}\right)^{\alpha}} \cdot g(1)$$

$i := 0 .. n - 3$

$$m_{i,i+1} := -(x_{i+1})^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{1,i} := -(x_{i+2})^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i + 2)$$

$hh := 1$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{2,i} := -(x_{i+2})^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i + 2)$$

$hh := 2$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{3,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 3$
$i := 0 .. hh$
$j := 1 .. (hh-1)$

$$m_{4,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 4$
$i := 0 .. hh$
$j := 1 .. (hh-1)$

$$m_{5,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 5$
$i := 0 .. hh$
$j := 1 .. (hh-1)$

$$m_{6,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 6$
$i := 0 .. hh$
$j := 1 .. (hh-1)$

$$m_{7,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 7$
$i := 0 .. hh$
$j := 1 .. (hh-1)$

$$m_{8,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$i := 0 .. n-2$
$b_i := s\left(x_{i+1}, t_1\right) \cdot k$
$u := m^{-1} \cdot b$

$$u = \begin{pmatrix} -7.365\times10^{-3} \\ -0.012 \\ -0.012 \\ -0.011 \\ -0.01 \\ -9.181\times10^{-3} \\ -7.651\times10^{-3} \\ -5.759\times10^{-3} \\ -3.331\times10^{-3} \end{pmatrix}$$

$i := 0 .. n-2$

$ue_i := x_{i+1}(x_{i+1}-1)\cdot\frac{1}{10}$

$$ue = \begin{pmatrix} -9\times10^{-3} \\ -0.016 \\ -0.021 \\ -0.024 \\ -0.025 \\ -0.024 \\ -0.021 \\ -0.016 \\ -9\times10^{-3} \end{pmatrix}$$

$n := 20$

$m := 10$

$L := 0$

$R := 1$

$T := 1$

$h := \frac{(R-L)}{n}$

$k := \frac{T}{m}$

$j := 0 .. m$

$t_j := 0 + j \cdot k$

$\alpha := \frac{1}{2}$

$$g(z) := \left| \begin{array}{l} p \leftarrow \left[(-1)^z \frac{1}{z!} \right] \\ \text{for } i \in 0 .. (z-1) \\ \quad p \leftarrow p \cdot (\alpha - i) \end{array} \right.$$

$i := 0 .. n$

$u_{i,0} := 0$

$i := 0 .. m$

$u_{0,i} := 0$

$h := \frac{1}{20}$

$j := 0 .. 20$

$x_j := \frac{j}{20}$

$z := 0 .. n - 2$

$$m_{z,z} := 1 - \left(x_{z+1}\right)^{\frac{4}{5}} \cdot \frac{\frac{1}{10}}{\left(\frac{1}{20}\right)^{\alpha}} \cdot g(1)$$

$i := 0 .. n - 3$

$$m_{i,i+1} := -\left(x_{i+1}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}}$$

$hh := 0$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{1,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i + 2)$$

$hh := 1$

$i := 0 .. hh$

$j := 1 .. (hh - 1)$

$$m_{2,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 2
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{3,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 3
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{4,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 4
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{5,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 5
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{6,i} := -\left(x_{i+2}\right)^{\frac{1}{2}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 6
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{7,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 7
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{8,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 8
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{9,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 9
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{10,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 10
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{11,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 11
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{12,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 12
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{13,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

hh := 13
i := 0 .. hh
j := 1 .. (hh − 1)

$$m_{14,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 14$
$i := 0 .. hh$
$j := 1 .. (hh - 1)$

$$m_{15,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 15$
$i := 0 .. hh$
$j := 1 .. (hh - 1)$

$$m_{16,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 16$
$i := 0 .. hh$
$j := 1 .. (hh - 1)$

$$m_{17,i} := -\left(x_{i+2}\right)^{\frac{4}{5}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$hh := 17$
$i := 0 .. hh$
$j := 1 .. (hh - 1)$

$$m_{18,i} := -\left(x_{i+2}\right)^{\frac{1}{2}} \cdot \frac{k}{(h)^{\alpha}} \cdot g(i+2)$$

$i := 0 .. n - 2$
$b_i := s\left(x_{i+1}, t_1\right) \cdot k$
$u := m^{-1} \cdot b$

u=

-3.97E-03
-9.68E-03
-0.015
-0.02
-0.024
-0.028
-0.031
-0.034
-0.037
-0.038
-0.039
-0.04
-0.04
-0.039
-0.038
-0.036
-0.032
-0.027
-0.018

$i := 0 .. n - 2$

$$ue_i := x_{i+1}(x_{i+1} - 1)\cdot\frac{1}{10}$$

ue=

-4.75E-03
-9.00E-03
-0.013
-0.016
-0.019
-0.021
-0.023
-0.024
-0.025
-0.025
-0.025
-0.024
-0.023
-0.021
-0.019
-0.016
-0.013
-9.00E-03
-4.75E-03

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics and Computer Applications.

Signature:

Name: Dr. Ahlam Jameel khaleel

Address: Assist. Prof.

Date: / / 2007

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Name: Dr. Akram M. Al-Abood

Address: Assist. Prof.

Head of the Department of Mathematics

and Computer Applications

Date: / / 2007

Examining Committee Certification

We certify that we have read this thesis entitled "***Some Finite Difference Methods for Solving Fractional Differential Equations***" and as examining committee examined the student (***Laylan Sidqi Mohamed Ghareeb***) in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)
Signature:
Name: Dr. Jehad R. Kider
Assist. Prof.
Date: / / 2008

(Member)
Signature:
Name: Dr. Radhi A. Zboon
Assist. Prof.
Date: / / 2008

(Member)
Signature:
Name: Dr. Fadhel S. Fadhel
Lecturer
Date: / / 2008

(Member and Supervisor)
Signature:
Name: Dr. Ahlam J. Khalel
Assist. Prof.
Date: / / 2008

Approved for the Dean of the College of Science, Al-Nahrain University.

Signature
Name: Assist Prof. Dr. Laith Abdul Aziz Al-Ani
Dean of the Collage of Science, Al-Nahrain University
Data: / / 2008

1 Some Basic Concepts of Fractional Calculus

Introduction:

Fractional calculus is commonly called generalized differ-integration, which means an arbitrary order (real or complex) derivatives and integrals. The fractional differintegrations of the function of single variable and that of the functions of many variables can be unified without any trouble. That is, the classical integer order's differintegral is the special case of the fractional calculus, [Nishimoto K., 1989].

In this chapter, we give simple information about the fractional order ordinary derivatives of functions with one and more than one independent variables, and fractional order ordinary integrals of functions with one independent variables.

This chapter consists of three sections:

In section one, we give some definitions of the fractional order ordinary derivatives.

In section two, we give some definitions of the fractional order ordinary integrations.

In section three, we give some definitions of the fractional order partial derivatives.

1.1 Fractional Order Ordinary Derivatives:

As seen before, there are many references, in which the fractional order derivatives of functions of single variable (or simply fractional order ordinary derivatives) are discussed. Also, many definitions on the fractional order derivatives of functions of single variable are reported. Some of them are shown below.

(1) Riemann-Liouville fractional order ordinary derivatives, [Oldham K. and Spanir J., 1974]:

The fractional order derivative of order q of a function f is defined to be:

$$D^{q}_{a,x}f(x)=\frac{1}{\Gamma(n-q)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}\frac{f(y)}{(x-y)^{q+1-n}}dy,\ x\geq a \quad \text{.................................... (1.1.1)}$$

where q is a positive fractional number and n is a natural number, such that $n-1<q\leq n$.

Note that, this derivative is said to be the left-handed fractional order derivative of order q of a function f at a point x since it depends on all function values to the left of the point x, that is, this derivative is a weighted average of such function values.

On the other hand, the right-handed fractional order derivative of order q of a function f is defined to be:

$$D^{q}_{x,b}f(x)=\frac{(-1)^{n}}{\Gamma(n-q)}\frac{d^{n}}{dx^{n}}\int_{x}^{b}\frac{f(y)}{(y-x)^{q+1-n}}dy,\ x\leq b \quad \text{.................................... (1.1.2)}$$

where q is a positive fractional number and n is a natural number, such that $n-1<q\leq n$.

Note that, this derivative is said to be the right-handed fractional order derivative of order q of a function f at a point x since it depends on all function values to the right of the point x. In general, the left-handed and right-handed derivatives are not equal unless q is an even integer. Moreover, if q is a positive integer, then the above definitions give the standard integer derivatives, that is,

$$(D_{a,x}^{q}f)(x)=\frac{d^{q}f(x)}{dx^{q}}$$

and

$$(D_{x,b}^{q}f)(x)=(-1)^{q}\frac{d^{q}f(x)}{dx^{q}}=\frac{d^{q}f(x)}{d(-x)^{q}}.$$

To illustrate this definition, consider the following example:

Example (1.1.1):

Consider f(x)=x, where $x\in[0,1]$ then

$$D_{0,x}^{1.8}x=\frac{1}{\Gamma(2-1.8)}\frac{d^{2}}{dx^{2}}\int_{0}^{x}\frac{y}{(x-y)^{1.8+1-2}}dy=\frac{1}{\Gamma(0.2)x^{0.8}}.$$

$$D_{x,1}^{1.8}x=\frac{1}{\Gamma(2-1.8)}\frac{d^{2}}{dx^{2}}\int_{x}^{1}\frac{y}{(y-x)^{1.8+1-2}}dy=\frac{1}{\Gamma(0.2)}\left(\frac{-1.8}{(1-x)^{0.8}}-\frac{0.8x}{(1-x)^{1.8}}\right).$$

$$D_{0,x}^{0.5}x=\frac{1}{\Gamma(1-0.5)}\frac{d}{dx}\int_{0}^{x}\frac{y}{(x-y)^{0.5+1-1}}dy=\frac{2\sqrt{x}}{\Gamma(0.5)}.$$

$$D_{x,1}^{0.5}x=\frac{-1}{\Gamma(1-0.5)}\frac{d}{dx}\int_{x}^{1}\frac{y}{(y-x)^{0.5+1-1}}dy=\frac{-1}{\Gamma(0.5)}\left(\sqrt{1-x}-\frac{x}{\sqrt{1-x}}\right).$$

(2) Grünwald fractional order ordinary derivatives, [Caputo M., 1971]:

The fractional order derivative of order q of a function f is defined to be:

$$\frac{d^q}{dx^q} f(x) = \lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\frac{x}{N}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.3)$$

where q is a positive fractional number.

To illustrate this definition, consider the following examples:

<u>Examples (1.1.2):</u>

(1) Consider f(x)=x, then

$$\frac{d^{1.8}}{dx^{1.8}} x = \lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-1.8)} \sum_{j=0}^{N} \frac{\Gamma(j-1.8)}{\Gamma(j+1)} \left(x - j\frac{x}{N}\right)\right] = \frac{-0.8}{\Gamma(0.2)x^{0.8}}.$$

(2) Consider $f(x) = x^m$, where m is any real number then

$$\frac{d^{1.5}}{dx^{1.5}} x^m = \lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-1.5}}{\Gamma(-1.5)} \sum_{j=0}^{N} \frac{\Gamma(j-1.5)}{\Gamma(j+1)} \left(x - j\frac{x}{N}\right)^m\right] = \frac{-0.5x^m}{\Gamma(0.5)x^{1.5}}.$$

$$\frac{d^{1.8}}{dx^{1.8}} x^m = \lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-1.8)} \sum_{j=0}^{N} \frac{\Gamma(j-1.8)}{\Gamma(j+1)} \left(x - j\frac{x}{N}\right)^m\right] = \frac{-0.8x^m}{\Gamma(0.2)x^{1.8}}.$$

(3) Consider $f(x)=\sin x$, then

$$\frac{d^{0.5}}{dx^{0.5}}\sin x=\lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-0.5}}{\Gamma(-0.5)}\sum_{j=0}^{N}\frac{\Gamma(j-0.5)}{\Gamma(j+1)}\sin\left(x-j\frac{x}{N}\right)\right]=\frac{\sin x}{\Gamma(0.5)x^{0.5}}.$$

(4) Consider $f(x)=\cos x$, then

$$\frac{d^{0.5}}{dx^{0.5}}\cos x=\lim_{N\to\infty}\left[\frac{\left(\frac{x}{N}\right)^{-0.5}}{\Gamma(-0.5)}\sum_{j=0}^{N}\frac{\Gamma(j-0.5)}{\Gamma(j+1)}\cos\left(x-j\frac{x}{N}\right)\right]=\frac{\cos x}{\Gamma(0.5)x^{0.5}}.$$

(3) Caputo fractional order ordinary derivatives, [Caputo M., 1971]:

The left-handed and the right handed fractional order derivatives of order q of a function f are defined to be:

$${}_aD_x^q f(x)=\frac{1}{\Gamma(n-q)}\int_a^x\frac{1}{(x-y)^{q-n}}\frac{d^n f(y)}{dy^n}dy,\ \ x\ge a \text{ ....(1.1.4)}$$

and

$${}_xD_b^q f(x)=\frac{(-1)^q}{\Gamma(n-q)}\int_x^b\frac{1}{(y-x)^{q-n}}\frac{d^n f(y)}{dy^n}dy,\ \ x\le b \text{ .... (1.1.5)}$$

where q is a positive fractional number and n is a natural number, such that $n-1<q\le n$.

To illustrate this definition, consider the following example:

Example (1.1.3):

Consider f(x)=x, then

$$ {}_0D_x^{1.8}x = \frac{1}{\Gamma(2-1.8)}\int_0^x \frac{1}{(x-y)^{1.8-2}}\frac{d^2y}{dy^2}dy $$

$$ = \frac{1}{\Gamma(0.2)}\int_0^x \frac{2y}{(x-y)^{-0.2}}dy = \frac{1.667}{\Gamma(0.2)}x^{1.2}. $$

$$ {}_xD_1^{1.8}x^2 = \frac{1}{\Gamma(2-1.8)}\int_x^1 \frac{1}{(y-x)^{1.8-2}}\frac{d^2y^2}{dy^2}dy $$

$$ = \frac{1}{\Gamma(0.2)}\int_x^1 \frac{2}{(x-y)^{-0.2}}dy = \frac{1.667}{\Gamma(0.2)}(1-x)^{1.2}. $$

$$ {}_0D_x^{0.5}x = \frac{1}{\Gamma(1-0.5)}\int_0^x \frac{1}{(x-y)^{0.5-1}}\frac{dy}{dy}dy = \frac{2x^{1.5}}{3\Gamma(0.5)}. $$

$$ {}_xD_2^{0.5}x = \frac{-1}{\Gamma(1-0.5)}\int_x^2 \frac{1}{(y-x)^{0.5-1}}\frac{dy}{dy}dy = \frac{-2(2-x)^{1.5}}{3\Gamma(0.5)}. $$

1.2 Fractional Order Ordinary Integrals:

Like the fractional order ordinary derivatives, there are many references, in which the fractional integrals of functions of single variable (or simply fractional order ordinary integrals) are introduced. Some of them are shown below.

(1) Riemann-Liouville fractional order ordinary integral, [Nishimoto K., 1983]:

The fractional order ordinary integral of order q of a function f is defined to be:

$$(I^{q}_{a,x}f)(x)=\frac{1}{\Gamma(q)}\int_{a}^{x}\frac{f(y)}{(x-y)^{1-q}}dy,\ x\geq a \quad \text{...(1.2.1)}$$

where q is a positive fractional number.

(2) Weyl fractional order ordinary integral, [Nishimoto K., 1983]:

The left-handed fractional order ordinary integral of order q of a function f is defined to be:

$$(I^{q}_{-\infty,x}f)(x)=\frac{1}{\Gamma(q)}\int_{-\infty}^{x}\frac{f(y)}{(x-y)^{1-q}}dy,\ x>-\infty \quad \text{.......................................(1.2.2)}$$

and the right-handed fractional order ordinary integral of order q of a function f is defined to be:

$$(I^{q}_{x,\infty}f)(x)=\frac{1}{\Gamma(q)}\int_{x}^{\infty}\frac{f(y)}{(y-x)^{1-q}}dy,\ x<\infty. \quad \text{.. (1.2.3)}$$

where q is a positive fractional number and f is a periodic function and its mean value for one period is zero.

(3) Erdely fractional order ordinary integral, [Nishimoto K., 1983]:

The fractional order ordinary integral of order q of a function f is defined either by:

$$I^{q}_{x}f(x)=\frac{1}{\Gamma(q)}\int_{0}^{x}(x-z)^{q-1}f(z)dz,\ x>0,\ I^{0}_{x}f(x)=f(x) \quad \text{........................... (1.2.4)}$$

or by

$$K^{q}_{x}f(x)=\frac{1}{\Gamma(q)}\int_{x}^{\infty}(z-x)^{q-1}f(z)dz,\ x<0,\ K^{0}_{x}f(x)=f(x) \quad \text{..................... (1.2.5)}$$

(4) Okikiolu fractional order ordinary integral, [Nishimoto K., 1983]:

The fractional order ordinary integral of order q of a function f is defined either by:

$$H_q(f)(x) = \frac{1}{\phi(q)} \int_{-\infty}^{\infty} \frac{|y-x|^q}{(y-x)} f(y)dy, -\infty < x < \infty \quad \text{.................................} (1.2.6)$$

or by

$$K_q(f)(x) = \frac{1}{\phi(q)} \int_{-\infty}^{\infty} |y-x|^q f(y)dy, -\infty < x < \infty \quad \text{..................................} (1.2.7)$$

where $\phi(q) = 2\Gamma(q)\sin\left(\frac{\pi q}{2}\right)$.

(5) Riesz fractional order ordinary integral, [Nishimoto K., 1983]:

The fractional order ordinary integral of order q of a function f is defined to be:

$$f_q(x) = \int_{-\infty}^{\infty} |x-y|^{q-1} f(y)dy, -\infty < x < \infty,\ 0 < q < 1 \quad \text{.............................} (1.2.8)$$

1.3 Fractional Order Partial Derivatives:

In this section, we extend the previous definitions of fractional order ordinary derivatives of a function of single variable to include functions of many variables.

(1) Riemann-Liouville fractional order partial derivatives:

The left-handed fractional partial derivative of order q of a function f is defined to be:

$$\partial_{a_i,x_i}^{q_i} f(x_1,x_2,...,x_m) = \frac{1}{\Gamma(n_i - q_i)} \frac{\partial^n}{\partial x_i{}^n} \int_{a_i}^{x_i} \frac{f(x_1,x_2,...,x_{i-1},y,x_{i+1},...,x_m)}{(x_i - y)^{q_i+1-n}} dy, x_i \geq a_i$$

.... (1.3.1)

where $i \in \{1,2,3,...,n\}$

On the other hand, the right-handed fractional order partial derivative of order q of a function f is defined to be:

$$\partial_{x_i,b_i}^{q_i} f(x_1,x_2,...,x_m) = \frac{(-1)^{n_i}}{\Gamma(n_i - q_i)} \frac{\partial^{n_i}}{\partial x_i{}^{n_i}} \int_{x_i}^{b_i} \frac{f(x_1,x_2,...,x_{i-1},y,x_{i+1},...,x_m)}{(y - x_i)^{q_i+1-n_i}} dy, x_i \leq b_i$$

.... (1.3.2)

where q_i is a positive fractional number and n_i is a natural number, such that $n_i - 1 < q_i \leq n_i$ for some $i \in \{1,2,...,m\}$.

To illustrate this definition, consider the following example:

<u>Example (1.3.1):</u>

Consider $f\ (x_1, x_2) = x_1 x_2$, then

$$\partial_{0,x_1}^{0.5} f(x_1,x_2) = \frac{x_2}{\Gamma(1-0.5)} \frac{d}{dx_1} \int_0^{x_1} \frac{y}{(x_1 - y)^{0.5+1-1}} dy$$

$$\partial_{x_1}^{0.5}(x_1 x_2) = 2x_2 \frac{\sqrt{x_1}}{\Gamma(0.5)}$$

(2) Grünwald fractional order partial derivatives:

The fractional order partial derivative of order q of a function f is defined to be:

$$\frac{\partial^{q_i}}{\partial x_i^{q_i}} f(x_1, x_2, \ldots, x_m) = \lim_{N\to\infty}\left[\frac{\left(\frac{x_i}{N}\right)^{-q_i}}{\Gamma(-q_i)} \sum_{j=0}^{N} \frac{\Gamma(j-q_i)}{\Gamma(j+1)} f\left(x_1, x_2, \ldots, x_{i+1}, x_i - j\frac{x_i}{N}, x_{i-1}, \ldots, x_m\right)\right]$$

................................(1.3.3)

for some $i \in \{1,2,\ldots,m\}$.

To illustrate this definition, consider this example:

<u>Example (1.3.2):</u>

Consider $f(x_1, x_2) = x_1 x_2$, then:

$$\frac{\partial^{0.5}}{\partial x_1^{0.5}} f(x_1, x_2) = x_2 \lim_{N\to\infty}\left[\frac{\left(\frac{x_1}{N}\right)^{-0.5}}{\Gamma(-0.5)} \sum_{j=0}^{N} \frac{\Gamma(j-0.5)}{\Gamma(j+1)} \left(x_1 - j\frac{x_1}{N}\right)\right]$$

$$\frac{\partial^{0.5}}{\partial x_1^{0.5}} (x_1 x_2) = 2x_2 \frac{\sqrt{x_1}}{\Gamma(0.5)}$$

$$\frac{\partial^{0.5}}{\partial x_2^{0.5}} f(x_1, x_2) = x_1 \lim_{N\to\infty}\left[\frac{\left(\frac{x_2}{N}\right)^{-0.5}}{\Gamma(-0.5)} \sum_{j=0}^{N} \frac{\Gamma(j-0.5)}{\Gamma(j+1)} \left(x_2 - j\frac{x_2}{N}\right)\right]$$

$$\frac{\partial^{0.5}}{\partial x_2^{0.5}} (x_1 x_2) = 2x_1 \frac{\sqrt{x_2}}{\Gamma(0.5)}$$

(3) Caputo fractional order partial derivatives:

The left-handed and the right handed fractional order partial derivatives of order q of a function f are defined to be:

$$ {}_{a_i}\partial^{q_i}_{x_i} f(x_1, x_2, \ldots, x_m) = $$

$$ \frac{1}{\Gamma(n_i - q_i)} \int_{a_i}^{x_i} \frac{1}{(x_i - y)^{q_i - n_i}} \frac{\partial^{n_i} f(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m)}{\partial y^{n_i}} dy,\ x_i \geq a_i \quad \ldots(1.3.4) $$

and

$$ {}_{x_i}\partial^{q_i}_{b_i} f(x_1, x_2, \ldots, x_m) = $$

$$ \frac{(-1)^{n_i}}{\Gamma(n_i - q_i)} \int_{x_i}^{b_i} \frac{1}{(y - x_i)^{q_i - n_i}} \frac{\partial^{n_i} f(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m)}{\partial y^{n_i}} dy, x_i \leq b_i \quad ..(1.3.5) $$

where q_i is a positive fractional number and n_i is a natural number, such that $n_i - 1 < q_i \leq n_i$ for some $i \in \{1,2,\ldots,m\}$.

To illustrate this definition, consider this example

<u>Example (1.3.3):</u>

Consider $f(x_1, x_2) = x_1^2 x_2^2$, then

$$ {}_0\partial^{0.5}_{x_1}(x^2{}_1 x^2{}_2) = \frac{1}{\Gamma(2 - 0.5)} \int_0^{x_1} \frac{2}{(x_1 - y)^{0.5-2}} dy $$

$$ = 0.902703\, x_2^2 x_1^{5/2}. $$

$$ {}_0\partial^{0.5}_{x_2}(x_1^2 x_2^2) = \frac{1}{\Gamma(2 - 0.5)} \int_0^{x_2} \frac{2}{(x_2 - y)^{0.5-2}} dy $$

$$ = 0.902703\, x_1^2 x_2^{5/2} $$

3 Numerical Methods for Solving Fractional Order Partial Differential Equations

Introduction:

As seen before, the fractional order ordinary differential equations are generalizations of classical ordinary differential equations. Therefore the fractional order partial differential equations are generalizations of classical partial differential equations. Increasingly, these models are used in real life applications such as fluid flow, finance and others, [Meerschaert M. and Tadjeran C., 2005].

The aim of this chapter is to use some special types of the numerical methods, which is the finite difference methods to solve classes of the linear initial-boundary value fractional order partial differential equations with variable coefficients on a finite domain. Also, stability of these methods are discussed.

This chapter consists of two sections:

In section one; we use the explicit finite difference method to solve the initial-boundary value problem of the parabolic and hyperbolic for one-sided and two-sided fractional order partial differential equations.

In section two, we use the implicit finite difference method to solve the initial-boundary value problem of the two-sided fractional order partial differential equations.

3.1 The Explicit Finite Difference Method:

In this section, we use the explicit finite difference method to solve the initial-boundary value problem of the fractional order partial parabolic and hyperbolic differential equations. Also, the stability of this method is discussed.

3.1.1 The Explicit Finite Difference Method for Solving Heat Partial Differential Equations, [Smith G., 1978]:

Consider the parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad L \le x \le R,\ 0 \le t \le T \quad \text{.......................................(3.1.a)}$$

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{aligned} u(x,0) &= f(x) && \text{for } L \le x \le R \\ u(L,t) &= 0 && \text{for } 0 \le t \le T \\ u(R,t) &= 0 && \text{for } 0 \le t \le T \end{aligned}\right\} \quad \text{.......................................(3.1.b)}$$

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $x=x_i$, $t=t_j$, into equation (3.1.a) and replacing the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ with their approximations to get:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2} \quad \text{...(3.2)}$$

where t_j=jΔt, j =0,1,…, m and m is the number of subintervals of the interval [0,T], x_i= L+iΔx, i=0,1,…,n and n is the number of subintervals of the interval [L, R].

The resulting equation can be explicitly solved for $u_{i,j+1}$ to give:

$u_{i,j+1} = \beta u_{i-1,j} + (1 - 2\beta)u_{i,j} + \beta u_{i+1,j}$, i = 1,2,…,n; j = 0,1,…,m−1 ....(3.3)

where $\beta=\frac{\Delta t}{(\Delta x)^2}$ and $u_{i,,j}$ is the numerical solution of equation (3.1) at each (x_i, t_j), i=0,1,….,n and j=0,1,…,m, such that $u_{i,0}$=f(x_i), for i=0,1,…,n and $u_{0,j}$=$u_{n,j}$=0 for j=0,1,…,m. By evaluating the above equation at each i=1,2,…,n−1 and j=0,1,…,m−1 one can get the numerical solution of equations (3.1).

Next, we prove that the explicit finite difference method given by equation (3.3) is conditionally stable.

Theorem (3.1.1), [Smith G., 1978]:

The explicit finite difference method given by equation (3.3) is stable if $\beta= \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$.

Proof:

The explicit finite difference approximation method given by equation (3.3) together with the zero Dirichlet boundary conditions can be written as a linear system of equations of the form:

$U_{j+1}=AU_j$

where $U_{j+1}=\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix}$ and $Uj=\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix}$. The matrix A can be written as:

$$A=\begin{bmatrix} 1-2\beta & \beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1-2\beta & \beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1-2\beta & \beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta & 1-2\beta & \beta \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1-2\beta \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}+\beta\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix}$$

That is

$$A=I+\beta T_{n-1}$$

Where T_{n-1} is an $(n-1)\times(n-1)$ matrix whose eigenvalues λ_s and eigenvectors v_s are given by

$$\lambda_s=-4\sin^2\left(\frac{s\pi}{2n}\right),\ s=1,2,...,n$$

$$v_s=\left(\sin\left(\frac{s\pi}{n}\right),\sin\left(\frac{2s\pi}{n}\right),\ldots,\sin\left(\frac{(n-1)s\pi}{n}\right)\right),\ s=1,2,...,n$$

Hence the eigenvalues of the matrix A are $1-4\beta\sin^2\left(\frac{s\pi}{2n}\right)$, $s=1,2,...,n$.

Therefore the condition for stability of the explicit scheme is

$$\left|1-4\beta\sin^2\left(\frac{s\pi}{2n}\right)\right| \le 1,\quad s=1,2,...,n.$$

The only useful inequality is

$$-1 \le 1-4\beta\sin^2\left(\frac{s\pi}{2n}\right),\quad s=1,2,...,n.$$

giving

$$\beta \le \frac{1}{2}\sin^2\left(\frac{s\pi}{2n}\right) > \frac{1}{2},\quad s=1,2,...,n.$$

Proving that the scheme is stable for $\beta=\frac{\Delta t}{(\Delta x)^2}\le\frac{1}{2}$. ■

Now, we give an example to illustrate this method.

Example (3.1.1):

Consider the parabolic partial differential equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\quad 0\le x\le 1,\ 0\le t\le 1$$

Together with the initial and two Dirichlet boundary conditions:

$u(x, 0) = \sin(\pi x)$ for $0 \le x \le 1$

$u(0, t) = 0$ for $0 \le t \le 1$

$u(1, t) = 0$ for $0 \le t \le 1$

We use the explicit finite difference method given by equation (3.3) to solve this example. To do this, we take $\Delta x = 0.1$ and $\Delta t = 0.001$.In this case $\beta = 0.1 < 0.5$. Therefore equation (3.3) becomes

$u_{i,j+1} = 0.1(u_{i-1,j} + u_{i+1,j}) + 0.8u_{i,j}$, $i = 1,2,\ldots,9$; $j = 0,1,\ldots,999$

by evaluating the above equation at each $i=1,2,\ldots,9$ and $j=0,1,\ldots,999$ one can get the numerical solution $u_{i,j}$ of this example. Some of these results are given in the following table with its comparison with the exact solution $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Table (3.1) Represents the numerical and the exact solutions for $n = 10$ and $m = 1000$ of example (3.1.1) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.1	0.005	0.2942	0.2941
0.2	0.01	0.5327	0.5325
0.3	0.01	0.6645	0.6641
0.4	0.02	0.7812	0.7807
0.5	0.02	0.8214	0.8209

3.1.2 The Explicit Finite Difference Method for Solving Parabolic Partial Differential Equations:

Consider the classical parabolic partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = c(x, t)\frac{\partial^2 u(x,t)}{\partial x^2} + s(x, t), \quad L \le x \le R, 0 \le t \le T \text{ .... (3.4.a)}$$

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{array}{ll} u(x,0) = f(x) & \text{for } L \le x \le R \\ u(L,t) = 0 & \text{for } 0 \le t \le T \\ u(R,t) = 0 & \text{for } 0 \le t \le T \end{array}\right\} \text{ .... (3.4.b)}$$

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we divide the x-interval [L,R] into n-subintervals $[x_i,x_{i+1}]$ such that $x_i = L + i\Delta x$, x=0,1,…,n and $\Delta x=\frac{R-L}{n}$. Also, we divide the t-interval [0,T] into m-subintervals $[t_j,t_{j+1}]$ such that $t_j = j\Delta t$, j = 0,1,…,m and $\Delta t = \frac{T}{m}$.

Let $u_{i,j}$ denote the numerical solution of equation (3.4) at each (x_i,t_j), where i=0,1,…,n and j=0,1,…,m. From the boundary conditions given by equation (3.4.b), on can have

$$\left.\begin{array}{l} u_{i,0} = f(x_i), \ i = 0,1,...,n \\ u_{0,j} = u_{n,j} = 0, \ \ j = 0,1,...,m \end{array}\right\} \text{ .... (3.5)}$$

Therefore, we use the explicit finite difference method to find the numerical solution $u_{i,j}$ of equation (3.4) at each i = 1, 2,…, n−1 and j = 0,1,…,m−1. To do this, we substitute $x=x_i$, $t=t_j$ into equation (3.4.a) and replacing the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ with their approximations to get

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = c_{i,j} \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + s_{i,j} \quad \text{.......................................(3.6)}$$

where $c_{i,j}$=c(x_i,t_j) and $s_{i,j}$=s(x_i,t_j), for each i=0,1,…,n and j=0,1,…,m. Thus

$$u_{i,j+1} = \beta c_{i,j} u_{i+1,j} + (1 - 2\beta c_{i,j}) u_{i,j} + \beta c_{i,j} u_{i-1,j} + \Delta t\, s_{i,j} \quad \text{...............................(3.7)}$$

where $\beta=\frac{\Delta t}{(\Delta x)^2}$ and i=1,2,…,n−1; j=0,1,…,m−1.

By evaluating the above equation at each i=1,2,…,n−1 and j=0,1,…,m−1 and using equation (3.5) one can get the numerical solutions of equations (3.4).

Next, we prove that the explicit finite difference approximate method given by equation (3.7) is conditionally stable. The proof of this fact is based on the idea that appeared in [Meerschaert M. and Tadjeran C., 2006].

Theorem (3.1.2):

The explicit finite difference method given by equation (3.7) is stable if $\beta=\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2c_{max}}$, where c_{max} is the maximum value of c(x, t) over the region $L \leq x \leq R$, $0 \leq t \leq T$.

Proof:

The explicit finite difference approximate method given by equation (3.7) together with the zero Dirichlt boundary conditions can be written as a linear system of equations of the form:

$U_{j+1} = AU_j + \Delta t s_j$

Where $U_{j+1} = \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix}$, $U_j = \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix}$ and the matrix A can be written as:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta c_{1,j} & 1-2\beta c_{1,j} & \beta c_{1,j} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta c_{2,j} & 1-2\beta c_{2,j} & \beta c_{2,j} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \beta c_{3,j} & 1-2\beta c_{3,j} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \beta c_{n-2,j} & 1-2\beta c_{n-2,j} & \beta c_{n-2,j} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta c_{n-1,j} & 1-2\beta c_{n-1,j} & \beta c_{n-1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then by according to Greschgorin's theorem, the eigenvalues λ of the matrix A lie in the union of the n-circles centered at $a_{i,i}$ with radius $r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} a_{i,k}$.

Here, we have $a_{i,i} = 1 - 2\beta c_{i,j}$, and $r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} a_{i,k} = 2\beta c_{i,j}$

Therefore

$a_{i,i} + r_i = 1 - 2\beta\, c_{i,j} + 2\beta\, c_{i,j} = 1$

and

$a_{i,i} - r_i = 1 - 4\beta\, c_{i,j}$

But $c_{i,j} \le c_{max}$, thus $-4\beta\, c_{i,j} \ge -4\beta\, c_{max}$. Hence

$a_{i,i} - r_i \ge 1 - 4\beta c_{max}$.

Therefore, for the spectral radius of the matrix A to be at most one, it suffices to have

$1 - 4\beta\, c_{max} \ge -1$

Hence

$$\beta = \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2c_{max}}. \quad \blacksquare$$

Now, we give an example to illustrate this method.

Example (3.1.2):

Consider the parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = e^{\pi t}\frac{\partial^2 u}{\partial x^2} - \pi x e^{-\pi t}\sin(\pi x) + \pi^2 x \sin(\pi x) - 2\pi\cos(\pi x),\ 0\le x\le 1, 0\le t\le 1$$

Together with the initial and two Dirichlet boundary conditions:

$u(x, 0) = x\sin(\pi x)$ for $0 \le x \le 1$

$u(0, t) = 0$ for $0 \le t \le 1$

$u(1, t) = 0$ for $0 \le t \le 1$

We use the explicit finite difference method given by equation (3.7) to solve this example. To do this, first, we take $\Delta x = 0.5$ and $\Delta t = 0.5$.

Therefore $c_{max} = e^{\pi}$ and hence $\frac{1}{2c_{max}} = 0.022$. But β = 2. Therefore, equation (3.7) becomes

$$u_{i,j+1} = 2e^{\pi t_j} u_{i+1,j} + (1 - 4e^{\pi t_j})u_{i,j} + 2e^{\pi t_j} u_{i-1,j} +$$
$$0.5(-\pi x_i e^{-\pi t_j} \sin(\pi x_i) + \pi^2 x_i \sin(\pi x_i) - 2\pi\cos(\pi x_i))$$

By evaluating the above equation at i=1 and j=0,1 one can get the numerical solution $u_{i,j}$ that are tabulated down with the comparison with the exact solution $u(x,t) = xe^{-\pi t}\sin(\pi x)$.

Table (3.2) Represents the numerical and the exact solutions for n = m = 2 of example (3.1.2).

x_i	t_j	*Numerical solution $u_{i,j}$*	*Exact solution $u(x_i,t_j)$*
0.5	0.5	0.182	0.104
0.5	1	-1.016	0.0212

Second, we take Δx = Δt = 0.1. Therefore β = 10. But $\frac{1}{2c_{max}} = 0.022$. Hence equation (3.7) becomes

$$u_{i,j+1} = 10e^{\pi t_j} u_{i+1,j} + (1 - 20e^{\pi t_j})u_{i,j} + 10e^{\pi t_j} u_{i-1,j} +$$
$$0.1(-\pi x_i e^{-\pi t_j} \sin(\pi x_i) + \pi^2 x_i \sin(\pi x_i) - 2\pi\cos(\pi x_i))$$

By evaluating the above equation at i=1,2,…,9 and j=0,1,…,9 one can get the numerical solution $u_{i,j}$ that are tabulated down with the comparison with the exact solution.

Table (3.3) Represents the numerical and the exact solutions for $n = m = 10$ of example (3.1.2) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.5	0.5	0.103	0.104
0.5	1	2.043×10^{4}	0.022
0.6	0.2	0.309	0.304
0.8	0.3	-2.465	0.183
0.9	0.4	-290.503	0.079
0.1	0.6	4.169×10^{6}	4.692×10^{-3}
0.5	0.7	0.032	0.055
0.6	0.8	1.207×10^{11}	0.046
0.4	0.9	6.169×10^{13}	0.023
0.9	1	-6.012×10^{16}	0.012

Third, we take $\Delta x = 0.1$ and $\Delta t = 0.0001$.

Therefore $\beta = 0.01 < \dfrac{1}{2c_{max}} = 0.022$. Hence equation (3.7) becomes

$$u_{i,j+1} = 0.01\,e^{\pi t_j}\,u_{i+1,j} + (1 - 0.02\,e^{\pi t_j})u_{i,j} + 0.01\,e^{\pi t_j}\,u_{i-1,j} +$$
$$0.0001(-\pi x_i e^{-\pi t_j}\sin(\pi x_i) + \pi^2 x_i \sin(\pi x_i) - 2\pi\cos(\pi x_i))$$

By evaluating the above equation at i=1,2,…,9 and j=0,1,…,9999 one can get the numerical solution $u_{i,j}$ that are tabulated in the appendix (see program (3.1.2)). Some of these results are tabulated down with the comparison with the exact solution.

Table (3.4) Represents the numerical and the exact solutions for $n = 10$ and $m = 10000$ of example (3.1.2) at specific points.

x_i	t_j	**Numerical solution $u_{i,j}$**	**Exact solution $u(x_i,t_j)$**
0.1	1×10^{-4}	0.031	0.031
0.2	2×10^{-4}	0.117	0.117
0.3	3×10^{-4}	0.242	0.242
0.4	4×10^{-4}	0.380	0.380
0.5	5×10^{-4}	0.499	0.499
0.6	6×10^{-4}	0.570	0.570
0.7	7×10^{-4}	0.565	0.565
0.8	8×10^{-4}	0.469	0.469
0.9	9×10^{-4}	0.277	0.277

From the above tables, one can conclude that the results given in table (3.4) are more accurate than the previous results given in the other tables since the condition of the stability is satisfied in this case.

3.1.3 The Explicit Finite Difference Method for Solving Fractional Parabolic Partial Differential Equations, [Meerschaert M. and Tadjeran C., 2006]:

Consider the fractional order partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = c(x, t)\frac{\partial^q u(x,t)}{\partial x^q} + s(x, t),\ L \leq x \leq R,\ 0 \leq t \leq T \quad \text{................(3.8.a)}$$

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{array}{ll} u(x,0) = f(x) & \text{for } L \leq x \leq R \\ u(L,t) = 0 & \text{for } 0 \leq t \leq T \\ u(R,t) = 0 & \text{for } 0 \leq t \leq T \end{array}\right\} \quad \text{...(3.8.b)}$$

where $\frac{\partial^q u(x,t)}{\partial x^q}$ denote the left-handed partial fractional derivative of order q of the function u with respect to x and $1 < q \leq 2$.

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $t = t_j$, in equation (3.8.a) and replacing the partial derivative $\frac{\partial u}{\partial t}$ with its approximation to get:

$$\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} = c(x, t_j)\frac{\partial^q u(x,t_j)}{\partial x^q} + s(x, t_j) \quad \text{................................(3.9)}$$

where $t_j = j\Delta t$, j=0,1,…, m and m is the number of subintervals of the interval [0, T].

Next, we recall the left-handed shifted Grünwald estimate to the left-handed derivative:

$$\frac{d^q f(x)}{dx^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{n} g_k f(x - (k-1)\Delta x)$$

where n is the number of subintervals of the interval [L, R] and q is the fractional number. Therefore

$$\frac{\partial^q u(x_i, t_j)}{\partial x_i^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u(x_i - (k-1)\Delta x, t_j)$$

$$= \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} \qquad (3.10)$$

where $g_0 = 1$ and $g_k = (-1)^k \frac{q(q-1)...(q-k+1)}{k!}$, $k = 1,2,\ldots$

By substituting equation (3.10) in equation (3.9) one can have:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{c_{i,j}}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + s_{i,j},\ i=1,2,\ldots, n-1;\ j=0,1,\ldots,m-1$$

The resulting equation can be explicitly solved for $u_{i,j+1}$ to give:

$$u_{i,j+1} = \beta c_{i,j} g_0 u_{i+1,j} + (1-\beta c_{i,j} g_1) u_{i,j} + \beta c_{i,j} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + \Delta t s_{i,j} \qquad (3.11)$$

where $i = 1,2,\ldots,n-1$; $j = 0,1,\ldots,m-1$, $\beta = \frac{\Delta t}{(\Delta x)^q}$, $c_{i,j}=c(x_i,t_j)$, $s_{i,j}=s(x_i, t_j)$ and $u_{i,j}$ is the numerical solutions of equations (3.8) at each (x_i,t_j), $i = 0, 1, \ldots, n$; $j = 0, 1, \ldots, m$ such that $u_{i,0} = f(x_i)$ for $i = 0, 1, \ldots, n$ and $u_{0,j}=u_{n,j}=0$ for $j = 0,1,\ldots,m$.

Then by evaluating the above equation at each $i = 1, 2, \ldots, n-1$ and $j = 0,1,\ldots,m-1$ one can get the numerical solution of equations (3.8).

Next, we prove that the explicit finite difference approximation method given by equation (3.11) is conditionally stable.

Theorem (3.1.3), [Meerschaert M. and Tadjeran C., 2006]:

The explicit finite difference method given by equation (3.11) is stable if $\beta = \dfrac{\Delta t}{(\Delta x)^q} \leq \dfrac{1}{qc_{max}}$, where c_{max} is the maximum value of c(x, t) over the region $L \leq x \leq R$, $0 \leq t \leq T$.

Proof:

The explicit finite difference method given by equation (3.11) together with the zero Dirichlet boundary conditions can be written as a linear system of equations of the form:

$U_{j+1} = AU_j + \Delta t s_j$

where $U_{j+1} = \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix}$, $U_j = \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix}$ and the matrix A can be written as:

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ g_2\beta c_{1,j} & 1+g_1\beta c_{1,j} & \beta c_{1,j} & 0 & \cdots & \cdots & 0 & 0 \\ g_3\beta c_{2,j} & g_2\beta c_{2,j} & 1+g_1\beta c_{2,j} & \beta c_{2,j} & \cdots & \cdots & 0 & 0 \\ g_4\beta c_{3,j} & g_3\beta c_{3,j} & g_2\beta c_{3,j} & 1+g_1\beta c_{3,j} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ g_n\beta c_{n-1,j} & g_{n-1}\beta c_{n-1,j} & g_{n-2}\beta c_{n-1,j} & g_{n-3}\beta c_{n-1,j} & \cdots & g_2\beta c_{n-1,j} & 1+g_1\beta c_{n-1,j} & \beta c_{n+1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that $g_1 = -q$ and for $1< q \le 2$ and $k \ne 1$, we have $g_k \ge 0$.

Then, by according to Greschgorin's theorem, the eigencvalues of the matrix A lie in the union of the n-circles centered at $a_{i,i}$ with radius $r_i = \sum_{\substack{k=0\\k\ne i}}^{n} a_{i,k}$.

Hence, we have

$a_{i,i} = 1 + g_1\beta c_{i,j} = 1 - q\beta c_{i,j}$

and

$$r_i = \sum_{\substack{k=0\\k\ne i}}^{n} a_{i,k} = \sum_{\substack{k=0\\k\ne i}}^{i+1} a_{i,k} = c_{i,j}\beta \sum_{\substack{k=0\\k\ne i}}^{i+1} g_k$$

But $\sum_{\substack{k=0\\k\ne i}}^{i+1} g_k \le -g_1 =-(-q) = q$

Therefore, $r_i \le q\beta c_{i,j}$ and hence

$a_{i,i} + r_i \le 1 - q\beta c_{i,j} + q\beta c_{i,j} = 1.$

Also, $a_{i,i}-r_i \ge 1-2q\beta c_{i,j}$, but, $c_{i,j} \le c_{max}$, thus $a_{i,i} - r_i \ge 1 - 2q\beta c_{max}$.

Therefore, for the spectral radius of the matrix A to be at most one, it suffices to have

$1 - 2\beta c_{max} \geq -1$

Hence

$$\beta = \frac{\Delta t}{(\Delta x)^q} \leq \frac{1}{q c_{max}}. \quad \blacksquare$$

To illustrate this method, consider the following example:

Example (3.1.3):

Consider the fractional parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = x^{\frac{4}{5}} \frac{\partial^{1.8} u}{\partial x^{1/8}} + x(x-1) - \frac{t}{\Gamma(0.2)}(10x-1),\ 0 \leq x \leq 1,\ 0 \leq t \leq 1$$

Together with the initial and zero Dirichlet boundary conditions:

u(x,0)=0 for $0 \leq x \leq 1$.

u(0,t)=0 for $0 \leq t \leq 1$.

u(1,t)=0 for $0 \leq t \leq 1$.

This example is constructed such that the exact solution of it is $u(x,t) = x(x-1)t$.

Here, we use the explicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = \frac{i}{2}$, i = 0, 1, 2 and the t-interval into 2 subintervals such that $t_j = \frac{j}{2}$, i = 0, 1, 2. In this case, β = 1.741, $c_{max} = 1$ and $\frac{1}{qc_{max}} = \frac{1}{1.8} = 0.556$. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i = 0, 1, 2.

$u(0, t_j) = 0$ for j = 0, 1, 2.

$u(1, t_j) = 0$ for j = 0, 1, 2

Moreover, equation (3.11) becomes

$$u_{i,j+1} = 2^{0.8} x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j} + \frac{1}{2}\left(x_i(x_i - 1) - \frac{t_j}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

where i=1 and j=0, 1.

By evaluating the above equation at each i=1 and j=0, 1 one can get the values that are tabulated down with the comparison with the exact solution.

Table (3.5) Represents the numerical and the exact solutions for n=m=2 of example (3.1.3).

x_i	t_j	***Numerical solution*** $u_{i,j}$	***Exact solution*** $u(x_i,t_j)$
0.5	0.5	–0.125	–0.125
0.5	1	–0.243	–0.25

Second, we divide the x-interval into 3 subintervals such that $x_i = \frac{i}{3}, \ i = 0,1,2,3$ and the t-interval into 4 subintervals such that $t_j = \frac{j}{4}, \ j = 0,1,2,3, 4$. In this case, β = 1.806. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for $i = 0, 1, 2, 3.$

$u(0, t_j) = 0$ for j = 0, 1, 2, 3, 4.

$u(1, t_j) = 0$ for j=0, 1, 2, 3,4..

Moreover, equation (3.11) becomes

$$u_{i,j+1} = 1.806 x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j} + \frac{1}{4}\left(x_i(x_i - 1) - \frac{t_j}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

where i=1,2 and j=0, 1, 2, 3.

By evaluating the above equation at each i=1,2 and j=0, 1, 2, 3 one can get the values that are tabulated down with the comparison with the exact solution.

Table (3.6) Represents the numerical and the exact solutions for n=3 and m=4 of example (3.1.3).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.333	0.25	–0.056	–0.056
0.667	0.25	–0.056	–0.056
0.333	0.5	–0.11	–0.111
0.667	0.5	–0.11	–0.111
0.333	0.75	–0.163	–0.167
0.667	0.75	–0.164	–0.167
0.333	1	–0.217	–0.222
0.667	1	–0.218	–0.222

Third, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{10}$, $i = 0, 1, \ldots, 10$ and the t-interval into 10 subintervals such that $t_j = \frac{j}{10}$, $j = 0, 1, \ldots, 10$. In this case, β=6.31. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for $i = 0, 1, \ldots, 10$.

$u(0, t_j) = 0$ for j=0, 1,…,10.

$u(1, t_j) = 0$ for j=0, 1,…,10.

Moreover, equation (3.11) becomes

$$u_{i,j+1} = 10^{0.8} x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j} + \frac{1}{10}\left(x_i(x_i - 1) - \frac{t_j}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

where i=1,2,…,9 and j=0, 1,…,9.

By evaluating the above equation at each i=1,2,…,9 and j=0, 1,…,9 one can get the values that are tabulated in the appendix (see program (3.1.3)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.7) Represents the numerical and the exact solutions for n=m=10 of example (3.1.3) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.5	0.5	–0.124	–0.125
0.5	1	2.104×10^{3}	–0.250
0.4	0.2	–0.048	–0.048
0.1	0.7	–0.064	–0.063
0.2	0.9	0.096	–0.144
0.3	1	42.861	–0.210
0.6	0.7	0.863	–0.168
0.7	0.3	–0.063	–0.063
0.8	0.8	–95.228	–0.128
0.9	1	1.612×10^{4}	–0.090

Fourth, we divide the x-interval into 10 subintervals such that 1×10^{-3} $i=0,1,\ldots,10$ and the t-interval into 1000 subintervals such that $t_j=\dfrac{j}{1000}$, $j=0,1,\ldots,1000$. In this case, β=0.063<$\dfrac{1}{qc_{max}}=\dfrac{1}{1.8}=0.556$. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for $i = 0, 1, \ldots, 10$.

$u(0, t_j) = 0$ for j=0, 1,…,1000.

$u(1, t_j) = 0$ for j=0, 1,…,1000.

Moreover, equation (3.11) becomes

$$u_{i,j+1} = 0.063 x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j} + \frac{1}{1000}\left(x_i(x_i - 1) - \frac{t_j}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

where i=1,2,…,9 and j=0, 1,…,1000.

By evaluating the above equation at each i=1,2,…,9 and j=0, 1,…,1000 one can get the values that are tabulated in the appendix (see program (3.1.3)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.8) Represents the numerical and the exact solutions for n=10 and m=1000 of example (3.1.3) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.1	1×10^{-3}	-9×10^{-5}	-9×10^{-5}
0.2	2×10^{-3}	-3.2×10^{-4}	-3.2×10^{-4}
0.3	3×10^{-3}	-6.3×10^{-4}	-6.3×10^{-4}
0.4	4×10^{-3}	-9.6×10^{-4}	-9.6×10^{-4}
0.5	5×10^{-3}	-1.25×10^{-3}	-1.25×10^{-3}
0.6	6×10^{-3}	-1.440×10^{-3}	-1.440×10^{-3}
0.7	7×10^{-3}	-1.470×10^{-3}	-1.470×10^{-3}
0.8	8×10^{-3}	-1.280×10^{-3}	-1.280×10^{-3}
0.9	9×10^{-3}	-8.098×10^{-4}	-8.100×10^{-4}

From the above tables, one can conclude that the results given in table (3.8) are more accurate than the previous results given in the other tables since the condition of the stability is satisfied in this case.

Remark (3.1.1):

Like the previous steps, the explicit finite difference method can be also used to solve the initial-boundary value problems of the right-handed fractional parabolic partial differential equations.

3.1.4 The Explicit Finite Difference Method for Solving the Two-Sided Fractional Parabolic Partial Differential Equations:

Consider the two-sided fractional order partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^q u(x,t)}{\partial_+ x^q} + d(x,t)\frac{\partial^q u(x,t)}{\partial_- x^q} + s(x,t) \quad \text{..................(3.12.a)}$$

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{array}{ll} u(x,0) = f(x) & \text{for } L \le x \le R \\ u(L,t) = 0 & \text{for } 0 \le t \le T \\ u(R,t) = 0 & \text{for } 0 \le t \le T \end{array}\right\} \quad \text{..(3.12.b)}$$

where L ≤ x ≤ R, 0 ≤ t ≤ T, $\frac{\partial^q u(x,t)}{\partial_+ x^q}$ and $\frac{\partial^q u(x,t)}{\partial_- x^q}$ denote the left-handed and the right-handed partial fractional derivatives of order q of the function u with respect to x and $1<q\le 2$.

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $t = t_j$, in equation (3.12) and replacing the partial derivative $\frac{\partial u}{\partial t}$ with its approximation, to get:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = c_{i,j}\frac{\partial^q u(x,t_j)}{\partial_+ x^q} + d_{i,j}\frac{\partial^q u(x,t_j)}{\partial_- x^q} + s(x, t_j) \quad \text{.......................(3.13)}$$

where $t_j=j\Delta t$, $j=0,1,\ldots, m$ and m is the number of subintervals of the interval [0, T].

Next, we recall the left-handed shifted and the right-handed shifted Grünwald estimate to the left-handed and right-handed derivatives:

$$\frac{d^q f(x)}{d_+ x^q} = \frac{1}{(\Delta x)^q}\sum_{k=0}^{n} g_k f(x - (k-1)\Delta x)$$

$$\frac{d^q f(x)}{d_- x^q} = \frac{1}{(\Delta x)^q}\sum_{k=0}^{n} g_k f(x + (k-1)\Delta x)$$

where n is the number of subintervals of the interval [L, R] and q is the fractional number. Therefore

$$\frac{\partial^q u(x_i,t_j)}{\partial_+ x^q} = \frac{1}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u(x_i - (k-1)\Delta x, t_j)$$

$$= \frac{1}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,j} \quad \text{...(3.14.a)}$$

and

$$\frac{\partial^q u(x_i,t_j)}{\partial_- x^q} = \frac{1}{(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k u(x_i + (k-1)\,\Delta x,\, t_j)$$

$$= \frac{1}{(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j} \quad \text{...(3.14.b)}$$

where $g_0 = 1$ and $g_k = (-1)^k \dfrac{q(q-1)...(q-k+1)}{k!}$, $k = 1,2,\ldots$

Then by substituting equations (3.14) into equation (3.13) one can have

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{h^q}\left[c_{i,j}\sum_{k=0}^{i+1} g_k u_{i-k+1,j} + d_{i,j}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j} \right] + s_{i,j}$$

where $i=1,2,\ldots,n-1$; $j=0,1,\ldots,m-1$

The resulting equation can be explicitly solved for $u_{i,j+1}$ to give

$$u_{i,j+1} = \beta\left[c_{i,j}\sum_{k=0}^{i+1} g_k u_{i-k+1,j} + d_{i,j}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j} \right] + \Delta t s_{i,j} + u_{i,j} \quad \text{..............(3.15)}$$

where $i=1,2,\ldots,n-1$; $j=0,1,\ldots,m-1$, $\beta=\dfrac{\Delta t}{(\Delta x)^q}$, $c_{i,j}=c(x_i,t_j)$, $d_{i,j}=d(x_i,t_j)$, $s_{i,j}=s(x_i, t_j)$ and $u_{i,j}$ is the numerical solution of equation (3.12) at each (x_i,t_j), $i = 0,1,\ldots,n$; $j = 0,1,\ldots,m$; such that $u_{i,0}= f(x_i)$, for $i = 0,1,\ldots,n$ and $u_{0,j} = u_{n,j} = 0$ for $j = 0,1,\ldots,m$.

Then by evaluating the above equation at each $i = 1,2,\ldots,n-1$ and $j = 0,1,\ldots,m-1$ one can get the numerical solution of equations (3.12).

Next, we prove that the explicit finite difference method given by equation (3.15) is conditionally stable.

Theorem (3.1.4), [Meerschaert M. and Tadjeran C., 2006]:

The explicit finite difference method given by equation (3.15) is stable if $\beta = \dfrac{\Delta t}{(\Delta x)^q} \le \dfrac{1}{q(c_{max} + d_{max})}$, where c_{max} is the maximum value of c(x, t) and d_{max} is the maximum value of d(x, t) over the region $L \le x \le R$, $0 \le t \le T$.

Proof:

The explicit finite difference method given by equation (3.15) together with zero Dirichlet boundary condition can be written as a linear system of equations of the form:

$$U_{j+1} = AU_j + \Delta t s_j$$

Where $U_{j+1} = \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix}$, $U_j = \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix}$ and the matrix A can be written as:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ g_2\beta(c_{1,j}+d_{1,j}) & 1+g_1\beta(c_{1,j}+d_{1,j}) & g_2\beta(c_{1,j}+d_{1,j}) & 0 & \cdots & 0 \\ g_3\beta(c_{2,j}+d_{2,j}) & g_2\beta(c_{2,j}+d_{2,j}) & 1+g_1\beta(c_{2,j}+d_{2,j}) & g_2\beta(c_{2,j}+d_{2,j}) & \cdots & 0 \\ g_4\beta(c_{3,j}+d_{3,j}) & g_3\beta(c_{3,j}+d_{3,j}) & g_2\beta(c_{3,j}+d_{3,j}) & 1+g_1\beta(c_{3,j}+d_{3,j}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n\beta(c_{n-1,j}+d_{n-1,j}) & g_{n-1}\beta(c_{n-1,j}+d_{n-1,j}) & g_{n-2}\beta(c_{n-1,j}+d_{n-1,j}) & g_{n-3}\beta(c_{n-1,j}+d_{n-1,j}) & \cdots & \beta(c_{n+1,j}+d_{n+1,j}) \end{bmatrix}$$

Note that $g_1 = -q$ and for $1 < q \le 2$ and $i \ne 1$, we have $g_1 \ge 0$. Then according to Greschgorin's theorem, the eigenvalues of the matrix A lie in the union of the n-circles centered at $a_{i,j}$ with radius $r_i = \sum_{\substack{k=0 \\ k \ne i}}^{n} a_{i,k}$.

Here, we have:

$a_{i,i} = 1 + g_1\beta(c_{i,j} + d_{i,j}) = 1 - q\beta(c_{i,j} + d_{i,j})$

and

$$r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} a_{i,k} = \sum_{\substack{k=0 \\ k\neq i}}^{i+1} a_{i,k} = (c_{i,j} + d_{i,j})\beta \sum_{\substack{k=0 \\ k\neq i}}^{i+1} g_k$$

but

$$\sum_{\substack{k=0 \\ k\neq i}}^{i+1} g_k \leq -g_1 = -(-q) = q$$

Therefore $r_i \leq q\beta(c_{i,j} + d_{i,j})$ and hence

$a_{i,i} + r_i \leq 1 - q\beta(c_{i,j} + d_{i,j}) + q\beta(c_{i,j} + d_{i,j}) = 1$

Also, $a_{i,i} - r_i \geq 1 - q\beta(c_{i,j} + d_{i,j}) - q\beta(c_{i,j} + d_{i,j}) = 1 - 2q\beta(c_{i,j} + d_{i,j})$

But, $c_{i,j} \leq c_{max}$, $d_{i,j} \leq d_{max}$, thus $-2q\beta(c_{i,j} + d_{i,j}) \geq -2q\beta(c_{max} + d_{max})$

Hence

$a_{i,i} - r_i \geq 1 - 2q\beta(c_{max} + d_{max})$

Therefore, for the spectral radius of the matrix A be at most one, it suffices to have:

$1 - 2\beta(c_{max} + d_{max}) \geq -1$

Hence

$$\beta = \frac{\Delta t}{(\Delta x)^q} \leq \frac{1}{q(c_{max} + d_{max})}. \quad \blacksquare$$

To illustrate this method, consider the following example:

Example (3.1.4):

Consider the fractional parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = (1-x)^{1.8} \frac{\partial^{1.8} u}{\partial_+ x^{1/8}} + t x^{0.8} \frac{\partial^{1.8} u}{\partial_- x^{1/8}} + x(1-x) - \frac{1}{\Gamma(0.2)}\left[t^2(-10x+1) - t(5.4 - 9.8x + 4.4x^2) \right], 0 \le x \le 1,\ 0 \le t \le 1$$

Together with the initial and zero Dirichlet boundary conditions:

u(x,0)=0 for $0 \le x \le 1$.

u(0,t)=0 for $0 \le t \le 1$.

u(1,t)=0 for $0 \le t \le 1$.

This example is constructed such that the exact solution of it is $u(x,t) = x(1-x)t$.

Here, we use the explicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = \frac{i}{2}$, i=0,1,2 and the t-interval into 2 subintervals such that $t_j = \frac{j}{2}$, j=0,1,2. In this case, β=1.741, $c_{max} = d_{max} = 1$ and $\frac{1}{q(c_{max} + d_{max})} = \frac{1}{3.6} = 0.278$. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i,0)=0$ for i=0,1,2.

$u(0,t_j)=0$ for j=0,1,2.

$u(1,t_j)=0$ for j=0,1,2.

Moreover, equation (3.15) becomes

$$u_{i,j+1}=1.741(1-x_i)^{1.8}\sum_{k=0}^{i+1} g_i u_{i-k+1,j}+1.741t_j x_i^{0.8}\sum_{k=0}^{2-i+1} g_i u_{i+k-1,j}+$$
$$0.5\left(x_i(1-x_i)-\frac{1}{\Gamma(0.2)}\left[t_j^{\,2}(-10x_i+1)-t_j(5.4-9.8x_i+4.4x_i^{\,2})\right]\right)+u_{i,j}$$

where i=1 and j=0, 1.

By evaluating the above equation at each i=1 and j=0, 1 one can get the values that are tabulated down with the comparison with the exact solution.

Table (3.9) Represents the numerical and the exact solutions for n=m=2 of example (3.1.4).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.5	0.5	0.125	0.125
0.5	1	–0.046	0.25

Second, we divide the x-interval into 3 subintervals such that $x_i=\frac{i}{3}$, i=0,1,2,3 and the t-interval into 4 subintervals such that $t_j=\frac{j}{4}$, j=0,1,2,3,4. In this case, β=1.806. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,2,3.

$u(0, t_j) = 0$ for j=0,1,2,3,4.

$u(1, t_j) = 0$ for j=0,1,2,3,4.

Moreover, equation (3.15) becomes

$$u_{i,j+1} = 1.806(1 - x_i)^{1.8} \sum_{k=0}^{i+1} g_i u_{i-k+1,j} + 1.806 t_j x_i^{0.8} \sum_{k=0}^{3-i+1} g_i u_{i+k-1,j} +$$
$$0.25\left(x_i(1 - x_i) - \frac{1}{\Gamma(0.2)}\left[t_j^{\,2}(-10x_i + 1) - t_j(5.4 - 9.8x_i + 4.4x_i^{\,2}) \right]\right) + u_{i,j}$$

where i=1,2 and j=0, 1, 2, 3.

By evaluating the above equation at each i=1,2 and j=0, 1, 2, 3 one can get the values that are tabulated down with the comparison with the exact solution.

Table (3.10) Represents the numerical and the exact solutions for n=3 and m=4 of example (3.1.4).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.333	0.25	0.056	0.056
0.667	0.25	5.24×10^{-3}	0.111
0.333	0.5	0.041	0.167
0.667	0.5	–0.063	0.222
0.333	0.75	0.056	0.056
0.667	0.75	0.066	0.111
0.333	1	0.028	0.167
0.667	1	0.134	0.222

Third, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{10}$, i=0,1,…,10 and the t-interval into 10 subintervals such that $t_j = \frac{j}{10}$, j=0,1,…,10. In this case, β=6.31. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,…,10.

$u(0, t_j) = 0$ for j=0,1,…,10.

$u(1, t_j) = 0$ for j=0, 1,…,10.

Moreover, equation (3.15) becomes

$$u_{i,j+1} = 6.31(1-x_i)^{1.8}\sum_{k=0}^{i+1} g_i u_{i-k+1,j} + 6.31 t_j x_i^{0.8} \sum_{k=0}^{10-i+1} g_i u_{i+k-1,j} +$$
$$0.1\left(x_i(1-x_i) - \frac{1}{\Gamma(0.2)}\left[t_j^{\,2}(-10x_i + 1) - t_j(5.4 - 9.8x_i + 4.4x_i^{\,2})\right]\right) + u_{i,j}$$

where i=1,2,…,9 and j=0, 1,…,9.

By evaluating the above equation at each i=1,2,…,9 and j=0, 1,…,9 one can get the values that are tabulated in the appendix (see program (3.1.4)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.11) Represents the numerical and the exact solutions for n=m=10 of example (3.1.4) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.5	0.5	0.013	0.125
0.5	1	-8.405×10^5	0.250
0.4	0.2	0.033	0.048
0.1	0.7	2.113×10^3	0.063
0.2	0.9	-5.758×10^5	0.144
0.3	1	-6.685×10^6	0.210
0.6	0.7	0.406	0.168
0.7	0.3	0.044	0.063
0.8	0.8	1.897	0.128
0.9	1	652.528	0.090

Fourth, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{10}$, $i = 0,1,\ldots,10$ and the t-interval into 1000 subintervals such that $t_j = \frac{j}{1000}$, $j = 0, 1, \ldots, 1000$. In this case, β=0.063 < $\frac{1}{q(c_{max} + d_{max})}$ = $\frac{1}{3.6} = 0.278$. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,…,10.

$u(0, t_j) = 0$ for j=0,1,…,1000.

$u(1, t_j) = 0$ for j=0,1,…,1000.

Moreover, equation (3.11) becomes

$$u_{i,j+1} = 0.063(1-x_i)^{1.8}\sum_{k=0}^{i+1} g_i u_{i-k+1,j} + 0.063 t_j x_i^{0.8}\sum_{k=0}^{10-i+1} g_i u_{i+k-1,j} +$$
$$0.001\left(x_i(1-x_i) - \frac{1}{\Gamma(0.2)}\left[t_j^{\,2}(-10x_i+1) - t_j(5.4-9.8x_i+4.4x_i^{\,2})\right]\right) + u_{i,j}$$

where i=1,2,…,9 and j=0, 1,…,1000.

By evaluating the above equation at each i=1,2,…,9 and j=0, 1,…,1000 one can get the values that are tabulated in the appendix (see program (3.1.4)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.12) Represents the numerical and the exact solutions for n=10 and m=1000 of example (3.1.4) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.1	1×10^{-3}	9×10^{-5}	9×10^{-5}
0.2	2×10^{-3}	3.179×10^{-4}	3.2×10^{-4}
0.3	3×10^{-3}	6.246×10^{-4}	6.3×10^{-4}
0.4	4×10^{-3}	9.512×10^{-4}	9.6×10^{-4}
0.5	5×10^{-3}	1.238×10^{-3}	1.25×10^{-3}
0.6	6×10^{-3}	1.427×10^{-3}	1.440×10^{-3}
0.7	7×10^{-3}	1.457×10^{-3}	1.470×10^{-3}
0.8	8×10^{-3}	1.269×10^{-3}	1.280×10^{-3}
0.9	9×10^{-3}	8.034×10^{-4}	8.100×10^{-4}

From the above tables, one can conclude that the results given in table (3.12) are more accurate than the previous results given in the other tables since the condition of the stability is satisfied in this case.

3.1.5 The Explicit Finite Difference Method for Solving the Fractional Hyperbolic Partial Differential Equations:

Consider the fractional order partial differential equation of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x, t)\frac{\partial^q u(x,t)}{\partial x^q} + s(x, t),\ L \le x \le R,\ 0 \le t \le T \quad \text{................(3.16.a)}$$

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{aligned} & u(x,0) = f(x) \quad \text{for } L \le x \le R \\ & \frac{\partial u(x,0)}{\partial t} = h(x) \quad \text{for } L \le x \le R \\ & u(L,t) = 0 \quad \text{for } 0 \le t \le T \\ & u(R,t) = 0 \quad \text{for } 0 \le t \le T \end{aligned}\right\} \quad \text{..(3.16.b)}$$

where $\frac{\partial^q u(x,t)}{\partial x^q}$ denote the left-handed partial fractional derivative of order q of the function u with respect to x and $1<q\le 2$.

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $t = t_j$, in equation (3.16.a) and replacing the partial derivative $\frac{\partial^2 u}{\partial t^2}$ with its central difference approximation to get:

$$\frac{u(x,t_{j+1})-2u(x,t_j)+u(x,t_{j-1})}{(\Delta t)^2}=c(x,t_j)\frac{\partial^q u(x,t_j)}{\partial x^q}+s(x,t_j) \quad \text{..........(3.17)}$$

where $t_j=j\Delta t$, j=0,1,…,m and m is the number of subintervals of the interval [0, T].

Next, we substitute equation (3.14.a) in equation (3.17) to obtain

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{(\Delta t)^2}=\frac{c_{i,j}}{(\Delta x)^q}\sum_{k=0}^{i+1}g_k u_{i-k+1,j}+s_{i,j}, i=1,2,...,n-1; j=0,1,...,m-1$$

................................(3.18)

On the other hand, the initial and boundary conditions given by equation (3.16.b) becomes

$$\left.\begin{array}{l} u_{i,0}=u(x_i,0)=f(x_i) \quad \text{for } i=0,1,...,n \\ \dfrac{\partial u(x_i,0)}{\partial t}=h(x_i) \quad \text{for } i=0,1,...,n \\ u_{0,j}=u(L,t_j)=0 \quad \text{for } j=0,1,...,m \\ u_{n,j}=u(R,t)=0 \quad \text{for } j=0,1,...,m \end{array}\right\}$$

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1}-u_{i,-1})=h_i\text{, i=0,1,…,n}$$

where $h_i=h(x_i)$ for i=0, 1,…, n. Hence

$$u_{i,1}=u_{i,-1}+2\Delta t h_i\text{, i=0,1,…, n}$$

Moreover, equation (3.18) becomes

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \frac{(\Delta t)^2 c_{i,j}}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + s_{i,j} (\Delta t)^2 \quad \text{.................}(3.19)$$

where i=1,2,…,n−1, j=0,1,…,m−1.

Therefore

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \frac{(\Delta t)^2 c_{i,0}}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} + s_{i,0} (\Delta t)^2$$

By substituting $u_{i,-1} = u_{i,1} - 2\Delta t h_i$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation:

$$u_{i,1} = f_i + \frac{(\Delta t)^2 c_{i,0}}{2(\Delta x)^q} \sum_{k=0}^{i+1} g_k f_{i-k+1} + \frac{(\Delta t)^2}{2} s_{i,0} + \Delta t g_i$$

where i=0,1,…,n−1.

By evaluating the above equation for each i=0,1,…,n−1, one can get the values $u_{i,1}$, $i = 1, 2, ..., n-1$. Then by evaluating equation (3.19) at each i=1,2,…,n−1 and j=2,3,…,m−1 one can get the numerical solution of equations (3.16).

To illustrate this method, consider the following example:

<u>Example (3.1.5):</u>

Consider the fractional order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\Gamma(0.5)} x^{\frac{1}{2}} \frac{\partial^{1.5} u}{\partial x^{1.5}} - 4x^2 + 2x^3 - 2.546x^2 t^2 + 2.546xt^2, 0 \le x \le 2, 0 \le t \le 1$$

Together with the initial and zero Dirichlet boundary conditions:

u(x,0)=0 for $0 \le x \le 2$.

$$\frac{\partial u(x,0)}{\partial t} = 0 \text{ for } 0 \le x \le 2.$$

u(0,t)=0 for $0 \le t \le 1$.

u(1,t)=0 for $0 \le t \le 1$.

This example is constructed such that the exact solution of it is $u(x,t) = x^2(x-2)t^2$. Here, we use the explicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = i$, i=0,1,2 and the t-interval into 2 subintervals such that $t_j = \frac{j}{2}$, j=0,1,2. Thus, the initial and zero Dirichlet boundary conditions becomes

$u(x_i,0) = 0$, for i=0,1,2.

$\frac{\partial u(x_i,0)}{\partial t} = 0$, for i=0,1,2.

$u(0,t_j) = 0$, for j=0,1,2.

$u(1,t_j) = 0$, for j=0,1,2.

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1} - u_{i,-1}) = 0$$

Hence

$u_{i,1} = u_{i,-1}$ for i=0,1,2.

Moreover, equation (3.19) becomes

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 0.25x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} +$$
$$0.25(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_j^2 + 2.546x_i t_j^2)$$

where i=1 and j=0, 1.

Therefore

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + 0.25x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} +$$
$$0.25(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_0^2 + 2.546xt_0^2)$$

By substituting $u_{i,-1} = u_{i,1}$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation

$$u_{i,1} = u_{i,0} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} +$$
$$0.125(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_0^2 + 2.546xt_0^2)$$

Thus

$$u_{1,1} = 0.125(-4x_1^2 + 2x_1^3) = -0.25.$$

Then

$$u_{1,2} = 2u_{1,1} - u_{1,0} + 0.25x_1^{\frac{1}{2}} \sum_{k=0}^{2} g_k u_{2-k,1} +$$
$$0.25(-4x_1^2 + 2x_1^3 - 2.546x_1^2 t_1^2 + 2.546x_1 t_1^2) = -0.947.$$

These values are tabulated down with the comparison with the exact solution.

Table (3.13) Represents the numerical and the exact solutions for n=m=2 of example (3.1.5).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
1	0.5	−0.25	−0.25
1	1	−0.947	−1

Second, we divide the x-interval into 3 subintervals such that $x_i = \frac{2i}{3}$, i=0,1,2,3 and the t-interval into 4 subintervals such that $t_j = \frac{j}{4}$, j=0,1,2,3,4. Thus, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$, for i=0,1,2,3.

$\frac{\partial u(x_i, 0)}{\partial t} = 0$, for i=0,1,2,3.

$u(0, t_j) = 0$, for j=0,1,2,3,4.

$u(1, t_j) = 0$, for j=0,1,2,3,4.

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1} - u_{i,-1}) = 0$$

Hence

$u_{i,1} = u_{i,-1}$ for i=0,1,2,3.

Moreover, equation (3.19) becomes

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} +$$
$$0.061(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_j^2 + 2.546xt_j^2)$$

where i=1, 2 and j=0, 1, 2, 3.

Therefore

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} +$$
$$0.063(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_0^2 + 2.546xt_0^2)$$

By substituting $u_{i,-1} = u_{i,1}$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation

$$u_{i,1} = u_{i,0} + 0.063x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} +$$
$$0.031(-4x_i^2 + 2x_i^3 - 2.546x_i^2 t_0^2 + 2.546xt_0^2)$$

Thus

$u_{1,1} = 0.031(-4x_1^2 + 2x_1^3) = -0.037$.

The other values are tabulated down with the comparison with the exact solution.

Table (3.14) Represents the numerical and the exact solutions for n=3 and m=4 of example (3.1.5).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.667	0.25	–0.037	–0.037
1.333	0.25	–0.074	–0.074
0.667	0.5	–0.147	–0.148
1.333	0.5	–0.293	–0.296
0.667	0.75	–0.326	–0.333
1.333	0.75	–0.65	–0.667
0.667	1	–0.568	–0.593
1.333	1	–1.13	–1.185

Third, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{5}$, i=0,1,…,10 and the t-interval into 10 subintervals such that $t_j = \frac{j}{10}$, j=0,1,…,10. Thus, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,…,10.

$\frac{\partial u(x_i, 0)}{\partial t} = 0$ for i=0,1,…,10.

$u(0, t_j) = 0$ for j=0,1,…,10.

$u(1, t_j) = 0$ for j=0,1,…,10.

By following the same previous steps one can get the values that are tabulated in the appendix (see program (3.1.5)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.15) Represents the numerical and the exact solutions for n=m=10 of example (3.1.5) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
1	0.5	–0.215	–0.250
1	1	–0.517	–1.000
0.8	0.2	–0.030	–0.031
0.2	0.7	–0.022	–0.035
0.4	0.9	–0.112	–0.207
0.6	1	–0.246	-5.040×10^{-3}
1.2	0.7	–0.419	–0.564
1.4	0.3	–0.101	–0.106
1.6	0.8	–0.496	–0.655
1.8	1	–0.586	–0.648

Fourth, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{5}$, i=0,1,…,10 and the t-interval into 1000 subintervals such that $t_j = \frac{j}{1000}$, i=0,1,…,1000. By running the same previous program one can get the values that are tabulated in the appendix (see program (3.1.5)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.16) Represents the numerical and the exact solutions for n=10 and m=1000 of example (3.1.5) at specific points.

x_i	t_j	**Numerical solution $u_{i,j}$**	**Exact solution $u(x_i,t_j)$**
0.2	1×10^{-3}	-7.200×10^{-8}	-7.200×10^{-8}
0.4	2×10^{-3}	-1.024×10^{-6}	-1.024×10^{-6}
0.6	3×10^{-3}	-4.536×10^{-6}	-4.536×10^{-6}
0.8	4×10^{-3}	-1.229×10^{-5}	-1.229×10^{-5}
1	5×10^{-3}	-2.500×10^{-5}	-2.500×10^{-5}
1.2	6×10^{-3}	-4.147×10^{-5}	-4.147×10^{-5}
1.4	7×10^{-3}	-5.762×10^{-5}	-5.762×10^{-5}
1.6	8×10^{-3}	-6.553×10^{-5}	-6.554×10^{-5}
1.8	9×10^{-3}	-5.249×10^{-5}	-5.249×10^{-5}

From the above tables, one can conclude that the results given in table (3.16) are more accurate than the previous results given in the other tables.

3.1.6 The Explicit Finite Difference Method for Solving the Two-Sided Fractional Hyperbolic Partial Differential Equations:

Consider the fractional order partial differential equation of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x,t)\frac{\partial^q u(x,t)}{\partial_+ x^q} + d(x,t)\frac{\partial^q u(x,t)}{\partial_- x^q} + s(x,t),\ L \le x \le R,\ 0 \le t \le T$$

.............................(3.20.a)

together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{aligned} & u(x,0) = f(x) \quad \text{for } L \le x \le R \\ & \frac{\partial u(x,0)}{\partial t} = h(x) \quad \text{for } L \le x \le R \\ & u(L,t) = 0 \quad \text{for } 0 \le t \le T \\ & u(R,t) = 0 \quad \text{for } 0 \le t \le T \end{aligned}\right\}$$

....(3.20.b)

where $\frac{\partial^q u(x,t)}{\partial_+ x^q}$, $\frac{\partial^q u(x,t)}{\partial_- x^q}$ denote the left-handed and the right-handed partial fractional derivatives of order q of the function u with respect to x and $1 < q \le 2$.

In this section, we use the explicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $t = t_j$, in equation (3.20.a) and replacing the partial derivative $\frac{\partial^2 u}{\partial t^2}$ with its central difference approximation to get:

$$\frac{u(x,t_{j+1}) - 2u(x,t_j) + u(x,t_{j-1})}{(\Delta t)^2} = c(x,t_j)\frac{\partial^q u(x,t_j)}{\partial_+ x^q} + d(x,t_j)\frac{\partial^q u(x,t_j)}{\partial_- x^q} + s(x,t_j)$$

................................(3.21)

where $t_j=j\Delta t$, $j=0,1,\dots,m$ and m is the number of subintervals of the interval [0, T].

Next, we substitute equations (3.14) in equation (3.21) to obtain

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{(\Delta t)^2}=\frac{c_{i,j}}{(\Delta x)^2}\sum_{k=0}^{i+1}g_k u_{i-k+1,j}+\frac{d_{i,j}}{(\Delta x)^2}\sum_{k=0}^{n-i+1}g_k u_{i+k-1,j}+s_{i,j}$$

................................(3.22)

where $i=1,2,\dots,n-1$, $j=0,1,\dots,m-1$

On the other hand, the initial and boundary conditions given by equation (3.20.b) becomes

$$\left.\begin{aligned}&u_{i,0}=u(x_i,0)=f(x_i)\quad \text{for } i=0,1,\dots,n\\&\frac{\partial u(x_i,0)}{\partial t}=h(x_i)\ \text{ for } i=0,1,\dots,n\\&u_{0,j}=u(L,t_j)=0\ \text{ for } j=0,1,\dots,m\\&u_{n,j}=u(R,t)=0\quad \text{for } j=0,1,\dots,m\end{aligned}\right\}$$

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1}-u_{i,-1})=h_i,\ i=0,1,\dots,n$$

where $h_i=h(x_i)$ for $i=0,1,\dots,n$. Hence

$$u_{i,1}=u_{i,-1}+2\Delta t h_i,\ i=0,1,\dots,n$$

Moreover, equation (3.18) becomes

$$u_{i,j+1}=2u_{i,j}-u_{i,j-1}+\frac{(\Delta t)^2 c_{i,j}}{(\Delta x)^q}\sum_{k=0}^{i+1}g_k u_{i-k+1,j}+\frac{(\Delta t)^2 d_{i,j}}{(\Delta x)^q}\sum_{k=0}^{n-i+1}g_k u_{i+k-1,j}+s_{i,j}(\Delta t)^2$$

................................(3.23)

where i=1,2,…,n–1, j=0,1,…,m–1.

Therefore

$$u_{i,1}=2u_{i,0}-u_{i,-1}+\frac{(\Delta t)^2 c_{i,0}}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,0}+\frac{(\Delta t)^2 d_{i,0}}{(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,0}+s_{i,0}(\Delta t)^2$$

By substituting $u_{i,-1}=u_{i,1}-2\Delta t h_i$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation:

$$u_{i,1}=f_i+\frac{(\Delta t)^2 c_{i,0}}{2(\Delta x)^q}\sum_{k=0}^{i+1} g_k f_{i-k+1}+\frac{(\Delta t)^2 d_{i,0}}{2(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k f_{i+k-1}\frac{(\Delta t)^2}{2}s_{i,0}+\Delta t g_i$$

where i=0,1,…,n–1.

By evaluating the above equation for each i=0,1,…,n–1, one can get the values $u_{i1},\ i=1,2,...,n-1$. Then by evaluating equation (3.23) at each i=1,2,…,n–1 and j=2,3,…,m–1 one can get one the numerical solution of equations (3.20).

To illustrate this method, consider the following example:

<u>Example (3.1.6):</u>

Consider the fractional order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{1}{\Gamma(0.5)}x^{\frac{1}{2}}\frac{\partial^{1.5}u}{\partial_+ x^{1.5}}+(2-x)^{\frac{1}{2}}\frac{\partial^{1.5}u}{\partial_- x^{1.5}}+2x^2(x-2)-$$

$$2.546xt^2(x-1)-(-4.514x^2+9.027x-2.257)t^2,\ 0\le x\le 2,\ 0\le t\le 1$$

Together with the initial and zero Dirichlet boundary conditions:

u(x,0)=0 for $0 \le x \le 2$.

$\frac{\partial u(x,0)}{\partial t} = 0$ for $0 \le x \le 2$.

u(0,t)=0 for $0 \le t \le 1$.

u(1,t)=0 for $0 \le t \le 1$.

This example is constructed such that the exact solution of it is $u(x,t) = x^2(x-2)t^2$. Here, we use the explicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = i$, i=0,1,2 and the t-interval into 2 subintervals such that $t_j = \frac{j}{2}$, j=0,1,2. Thus, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,2.

$\frac{\partial u(x_i,0)}{\partial t} = 0$ for i=0,1,2.

$u(0, t_j) = 0$ for j=0,1,2.

$u(1, t_j) = 0$ for j=0,1,2.

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1} - u_{i,-1}) = 0$$

Hence

$u_{i,1} = u_{i,-1}$ for i=0,1,2.

Moreover, equation (3.23) becomes

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 0.25x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + 0.25(2 - x_i)^{\frac{1}{2}} \sum_{k=0}^{2-i+1} g_k u_{i+k-1,j} +$$

$$0.25\left[2x_i^2(x_i - 2) - 2.546x_i t_j^2(x_i - 1) - (-4.514x_i^2 + 9.027x_i - 2.257)t_j^2\right]$$

where i=1 and j=0, 1.

Therefore

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + 0.25x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} + 0.25(2 - x_i)^{\frac{1}{2}} \sum_{k=0}^{2-i+1} g_k u_{i+k-1,0} +$$
$$0.25\left[2x_i^2(x_i - 2) - 2.546x_i t_j^2(x_i - 1) - (-4.514x_i^2 + 9.027x_i - 2.257)t_j^2\right]$$

By substituting $u_{i,-1} = u_{i,1}$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation

$$u_{i,1} = u_{i,0} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} + 0.125(2 - x_i)^{\frac{1}{2}} \sum_{k=0}^{2-i+1} g_k u_{i+k-1,0} +$$
$$0.125\left[2x_i^2(x_i - 2) - 2.546x_i t_j^2(x_i - 1) - (-4.514x_i^2 + 9.027x_i - 2.257)t_j^2\right]$$

Thus

$$u_{1,1} = 0.125(2x_1^2(x_1 - 2)) = -0.25.$$

Then

$$u_{1,2} = 2u_{1,1} - u_{1,0} + 0.25x_1^{\frac{1}{2}} \sum_{k=0}^{2} g_k u_{2-k,1} + 0.25(2 - x_1)^{\frac{1}{2}} \sum_{k=0}^{2-i+1} g_k u_{k,1} +$$
$$0.125\left[2x_1^2(x_1 - 2) - 2.546x_1 t_1^2(x_1 - 1) - (-4.514x_1^2 + 9.027x_1 - 2.257)t_1^2\right]$$

$$= -0.994.$$

These values are tabulated down with the comparison with the exact solution.

Table (3.17) Represents the numerical and the exact solutions for n=m=2 of example (3.1.6).

x_i	t_j	**Numerical solution $u_{i,j}$**	**Exact solution $u(x_i,t_j)$**
1	0.5	−0.25	−0.25
1	1	−0.994	−1

Second, we divide the x-interval into 3 subintervals such that $x_i = \frac{2i}{3}$, i=0,1,2,3 and the t-interval into 4 subintervals such that $t_j = \frac{j}{4}$, j=0,1,2,3,4. Thus, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,2,3.

$\frac{\partial u(x_i, 0)}{\partial t} = 0$ for i=0,1,2,3.

$u(0, t_j) = 0$ for j=0,1,2,3,4.

$u(1, t_j) = 0$ for j=0,1,2,3,4.

By using the central difference approximation to the initial derivative condition one can get:

$$\frac{1}{2\Delta t}(u_{i,1} - u_{i,-1}) = 0$$

Hence

$u_{i,1} = u_{i,-1}$ for i=0,1,2,3.

Moreover, equation (3.19) becomes

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} +$$
$$0.125(2-x_i)^{\frac{1}{2}} \sum_{k=0}^{3-i+1} g_k u_{i+k-1,j} + 0.063\Big[2x_i^2(x_i-2) - 2.546x_i t_j^2(x_i-1) -$$
$$(-4.514x_i^2 + 9.027x_i - 2.257)t_j^2\Big]$$

where i=1, 2 and j=0, 1, 2, 3.

Therefore

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 0.125x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + 0.125(2-x_i)^{\frac{1}{2}} \sum_{k=0}^{3-i+1} g_k u_{i+k-1,j} +$$
$$0.063\Big[2x_i^2(x_i-2) - 2.546x_i t_j^2(x_i-1) - (-4.514x_i^2 + 9.027x_i - 2.257)t_j^2\Big]$$

By substituting $u_{i,-1} = u_{i,1}$ in the above equation one can shows that $u_{i,1}$ can be calculated from the equation

$$u_{i,1} = u_{i,0} + 0.063x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} + 0.063(2-x_i)^{\frac{1}{2}} \sum_{k=0}^{3-i+1} g_k u_{i+k-1,0} +$$
$$0.063\Big[2x_i^2(x_i-2) - 2.546x_i t_0^2(x_i-1) - (-4.514x_i^2 + 9.027x_i - 2.257)t_0^2\Big]$$

Thus

$u_{1,1} = -0.037$.

The other values are tabulated down with the comparison with the exact solution.

Table (3.18) Represents the numerical and the exact solutions for n=3 and m=4 of example (3.1.6).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.667	0.25	−0.037	−0.037
1.333	0.25	−0.074	−0.074
0.667	0.5	−0.146	−0.148
1.333	0.5	−0.29	−0.296
0.667	0.75	−0.307	−0.333
1.333	0.75	−0.63	−0.667
0.667	1	−0.499	−0.593
1.333	1	−1.069	−1.185

Third, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{5}$, i=0,1,…,10 and the t-interval into 10 subintervals such that $t_j = \frac{j}{10}$, j=0,1,…,10. Thus, the initial and zero Dirichlet boundary conditions becomes:

$u(x_i, 0) = 0$ for i=0,1,…,10.

$\frac{\partial u(x_i, 0)}{\partial t} = 0$ for i=0,1,…,10.

$u(0, t_j) = 0$ for j=0,1,…,10.

$u(1, t_j) = 0$ for j=0,1,…,10.

By following the same previous steps one can get the values that are tabulated in the appendix (see program (3.1.6)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.19) Represents the numerical and the exact solutions for n=m=10 of example (3.1.6) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
1	0.5	–0.180	–0.250
1	1	–0.264	–1.000
0.8	0.2	–0.030	–0.031
0.2	0.7	–0.024	–0.035
0.4	0.9	–0.086	–0.207
0.6	1	-5.040×10^{-3}	-5.040×10^{-3}
1.2	0.7	–0.293	–0.564
1.4	0.3	–0.095	–0.106
1.6	0.8	–0.320	–0.655
1.8	1	–0.311	–0.648

Fourth, we divide the x-interval into 10 subintervals such that $x_i = \frac{i}{5}$, i=0,1,…,10 and the t-interval into 1000 subintervals such that $t_j = \frac{j}{1000}$, i=0,1,…,1000. By running the same program one can get the values that are tabulated in the appendix (see program (3.1.6)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (3.20) Represents the numerical and the exact solutions for n=10 and m=1000 of example(3.1.6) at specific points.

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.2	1×10^{-3}	-7.200×10^{-8}	-7.200×10^{-8}
0.4	2×10^{-3}	-1.024×10^{-6}	-1.024×10^{-6}
0.6	3×10^{-3}	-4.536×10^{-6}	-4.536×10^{-6}
0.8	4×10^{-3}	-1.229×10^{-5}	-1.229×10^{-5}
1	5×10^{-3}	-2.500×10^{-5}	-2.500×10^{-5}
1.2	6×10^{-3}	-4.147×10^{-5}	-4.147×10^{-5}
1.4	7×10^{-3}	-5.762×10^{-5}	-5.762×10^{-5}
1.6	8×10^{-3}	-6.553×10^{-5}	-6.554×10^{-5}
1.8	9×10^{-3}	-5.249×10^{-5}	-5.249×10^{-5}

From the above tables, one can conclude that the results given in table (3.20) are more accurate than the previous results given in the other tables.

3.2 The Implicit Finite Difference Method:

In this section, we use the implicit finite difference method to solve the initial-boundary value problems of the partial fractional parabolic differential equations. Also, the stability of this method is discussed.

3.2.1 The Implicit Finite Difference Method for Solving Fractional Parabolic Partial Differential Equations:

Consider the initial-boundary value problem given by equations (3.8). In this section, we use the implicit finite difference method to solve this initial-boundary value problem. To do this, we substitute $t = t_{j+1}$, in equation (3.8.a) and replacing the partial derivative $\frac{\partial u}{\partial t}$ with its approximation to get:

$$\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t} = c(x,\, t_{j+1})\frac{\partial^q u(x,t_{j+1})}{\partial x^q} + s(x,t_{j+1})$$

where $t_j = j\Delta t$, j=0,1,…, m and m is the number of subintervals of the interval [0, T].

Next, we substitute the left-handed shifted Grünwald estimate given by equation (3.10) in the above equation to get:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = \frac{c_{i,j+1}}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,j+1} + s_{i,j+1},\ i=1,2,\ldots,\ n-1;\ j=0,1,\ldots,m-1$$

................................(3.24)

Then for each j=0,1,…,m−1, we evaluate the above equation at each i=1,2,…,n−1 to get a system of n−1 equations with n−1 unknowns that can be solved by any suitable method.

Next, we prove that the implicit finite difference method given by equation (3.24) is unconditionally stable. The proof of this theorem depends on the idea that appeared in [Meerschaert M. and Tadjeran C., 2006].

Theorem (3.2.1):

The implicit finite difference method given by equation (3.24) is unconditionally stable.

Proof:

The system of equations defined by equation (3.24) ,together with the Dirichlet boundary conditions, define a linear system

This system can be written as AU=B, where

$$A=\begin{bmatrix} 1-\beta c_{1,j+1}g_1 & -\beta c_{1,j+1} & 0 & 0 & \cdots & 0 & 0 \\ -\beta c_{2,j+1}g_2 & 1-\beta c_{2,j+1}g_1 & -\beta c_{2,j+1} & 0 & \cdots & 0 & 0 \\ -\beta c_{3,j+1}g_3 & -\beta c_{3,j+1}g_2 & 1-\beta c_{3,j+1}g_1 & -\beta c_{3,j+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\beta c_{n-2,j+1}g_{n-2} & -\beta c_{n-2,j+1}g_{n-3} & -\beta c_{n-2,j+1}g_{n-4} & -\beta c_{n-2,j+1}g_{n-5} & \cdots & 1-\beta c_{n-2,j+1}g_1 & \beta c_{n-2,j+1} \\ -\beta c_{n-1,j+1}g_n & -\beta c_{n-1,j+1}g_{n-1} & -\beta c_{n-1,j+1}g_{n-2} & -\beta c_{n-1,j+1}g_{n-3} & -\beta c_{n-1,j+1}g_{n-4} & -\beta c_{n-1,j+1}g_{n-1} & 1-\beta c_{n-1,j+1}g_1 \end{bmatrix}$$

$$U=\begin{bmatrix} u_{1j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} \text{ and } B=\begin{bmatrix} \Delta t s_{1j+1}+u_{1,j} \\ \Delta t s_{2,j+1}+u_{2,j} \\ \vdots \\ \Delta t s_{n-2,j+1}+u_{n-2,j} \\ \Delta t s_{n-1,j+1}+u_{n-1,j} \end{bmatrix}.$$

where $i = 1, 2, \ldots, n-1$; $j = 0, 1, \ldots, m-1$, $\beta = \frac{\Delta t}{(\Delta x)^q}$, $c_{i,j+1} = c(x, t_{j+1})$, $s_{i,j+1} = s(x, t_{j+1})$.

Next, that $g_1 = -q$ and for $1 < q \leq 2$ and $i \neq 1$, we have $g_i \geq 0$.

Then, according to Greschgorin's theorem, the eigenvalues λ of the matrix A lie in the union of the n-circles centered at $a_{i,i}$ with radius

$$r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} a_{i,k}$$

Hence, we have:

$$a_{i,i} = 1 - g_1 \beta c_{i,j+1} = 1 + q\, \beta c_{i,j+1}$$

and

$$r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} a_{i,k} = \beta c_{i,j+1} \sum_{\substack{k=0 \\ k\neq i}}^{i+1} g_k$$

But $\sum_{\substack{k=0 \\ k\neq i}}^{i+1} g_k \leq -g_1 = -(-q) = q$

with strict inequality holding true when q is not an integer. This implies that the eigenvalues λ of the matrix A are all no less than 1 in magnitude. Thus any error in U_j is not magnified, and therefore the method is stable. ■

To illustrate this method, consider the following example:

Example (3.2.1):

Consider example (3.1.3). Here, we use the implicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = \frac{i}{2}$, i=0,1,2 and the t-interval into 2 subintervals such that $t_j = \frac{j}{2}$, j=0,1,2. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,2.

$u(0, t_j) = 0$ for j=0,1,2.

$u(1, t_j) = 0$ for j=0,1,2.

Moreover, equation (3.20) becomes

$$u_{i,j+1} = 2^{0.8} x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j+1} + \frac{1}{2}\left(x_i(x_i - 1) - \frac{t_{j+1}}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

where i=1 and j=0, 1.

By evaluating the above equation at each i=1 and j=0,1 one can get the values that are tabulated down with the comparison with the exact solution.

Table (3.21) Represents the numerical and the exact solutions for n=m=2 of example (3.2.1).

x_i	t_j	Numerical solution $u_{i,j}$	Exact solution $u(x_i,t_j)$
0.5	0.5	–0.122	–0.125
0.5	1	–0.244	–0.25

Second, we divide the x-interval into 3 subintervals such that $x_i = \frac{i}{3}$, i=0,1,2,3 and the t-interval into 4 subintervals such that $t_j = \frac{j}{4}$, j=0,1,2,,3,4. On the other hand, the initial and zero Dirichlet boundary conditions becomes

$u(x_i, 0) = 0$ for i=0,1,2,3.

$u(0, t_j) = 0$ for j=0,1,2,,3,4.

$u(1, t_j) = 0$ for j=0,1,2,,3,4.

Moreover, equation (3.24) becomes

$$u_{i,j+1} = 1.806 x_i^{\frac{4}{5}} \sum_{k=0}^{i+1} g_i u_{i-k+1,j+1} + \frac{1}{4}\left(x_i(x_i - 1) - \frac{t_{j+1}}{\Gamma(0.2)}(10x_i - 1) \right) + u_{i,j}$$

................................(3.25)

where i=1,2 and j=0, 1, 2, 3.

By evaluating equation (3.25) at j=0 and i=1,2 one can get the following system:

$$\begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} = \begin{pmatrix} -0.087 \\ -0.133 \end{pmatrix}$$

The solution of this system is given in the following table with the comparison with the exact solution. Then, we evaluate equation (3.25) at j = 1 and i = 1,2 to get the following system:

$$\begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}\begin{pmatrix} u_{1,2} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} -0.174 \\ -0.265 \end{pmatrix}$$

The solution of this system is given in the following table with the comparison with the exact solution. Then, we evaluate equation (3.25) at j = 2 and i = 1,2 to get the following system:

$$\begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}\begin{pmatrix} u_{1,3} \\ u_{2,3} \end{pmatrix} = \begin{pmatrix} -0.26 \\ -0.397 \end{pmatrix}$$

The solution of this system is given in the following table with the comparison with the exact solution. Then, we evaluate equation (3.25) at j = 3 and i = 1,2 to get the following system:

$$\begin{pmatrix} 2.35 & -0.75 \\ -0.94 & 3.35 \end{pmatrix}\begin{pmatrix} u_{1,4} \\ u_{2,4} \end{pmatrix} = \begin{pmatrix} -0.346 \\ -0.528 \end{pmatrix}$$

The solution of this system is given in the following table with the comparison with the exact solution.

Table (3.22) Represents the numerical and the exact solutions for n=3 and m=4 of example (3.2.1).

x_i	t_j	*Numerical solution $u_{i,j}$*	*Exact solution $u(x_i,t_j)$*
0.333	0.25	–0.055	–0.056
0.667	0.25	–0.055	–0.056
0.333	0.5	–0.109	–0.111
0.667	0.5	–0.11	–0.111
0.333	0.75	–0.163	–0.167
0.667	0.75	–0.164	–0.167
0.333	1	–0.217	–0.222
0.667	1	–0.219	–0.222

From the above tables, one can conclude that the results given in table (3.22) are more accurate than the previous results given in the other tables. And one can get the values that are tabulated in the appendix (see program (3.2.1)) with the comparison with the exact solution.

Remarks (3.2.1):

(1) Like the previous steps, the implicit finite difference method can be also used to solve the initial-boundary value problems of the right-handed fractional parabolic partial differential equations. In this case equation (3.24) becomes:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\frac{c_{i,j+1}}{(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j+1}+s_{i,j+1}$$

where i=1,2,…, n−1; j=0,1,…,m−1.

(2) In a similar manner, the implicit finite difference method can be also used to solve the initial-boundary value problems of the two-sided fractional parabolic partial differential equations given by equations (3.12). In this case equation (3.24) becomes

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\frac{c_{i,j+1}}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,j+1}+\frac{d_{i,j+1}}{(\Delta x)^q}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j+1}+s_{i,j+1}$$

where i=1,2,…, n−1; j=0,1,…,m−1.

2 Solutions of the Fractional Order Ordinary Differential Equations

Introduction:

In this chapter, we discuss the existence and uniqueness theorems of the solutions for the initial value problem of the fractional order ordinary differential equations. Also, we give some necessary conditions that ensure the existence of the extremal solutions for the initial value problem of the fractional order ordinary differential equations. Moreover, we use the finite difference method to solve the fractional order ordinary differential equations.

This chapter consists of three sections.

The section one consist of two theorems, one for the existence and the other for the existence of the unique solution for the initial value problem of the fractional order ordinary differential equations

The section two discusses the existence of the extremal solutions for the initial value problem of the fractional order ordinary differential equations.

The section three gives the numerical solutions via the finite difference method for the initial value problem of the fractional order ordinary differential equations.

2.1 Existence and Uniqueness Theorems of the Solution for the Fractional Order Ordinary Differential Equations:

In this section, we discuss the existence and the uniqueness theorems of special types of the initial value problems of the fractional order ordinary differential equations.

Now, start this section by the following theorem. This theorem gives necessary conditions for the existence of a solution of the initial value problem that consists of the fractional order ordinary differential equation:

$y^{(q)}(x) = f(x, y(x)),\ 0 < q \leq 1$ ....(2.1.a)

together with the initial condition

$y^{(q-1)}(x_0) = y_0$ ....(2.1.b)

This theorem is appeared in [Hadid S. and Momani S., 1996]. Here, present the details of its proof. But before that, we need the following lemma.

Lemma (2.1) (Arzela-Ascoli), [S.B. Hadid, S. M. Momani, 1996]:

Let F be a family of functions bounded and equicontinuous at every point of an interval I. Then, every sequence of functions $\{f_n\}$ in F contains a subsequence uniformly converge on every compact sub-interval of I.

Theorem (2.1), [Hadid S. and Momani S., 1996]:

Let $f \in C[B_0, \mathbb{R}^n]$, where

$B_0 = \{(x, y) \in \Omega \mid 0 < x_0 \leq x \leq x_0 + a, \|y - y_0(x - x_0)^{q-1}\| \leq b\}, 0 < q \leq 1$

and Ω is an open set in $\mathbb{R}^{n+1}$. Assume

$\|f(x, y)\| \leq \mu$ on B_0

Then the initial value problem given by equations (2.1) has at least one solution g(x) on $0 < x_0 \leq x \leq x_0 + B$, where

$$B^q = \min\left\{a, \frac{bq\Gamma(q)}{\mu}\right\}.$$

Proof:

Let y_0 be a continuously differentiable function on $[x_0-\lambda, x_0]$, $\lambda > 0$ satisfying:

$y_0^{(q)}(x) = f(x, y_0(x)), \ y_0(x_0) = y_0,$

$\|y_0(x) - y_0(x - x_0)^{q-1}\| \leq b$

and

$\| y_0^{(q)}(x)\| \leq \mu.$

For $0 < \varepsilon \leq \lambda$, we define a function $y_\varepsilon(x)$ on $[x_0 - \lambda, x_0 + B]$ by

$$y_\varepsilon(x) = \begin{cases} y_0(x), & x \in [x_0 - \lambda, x_0] \\ y_0(x - x_0)^{q-1} + \dfrac{1}{\Gamma(q)} \displaystyle\int_{x_0}^{x} (x - t)^{q-1} f(t, y_\varepsilon(t - \varepsilon))dt, & x \in [x_0, x_0 + B] \end{cases}$$

..................................(2.2)

For the above equation, we observed that $y_\varepsilon(x)$ define a continuous function on $[x_0 - \lambda, x_0 + B_1]$, where $B_1 = \min\{B, \varepsilon\}$. Moreover:

$$\left\| y_\varepsilon(x) - y_0(x - x_0)^{q-1} \right\| = \frac{1}{\Gamma(q)} \left\| \int_{x_0}^{x} (x-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt \right\|$$

$$\leq \frac{\mu}{\Gamma(q)} \int_{x_0}^{x} (x-t)^{q-1}\, dt = \frac{\mu}{q\Gamma(q)} B_1^q$$

$$\leq \frac{\mu}{q\Gamma(q)} B^q \leq \frac{\mu}{q\Gamma(q)} \frac{b\Gamma(q)}{\mu} = b$$

Therefore, $\| y_\varepsilon(x) - y_0(x - x_0)^{q-1} \| \leq b$, $\forall\, x \in [x_0 - \lambda, x_0 + B]$.

Moreover, we can use equation (2.2) to extend $y_\varepsilon(x)$ as a continuous function over $[x_0 - \lambda, x_0 + B_2]$, where $B_2 = \min [B, 2\varepsilon]$, such that:

$\| y_\varepsilon(x) - y_0(x - x_0)^{q-1} \| \leq b$, $\forall\, x \in [x_0 - \lambda, x_0 + B_2]$

By continuing in this process, we can define $y_\varepsilon(x)$ as a continuous function over the interval $[x_0 - \lambda, x_0 + B]$, so as to satisfy the following inequality:

$\| y_\varepsilon(x) - y_0(x - x_0)^{q-1} \| \leq b$, $\forall\, x \in [x_0 - \lambda, x_0 + B]$

Moreover, from the above inequality, one can deduce that the sequence $\{y_\varepsilon(x)\}$, $0 < \varepsilon \leq \lambda$ forms a family of uniformly bounded functions. It remains only to show that the sequence $\{y_\varepsilon(x)\}$ forms a family of equicontinuous functions. To do this, it is enough to show that the second term of equation (2.2) defined by:

$$z_\varepsilon(x) = \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt$$

Forms a family of equicontinuous functions, so consider:

$$\left\| z_\varepsilon(x_2) - z_\varepsilon(x_1) \right\| = \frac{1}{\Gamma(q)} \left\| \int_{x_0}^{x_2} (x_2-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt - \int_{x_0}^{x_1} (x_1-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt \right\|$$

$$= \left\| \int_{x_0}^{x_1} (x_2-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt + \int_{x_1}^{x_2} (x_2-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt - \int_{x_0}^{x_1} (x_1-t)^{q-1} f(t, y_\varepsilon(t-\varepsilon))\, dt \right\|$$

$$\le \frac{\mu}{\Gamma(q)} \left[\int_{x_0}^{x_1} \left\{ (x_1-t)^{q-1} - (x_2-t)^{q-1} \right\} dt + \int_{x_1}^{x_2} (x_2-t)^{q-1}\, dt \right]$$

$$= \frac{\mu}{q\Gamma(q)} \left[(x_1-x_0)^q - (x_2-x_0)^q + (x_2-x_1)^q + (x_2-x_1)^q \right]$$

$$\le \frac{\mu}{q\Gamma(q)} \left[2(x_2-x_1)^q + \left| (x_2-x_0)^q - (x_1-x_0)^q \right| \right]$$

$$\le \frac{\mu}{\Gamma(q)} \left[2(x_2-x_1)^q + \left| x_2-x_1 \right|^q \right]$$

for all $0 < x_0 < x_1 < x_2 \le x_0 + B$.

Hence, as $x_2 \longrightarrow x_1$, then $x_2^q \longrightarrow x_1^q$. Therefore, the family $\{z_\varepsilon(x)\}$ is equicontinuous functions. Thus, the sequence $\{y_\varepsilon(x)\}$, $0 < \varepsilon \leq \lambda$ forms a family of equcontinuous and uniformly bounded functions. Hence, by using Arzela-Ascoli lemma, there exists a sequence $\{\varepsilon_n\}$, such that $\varepsilon_1 > \varepsilon_2 > \ldots, \varepsilon_n \longrightarrow 0$, as $n \longrightarrow \infty$ and $\lim_{n\to\infty} y_{\varepsilon_n}(x)=y(x)$ exists uniformly on $[x_0-\lambda, x_0+\beta]$, this converge is uniform. The continuity of f on β_0 implies that $f(x, y_\varepsilon(x - \varepsilon_n))$ converge uniformly of $f(x, y)$ as $n \longrightarrow \infty$. Hence, as $n \longrightarrow \infty$ equation (2.2) with $\varepsilon = \varepsilon_n$ yields:

$$y(x) = y_0(x - x_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-t)^{q-1} f(t, y(t))\, dt$$

This shows that y is a solution of the initial value problem given by equations (2.1). ■

Next, we give another theorem that gives necessary conditions for the existence of the unique solution of the initial value problem given by equation (2.1). This theorem is appeared in [Hadid S. and Momani S., 1986]. Here, we give the details of its proof. But before that, we need the following lemma.

Lemma (2.2), [Hochstadt H., 1973]:

If S is a closed subset of a Banach space B, then any contraction mapping T of S into itself has a unique fixed point.

Theorem (2.2), [Hadid S. and Momani S., 1996]:

Consider the initial value problem given by equations (2.1). Assume that $f:[0, \infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and bounded in the region $D=\{(x, y) \mid 0 < x_0 \leq x < x_0 + a, \|y\| < \infty\}$ and satisfies the Lipschitz condition:

$$\|f(x, y_1) - f(x, y_2)\| \leq L\|y_1 - y_2\|, \forall (x, y_1), (x, y_2) \in D$$

for some positive constant L. If $\frac{La^q}{q\Gamma(q)}<1$, then the initial value problem given by equations (2.1) has a unique solution in $x_0 < x \leq x_0 + a$.

Proof:

Let $B = \{y \mid y: (x_0, x_0 + a] \longrightarrow \mathbb{R}^n, y \text{ is continuous}\}$.

Define the norm of any function y in B, as:

$$\|y\|_B = \sup_{x_0<x\leq x_0+a} \|y(x)\|$$

then $(B, \|\cdot\|_B)$ is a Banach space.

Define the set S(ρ) as:

$$S(\rho) = \{y \in B: \|y\|_B \leq \rho\}$$

and the operator T by:

$$(T_y)(x) = y_0(x - x_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{x_0}^{x} (x-t)^{q-1} f(t, y(t))\, dt \qquad \text{.......................(2.3)}$$

for $y \in S(\rho)$, where $\rho = \|y_0\|a^{q-1} + \frac{\mu a^q}{q\Gamma(q)}$ and $\|f(x, y)\| \leq \mu$.

Next, we shall prove that S(ρ) is closed. To do this, $f \in \overline{S(\rho)}$, then there exists $f_n \in S(\rho)$, such that $f_n \longrightarrow f$. Therefore

$$\|f\|_B = \|f - f_n + f_n\|_B \leq \|f_n\|_B + \|f - f_n\|_B \leq \rho$$

Hence S(ρ) is closed subset of the Banach space B.

Now, we prove that $T: S(\rho) \longrightarrow S(\rho)$. To do this, let $y \in S(\rho)$, then from equation (2.3), one can deduce that $\|T_y\|_B \leq \| y_0 \| a^{q-1} + \frac{\mu}{q\Gamma(q)} a^q = \rho$

Therefore, $T_y \in S(\rho)$.

To prove $T:S(\rho) \longrightarrow S(\rho)$ be contraction. Let $y_1, y_2 \in S(\rho)$, then one can have

$$\|(Ty_1)(x) - (Ty_2)(x)\| = \Bigg\| y_0 (x - x_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - t)^{q-1} f(t, y_1(t))\, dt - y_0 (x - x_0)^{q-1} - \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - t)^{q-1} f(t, y_2(t)) dt \Bigg\|$$

$$= \frac{1}{\Gamma(q)} \left\| \int_{x_0}^{x} (x - t)^{q-1} [f(t, y_1(t)) - f(t, y_2(t))]\, dt \right\|$$

$$\leq \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - t)^{q-1} \|f(t, y_1(t)) - f(t, y_2(t))\|\, dt$$

$$\leq \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - t)^{q-1} L \|y_1(t) - y_2(t)\|\, dt \leq \frac{La^q}{q\Gamma(q)} \|y_1 - y_2\|_B$$

But $\dfrac{La^q}{q\Gamma(q)} < 1$, then T is a contraction, and hence by using lemma (2.1), one can get T has a unique fixed point which is the solution of the initial value problem given by equations (2.1). ■

<u>2.2 *Existence Theorem of the Extermal Solutions for the Fractional Order Ordinary Differential Equations:*</u>

In this section, we discuss the existence for the extremal solutions of the initial value problem given by equations (2.1). But before that, we need to recall the definitions of the maximal and minimal solutions (extremal solutions) of the initial value problem given by equations (2.1).

<u>Definition (2.1), [Hadid S. and Momani S., 1996]:</u>

The maximal solution M(x) of the initial value problem given by equations (2.1) is a solution for equation (2.1) such that the following inequality is satisfied for any solution y(x):

$$y(x) \le M(x), \quad \forall x \ge x_0$$

<u>Definition (2.2), [Hadid S. and Momani S., 1996]:</u>

The minimal solution m(x) of the initial value problem given by equations (2.1) is a solution for equation (2.1) such that the following inequality is satisfied for any solution y(x):

$$m(x) \le y(x), \quad \forall x \ge x_0$$

Remark (2.1):

It is easy to check that the maximal (or the minimal) solution of the initial value problem given by equations (2.1) is unique.

Now, we give some conditions that guarantee the existence of the extremal solutions of the initial value problem given by equations (2.1). This theorem appeared in [Hadid S. and Momani S., 1996]. Here, we give the details of its proof.

Theorem (2.3), [Hadid S. and Momani S., 1996]:

Consider the initial value problem given by equations (2.1). Assume $f: [0, \infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$, is a continuous function in the region:

$$R(a, b) = \{(x, y) \mid 0 < x_0 < x \leq x_0 + a, |y - y_0(x - x_0)^{q-1}| \leq b\}$$

Then there exists a maximal (minimal) solution of equations (2.1) on $[x_0, x_0 + \beta]$, for some $\beta > 0$.

Proof:

We shall prove the existence of the maximal solution of equations (2.1). To do this, let $\varepsilon > 0$ be given such that $0 < \varepsilon \leq \frac{b}{2}$. Since f is continuous in R(a, b), then there exists a positive constant μ, such that:

$$|f(x, y(x))| \leq \mu + \frac{b}{2}, \text{ for } (x, y) \in R(a, b)$$

Consider the initial value problem:

$$\left.\begin{aligned} & y_\varepsilon^{(q)}(x) = f_\varepsilon(x, y), \ 0 < q \le 1 \\ & y_\varepsilon^{(q-1)}(x_0) = y_0 + \varepsilon \end{aligned}\right\} \quad \text{...} (2.4)$$

where $f_\varepsilon(x, y) = f(x, y) + \varepsilon$.

Then it can be easily see that the function $f_\varepsilon(x, y(x))$ is defined and continuous in the region:

$R(a, b, \varepsilon) = \{(x, y) \mid 0 < x_0 < x \le x_0+a, |y-y_0(x-x_0)^{q-1} - \varepsilon| \le \frac{b}{2}\}$

Therefore, from theorem (2.1), it follows that the initial value problem given by equations (2.4) has a solution $y_\varepsilon(x)$ on the interval $[x_0, x_0 + B]$, where:

$$B^q = \min\left\{a, \frac{2bq\Gamma(q)}{2\mu + b}\right\}$$

Let ε_1 and ε_2 be two real numbers such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, then from equations (2.4) and $f_\varepsilon(x, y) = f(x, y) + \varepsilon$.

$$y_{\varepsilon_2}(x_0) < y_\varepsilon(x_1)$$

$$y_{\varepsilon_2}^{(q)}(x) = f(x, y_{\varepsilon_2}(x)) + \varepsilon_2$$

$$y_{\varepsilon_1}^{(q)} > f(x, y_{\varepsilon_1}(x)) + \varepsilon_2$$

Therefore,

$$y_{\varepsilon_2}(x) < y_{\varepsilon_1}(x), \ \text{for } x \in [x_0, x_0 + B]$$

Moreover, from the hypothesis, the family of functions $y_\varepsilon(x)$ is equicontinuous and uniformly bounded on $[x_0, x_0+\beta]$. Hence, by Arzela-Ascoli lemma, there exists a decreasing sequence $\{\varepsilon_n\}$, such that

$\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, and $\lim\limits_{n\to\infty} y_{\varepsilon_n}(x)$ exists uniformly in $x \in [x_0, x_0+\beta]$, we denote this limiting value by r(x). Obviously, $r(x_0)=x_0$, and

$$y_{\varepsilon_n}(x) = y_0(x-x_0)^{q-1} + \varepsilon_n + \frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-t)^{q-1} f(t, y_{\varepsilon_n}(t))\,dt \quad \text{.............(2.5)}$$

yields r(x) as a solution of equations (2.1).

Finally, we show that the solution r(x) is the maximal solution of the initial value problem given by equations (2.1). To do this, let y(x) be any other solution of the initial value problem given by equations (2.1) existing on the interval $[x_0, x_0 + \beta]$. Then

$$y^{(q-1)}(x_0) = y_0 < y_0 + \varepsilon = y_\varepsilon^{(q-1)}(x_0)$$

Further, for $x \in [x_0, x_0 + \beta]$, we have

$$y^{(q)}(x) < f(x, y(x)) + \varepsilon$$

$$y_\varepsilon^{(q)}(x) = f(x, y(x)) + \varepsilon$$

Hence, we have

$$y^{(q)}(x) < y_\varepsilon^{(q)}(x) \quad \text{for } x \in [x_0, x_0 + \beta]$$

and therefore

$$y(x) < y_\varepsilon(x) \quad \text{for } x \in [x_0, x_0 + \beta]$$

Since the maximal solution is unique, it is clear that $y_\varepsilon(x)$ tends to r(x) uniformly in $x \in [x_0, x_0+\beta]$ as $\varepsilon \longrightarrow 0$. A similar argument holds for the minimal solution. ■

Next, the following theorem involves estimating a function satisfying a differential inequality by the maximal solution of the equation corresponding to the inequality. This theorem appeared in [Hadid S. and Momani S., 1996]. Here, we give the details of its proof.

Theorem (2.4), [Hadid S. and Momani S., 1996]:

Let $f \in C[\Omega, R]$, Ω being an open set in $\mathbb{R}^2$ and let r(x) be a maximal solution of the initial value problem given by equations (2.1) on $[x_0, x_0 + \beta]$, for some $\beta > 0$. Also, let m be a continuous function on $[x_0, x_0+\beta]$, such that $m(x_0) \leq y_0$ and $(x, m(x)) \in \Omega$ satisfying the differential inequality:

$$m^{(q)}(x) \leq f(x, m(x)), \; x \in [x_0, x_0 + \beta]$$

Then, on the common interval of existence of m(x) and r(x) the inequality

$$m(x) \leq r(x)$$

holds.

Proof:

Let $x_1 \in (x_0, x_0+\beta)$. An argument similar to that for theorem (2.3) shows that there exists a maximal solution $r(x, \varepsilon)$ of the initial value problem given by equations (2.4) on $[x_0, x_1]$ for all sufficiently small $\varepsilon > 0$ and

$$\lim_{\varepsilon \to 0} r(x, \varepsilon) = r(x) \quad \text{.......(2.6)}$$

Uniformly in $x \in [x_0,x_1]$. From equations (2.5), $f_\varepsilon(x, y) = f(x, y) + \varepsilon$ and $m^{(q)}(x) \le f(x,m(x))$, one can have

$m(x) \le r(x, \varepsilon),\ x \in [x_0, x_1]$..(2.7)

Inequality (2.7), together with the convergence property given by equation (2.6), leads to $m(x) \le r(x)$. ■

Remark (2.2):

It is easy to check that if $m^{(q)}(x) \ge f(x, m(x)),\ x \in [x_0, x_0 + \beta]$ and $m(x_0) \ge y_0$, then $m(x) \ge s(x)$, where $s(x)$ is the minimal solution of the initial value problem given by equations (2.1).

Next, the following theorem gives the same result as in the previous theorem but requires for its proof the monotonicity of f with respect to x. This theorem appeared in [Hadid S. and Momani S., 1996]. Here, we give the details of its proof.

Theorem (2.5), [Hadid S. and Momani S., 1996]:

Assume that:

1. $f(x, y(x))$ is continuous in the region $0 < x_0 < x \le x_0 + \beta$, $|y| < \infty$.
2. $f(x, y(x))$ is nondecreasing in y for each fixed x.
3. The maximal solution $r(x)$ of the initial value problem given by equations (2.1) exists on the interval $0 < x_0 < x \le x_0 + \beta$ for some $\beta > 0$
4. m is a continuous function satisfying the integral inequality:

$$m(x) \le m(x_0)(x - x_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-t)^{q-1} f(t, m(t))\, dt \quad \ldots\ldots\ldots\ldots\ldots(2.8)$$

for $x \in [x_0, x_0+\beta]$. Then $m(x) \le r(x)$ for $x \in [x_0, x_0+\beta]$.

Proof:

Let z be a function defined by the right-hand side of (2.8). That is:

$$z(x) = m(x_0)(x - x_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-t)^{q-1} f(t, m(t))\, dt$$

Then, we have

$m(x) \le z(x)$ and $z^{(q)}(x) = f(x, m(x))$

Hence, from nondecreasing property on f, it follows that:

$z^{(q)}(x) \le f(x, z(x))$, $x \in [x_0, x_0+\beta]$.

Thus, from theorem (2.4) one can have

$z(x) \le r(x)$, for $x \in [x_0, x_0+\beta]$.

Hence

$m(x) \le r(x)$ for $x \in [x_0, x_0+\beta]$. ■

<u>*2.3 Finite Difference Method for Solving the Fractional Order Ordinary Differential Equations:*</u>

Consider the initial value problem that consists of the fractional differential equation:

$$y^{(q)}(x) = f(x, y(x)), \quad L \le x \le R \qquad \text{(2.9.a)}$$

Together with the initial condition:

$$y(L) = \alpha \qquad \text{(2.9.b)}$$

where $y^{(q)}(x)$ denote fractional derivative of order q of the function y and q is a positive fractional number.

In this section, we use the explicit finite difference method to solve this initial value problem. To do this, we substitute $x=x_i$ into equation (2.9.a) to get:

$$y^{(q)}(x_i) := f(x_i, y(x_i)), \qquad \text{(2.10)}$$

Where $x_i = L + i\Delta x$, i=0,1,…,n, $\Delta x = \frac{(R-L)}{n}$ and n is the number of subintervals of the interval [L,R]

Next, we recall that the left-handed shifted Grünwald estimate to the left-handed derivative is

$$\frac{d^q f(x)}{dx^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{n} g_k f(x-(k-1)\Delta x)$$

Where n is the number of subintervals of the interval [L, R], $g_0=1$ and $g_k=(-1)^k \frac{q(q-1)...(q-k+1)}{k!}$, k = 1,2,… . Therefore

$$y^{(q)}(x_i) = \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k y(x_i - (k-1)\Delta x)$$

$$= \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k y_{i-k+1} \quad \text{...(2.11)}$$

where y_i is the numerical solution of equations (2.9), [Meerschaert M. and Tadjeran C., 2005].

By substituting equation (2.11) in equation (2.10) one can have

$$\frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k y_{i-k+1} = f(x_i, y_i), i = 0.1, \ldots, n-1$$

The resulting equation can be explicitly solved for y_{i+1} to give:

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + (\Delta x)^q f(x_i, y_i), \quad i = 0.1, \ldots, n-1 \text{.......................(2.12)}$$

Then by evaluating the above equation at each i=1, 2,…,n−1 and using the initial condition $y_0 = \alpha$ one can get the numerical solution of the initial value problem given by equations (2.9).

To illustrate this method, consider the following example:

Example (2.1):

Consider the fractional order ordinary differential equation:

$$y^{(0.5)}(x) := xy^2(x) + \frac{2.667}{\Gamma(0.5)} x^{1.5} - x^5, \quad 0 \le x \le 1$$

Together with the initial condition:

$y(0) = 0$

This example is constructed such that the exact solution of it is $y(x) = x^2$ via Riemann-Liouville definition of the left-handed fractional derivative. Here, we use the explicit finite difference method to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i = \frac{i}{2},\ i = 0,\ 1,\ 2$. Thus, the initial condition becomes $y_0 = 0$. Moreover, equation (2.12) becomes

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + 0.707\left(x_i y_i^2 + \frac{2.667}{\Gamma(0.5)} x_i^{1.5} - x_i^5 \right),\ i = 0, 1$$

By evaluating the above equation at each i=0, 1 one can get the values that are tabulated down with the comparison with the exact solution.

Table (2.1) Represents the numerical and the exact solutions for n=2 of example (2.1).

x_i	Numerical solution y_i	Exact solution $y(x_i)$
0.5	0	0.25
1	0.354	1

Second, we divide the interval [0,1] into 10 subintervals such that $x_i = \frac{i}{10},\ i = 0, 1, \ldots, 10$. Moreover, equation (2.12) becomes

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + 0.316\left(x_i y_i^2 + \frac{2.667}{\Gamma(0.5)} x_i^{1.5} - x_i^5 \right),\ \ i = 0,\ 1, \ldots, 9\,.$$

By evaluating the above equation at each i=0, 1,…,9 one can get the values that are tabulated down with the comparison with the exact solution.

Table (2.2) Represents the numerical and the exact solutions for n=10 of example (2.1).

x_i	*Numerical solution* y_i	*Exact solution* $y(x_i)$
0.1	0	0.01
0.2	0.015	0.04
0.3	0.05	0.09
0.4	0.105	0.16
0.5	0.178	0.25
0.6	0.269	0.36
0.7	0.376	0.49
0.8	0.495	0.64
0.9	0.621	0.81
1	0.744	1

Third, we divide the interval [0,1] into 100 subintervals such that $x_i = \frac{i}{100}$, $i = 0, 1, \ldots, 10$. In this case, equation (2.12) becomes

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + 0.1\left(x_i y_i^2 + \frac{2.667}{\Gamma(0.5)} x_i^{1.5} - x_i^5 \right),\ i = 0, 1, ..., 99$$

By evaluating the above equation at each i=1,2,…,99 one can get the values that are tabulated in the appendix (see program (2.1)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (2.3) Represents the numerical and the exact solutions for n=100 of example (2.1).

x_i	Numerical solution y_i	Exact solution $y(x_i)$
0.1	8.55×10^{-3}	0.01
0.2	0.037	0.04
0.3	0.086	0.09
0.4	0.154	0.16
0.5	0.242	0.25
0.6	0.35	0.36
0.7	0.477	0.49
0.8	0.622	0.64
0.9	0.783	0.81
1	0.957	1

Fourth, we divide the interval [0,1] into 170 subintervals such that $x_i = \frac{i}{170}$, $i = 0, 1, \ldots, 170$. In this case, equation (2.4) becomes

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + 0.077\left(x_i y_i^2 + \frac{2.667}{\Gamma(0.5)} x_i^{1.5} - x_i^5 \right),\ i = 0, 1, \ldots, 169$$

By evaluating the above equation at each i =1, 2,…,169 one can get the values that are tabulated in the appendix (see program (2.1)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (2.4) Represents the numerical and the exact solutions for n=170 of example (2.1).

x_i	*Numerical solution y_i*	*Exact solution $y(x_i)$*
0.1	9.136×10^{-3}	0.01
0.2	0.038	0.04
0.3	0.087	0.09
0.4	0.156	0.16
0.5	0.245	0.25
0.6	0.354	0.36
0.7	0.482	0.49
0.8	0.629	0.64
0.9	0.794	0.81
1	0.973	1

Fifth, we divide the interval [0,1] into 200 subintervals such that $x_i = \frac{i}{200},\ i = 0, 1, \ldots, 200$. In this case, equation (2.4) becomes

$$y_{i+1} = -\sum_{k=1}^{i+1} g_k y_{i-k+1} + 0.071\left(x_i y_i^2 + \frac{2.667}{\Gamma(0.5)} x_i^{1.5} - x_i^5 \right),\ i = 0, 1, \ldots, 199$$

By evaluating the above equation at each i=1,2,…,199 one can get the values that are tabulated in the appendix (see program (2.1)) with the comparison with the exact solution. Some of these results are tabulated down with the comparison with the exact solution.

Table (2.5) Represents the numerical and the exact solutions for n=200 of example (2.1).

x_i	Numerical solution y_i	Exact solution $y(x_i)$
0.1	9.264×10^{-3}	0.01
0.2	0.039	0.04
0.3	0.088	0.09
0.4	0.157	0.16
0.5	0.246	0.25
0.6	0.355	0.36
0.7	0.483	0.49
0.8	0.631	0.64

From the above tables, one can conclude that the results given in table (2.5) are more accurate than the previous results given in the other tables. Moreover, if we increase the value of n, then one can get more accurate results. But the MathCAD software package fails to be used if the value of n is greater than 170.

Conclusions and Recommendations

From the present study, we can conclude the following:

1. Many definitions for the fractional order ordinary derivatives can be extended to be valid for the fractional partial derivatives.
2. The existence of the extremal solutions ensures that the ordinary differential equation does not have a unique solution.
3. Finding analytic solutions for the fractional partial differential equations are so difficult in many cases.
4. The finite difference method gave the numerical solutions of the fractional differential equations and it depends on the Grünwald estimate for the fractional derivatives.
5. The explicit finite difference method is better than the implicit finite difference method, but it is conditionally stable.

For future work the following problems could be recommended:

1. Study the stability of the finite difference method given by equation (3.22).
2. Solve the initial value problem given by equations (3.22) via the implicit finite difference method.
3. Use the finite difference method for solving non-linear fractional order partial differential equation.
4. Discuss the extremal solution for the fractional order partial differential equation.
5. Devote other types of the fractional order partial differential equation towards this work.
6. Use other definitions for fractional derivatives.

Contents

Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. In recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. It is well believed today that fractional calculus is quite an irreplaceable means for description and investigation of classical and quantum complex dynamical system. In simple words, the fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical system. Dealing with fractional derivatives is not more complex than with usual differential operators, [Ross B., 1974].

Many authors and researchers studied the fractional differential equations such as Greer H. in 1859, studied the fractional differentiation, Letnikov A. in 1868, presented the theory of differentiation of fractional order, Hardy G. in 1918, gave some properties of fractional integrals, Davis H. in 1924, gave fractional operations as applied to a class of Volterra integral equation, Fabian W. in 1936, studied fractional calculus, and Love E. and Young L. in 1938, presented fractional integration by parts [Nishimoto K.,1983], Kober H. in 1940 discussed fractional integrals and derivatives, Zygmund A. in 1945, gave some theorems for fractional derivatives, Erdélyi A. and Snddon I. in 1964, gave some applications of fractional integrals, Ross

B. and Northover F. in 1976, presented the fractional calculus of the derivatives of complex order and Nishimoto K. in 1983, gave some definitions of fractional integration and derivation with some properties.

Also, many authors and researchers concerned with the fractional differential equations such as Al-Shather A. in 2003, presented some approximated solutions for the fractional delay integro-differential equations, Abdul-Razzak B. in 2004, gave some algorithms for solving fractional order Fredholm integro-differential equations, Al-Azawi S., in 2004, gave some results in fractional calculus, Al-Rahhal D. in 2005, used the numerical solutions for the fractional integro-differential equation, Gorial I. in 2005, used the finite difference method to solve the eigenvalue problems for the partial fractional differential equations, Abdul-Jabbar R. in 2005, studied the inverse problems of some fractional order integro-differential equations, Khalil E. in 2006, used the linear multi-step methods to solve some fractional order ordinary differential equations, Aziz S. in 2006 used some approximated methods for solving fractional order partial differential equations, and Al-Husseiny R. in 2006, discussed the existence of uniqueness of the solution for some fuzzy fractional order ordinary differential equations.

The purpose of this work is to generalize some definitions of the fractional order derivatives of a function with one independent variable to include functions with many independent variables.

Also, the existence of the extremal solutions of the fractional ordinary differential equations is discussed. Moreover, some numerical methods are used to solve classes of the linear fractional order ordinary and partial differential equations.

This thesis consists of three chapters:

In chapter one, we give three definitions of the fractional order ordinary derivatives, and extended them to be valid for the fractional order partial derivatives with some illustrative examples. Moreover, we present five definitions of the fractional order ordinary integrations.

In chapter two, we give some theorems for the existence and the uniqueness of the solution for the initial value problem of the fractional order ordinary differential equations and discuss the existence of the extremal solutions for the initial value problem of the fractional order ordinary differential equations and solving them via the finite difference method.

In chapter three, we examine some special types of the numerical methods; such as the finite difference method (explicit and implicit) to solve the initial-boundary value problem of the parabolic and hyperbolic for one-sided and two-sided fractional order partial differential equations.

For each method, some numerical examples are solved and computer programs are written in MathCAD (professional 2001i) software packages.

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Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science

SOME FINITE DIFFERENCE METHODS FOR SOLVING Fractional Differential Equations

A Thesis

Submitted to the College of Science, AL-Nahrain University
as a Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics

By
Laylan Sidqi Mohamed Ghareeb
(B.Sc., Al-Nahrain University, 2004)

Supervised by
Dr. Ahlam Jameel Khalil

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جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

طرق الفروقات المنتهية لحل بعض المعادلات التفاضلية الكسورية

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل

ليلان صدقي محمد غريب

(بكلوريوس علوم، جامعة النهرين، ٢٠٠٤)

بإشراف

د.احلام جميل خليل

تشرين الثاني ٢٠٠٧ رمضـان ١٤٢٨

الأهداء

الى من لم يبخلا علي بسنين عمرهم ويستنير دربي بدعائهم
الى أمي وأبي

الى زوجي العزيز الذي ساندني وشجعني

الى من شجعني وغمرني بحنانه وهم اعز الناس الى قلبي
الى أختاي و أخي

الى كل الاصدقاء الذين وقفوا بجانبي

الى مشرفتي الفاضلة واساتذتي الاعزاء

اهدي لكم هذا الجهد عرفانا ووفاءا

ليلان صدقي محمد

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بَســـَم اللهِ الـــــَرحَمنِ الـــــَرحِيمِ

وَ عَلمَ ءَادَمَ الاسمَآءَ كُلهَا ثُمَ عَرضَهُم عَلى

المَلائِكَة ِفقَاَل أنبئُونـِي بِأسمَاءَ هَؤُلاءِ ان كُنتُم

صَادِقِينَ ۞ قَاَلُو سبحانك لاعِلمَ لَنَا الا مَا عَلمتَنَا انَكَ

انتَ العَلِيمُ الحَكِيمُ ۞

صَدَقَ اللهُ العَظيم
سورة البقرة ٣٠-٣١

المستخلص

الهدف الرئيسي من هذا العمل هو دراسة الحلول العددية لانواع خاصة من المعادلات التفاضلية باستخدام طرق الفروقات المنتهية مع استقراريتها. هذه الدراسة شملت المحاور التالية:

١. طرح بعض التعاريف للمشتقات الكسورية الاعتيادية مع تعميمها للمشتقات الكسورية الجزئية.

٢. دراسة وجود الحلول والحلول المتطرفة لانواع خاصة من المعادلات التفاضلية الكسورية الاعتيادية.

٣. أستعمال طرق الفروقات المنتهية الصريحة والضمنية مع استقراريتها لحل انواع خاصة من المعادلات التفاضلية الكسورية الاعتيادية والجزئية ذات الجهة الواحدة وذات الجهتين.