Abstract

The main purpose of this work can be divided in to three aspects. First, a study of the existence and uniqueness of the solution for special types of linear operator equations, namely the Lyapunov equation.
Second, a discussion of the range for the quaii-Jordan*-derivation.
Third, some special types of Lyapunov equation, namely stein equation.

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CHAPTER ONE

The Continuous-Time Lyapunov Equation

Introduction

The purpose of this chapter is to recall some definitions, basic concepts and some properties which are important for the discussion of our later results. Also, we study the nature of the solution of the continuous-time Lyapunov equation for special types of operators as well as the study of the range of τ_A, where $\tau_A(X) = A^*X + XA, X \in \beta(H)$ and A is fixed.

This chapter consists of five sections. In section one, we recall the definition of an operator equation and introduce some types of operator equations namly the Sylvestor operator equation, the discrete and continuous time Lyapunov equations with their generalization.

In section two, we study the nature of the solution of the continuous-time Lyapunov equation $A^*X + XA = W$ for special types of operators.

In section three, we recall the definition of an invariant subspace. Also give some remarks and examples.

In section four, we study the range of τ_A, where $\tau_A(X) = A^*X + XA, \quad X \in \beta(H)$ and A is fixed and we prove that τ_A is not a derivation and not a Jordan $*$-derivation. Also, we give some new theorems, corollaries, remarks and examples on the range of τ_A.

In section five, we study the nature of the solution of more general continuous-time Lyapunov equations for special types of operators.

1.1 Some types of Operator Equations

In this section, we give the definition of an operator equation. Also, some types of operator equations are introduced.

We start this section by the following definition

Definition (1.1.1), [12]:

An equation of the form

$$L(X) = C \tag{1.1}$$

is said to be an operator equation, where L and C are known operators defined on a Hilbert space H and X is the unknown operator that must be determined.

Remark (1.1.1), [12]:

In eq.(1.1), if the operator L is linear then this equation is said to be linear operator equation. Otherwise, it is non–linear operator equation.

Now, we introduce some kinds of linear operator equations

(1) A special type of the linear operator equation takes the form

$$AX - XB = Y \tag{1.2}$$

where A, B and Y are given operators defined on a Hilbert space H and X is the unknown operator that must be determined. This equation is called the Sylvester operator equation, [6] and [22].

The author in [6] discussed the necessary and sufficient conditions for the solvability of this equation. Furthermore, he gave equivalent conditions for the solvability of this equation for special types of operators A and B.

(2) The operator equation of the form

$$X - F^{*}XF = Q \tag{1.3}$$

is called the discrete–time Lyapunov equation, or the Stein equation, where F and Q are known operators defined on a Hilbert space H and X is the unknown operator that must be determined, [5].

(3) The operator equation of the form

$$A^{*}X + XA = W \qquad (1.4)$$

is called the continuous-time Lyapunov equation, where A and W are given operators defined on a Hilbert space H and X is the unknown operator that must be determined, [4] and [23].

(4) The author in [7] studied the necessary and sufficient conditions for the solvability of the operator equation of the form

$$AX + XA = W \qquad (1.5)$$

where A and W are known operators defined on a Hilbert space H and X is the unknown operator that must be determined.

(5) The operator equations of the forms

$$A^{*}X + tXA = W \qquad (1.6)$$

$$A^{*}X + XA + tA^{*^{1/2}}XA^{1/2} = W \,, \qquad (1.7)$$

and

$$A^{2}X + XA^{2} + tAXA = W \,, \qquad (1.8)$$

are generalization of the continuous – time Lyapunov equations, where t is any scalar,[5] and [7].

(6) The operator equation of the form

$$X^{*}A + AX = W \qquad (1.9)$$

where A and W are given operators defined on a Hilbert space H, and X is the unknown operator that must be determined.

1.2 Continuous–Time Lyapunov Equations:

The continuous-time Lyapunov equations, are much studied because of its importance in differential equations and control theory,[4]. Therefore, devote the studying of the continuous-time Lyapunov equations.

The question now is pertinent, does eq.(1.4) have a solution? If yes, is it unique?.

To answer this question, recall the Sylvester–Rosenblum theorem, [6].

Sylvester - Rosenblum Theorem (1.2.1):

If A and B are operators in $\beta(H)$ such that $\sigma(A)\cap\sigma(B)=\phi$, then eq.(1.2) has a unique solution X for every operator Y.

According to the above theorem, we have the following corollaries.

Corollary (1.2.1):

If A is an operator such that $\sigma(A^*)\cap\sigma(-A)=\phi$, then eq.(1.4) has a unique solution X for every operator W.

Corollary (1.2.2):

If $\sigma(A^*)\cap\sigma(-A)=\phi$ then the operator $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$ is defined on $H\oplus H$ is similar to the operator $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$

Proof:

Since $\sigma(A^*)\cap\sigma(-A)=\phi$ then by Sylvester–Rosenblum theorem, eq.(1.4) has a unique solution X. Also

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}=\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

But $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ is invertible so $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$.♦

The converse of corollary (1.2.2) is not true in general as we see in the following example.

Example (1.2.1):

Let H=ℓ_2(C), that is, $\ell_2(C)=\left\{X=(x_1,x_2,\ldots) \,\middle|\, \sum_{i=1}^{\infty}|x_i|^2<\infty, x_i\in C\right\}$ Define

A:H$\longrightarrow$H by A(x_1,x_2,…)=(x_1,0,0,…). Thus $A^*=A$.

Consider eq.(1.4), where $W(x_1,x_2,...)=(0,x_1,0,...)$. Then $X=U$ is a solution of this equation since

$$(A^*X+XA)(x_1,x_2,...)=(A^*U+UA)(x_1,x_2,...)$$

$$\begin{aligned} A^*(0,x_1,x_2,...)+U(x_1,0,0,...) &= (0,0,0,...)+(0,x_1,0,0,...) \\ &= (0,x_1,0,0,...)=Wx \end{aligned}$$

On the other hand, U is the solution of eq.(1.4) and

$$\begin{bmatrix} I & U \\ 0 & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}=\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}\begin{bmatrix} I & U \\ 0 & I \end{bmatrix}$$

Therefore, $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$, Moreover 0 is an eigenvalue of A and $X=(0,x_2,...)$ is the associated eigenvector. Therefore, $0\in\sigma(A^*)\cap\sigma(-A)$ and hence $\sigma(A^*)\cap\sigma(-A)\neq\phi$.♦

Remark (1.2.1):

If the condition $\sigma(A^*)\cap\sigma(-A)=\phi$ fails to be satisfied then eq.(1.4) may have one solution, an infinite number of solutions or it may have no solution.

To understand this, consider the following examples.

Example (1.2.2):

Consider eq.(1.4), where A=iI and W=0. Clearly $\sigma(A^*)\cap\sigma(-A)\neq\phi$. Moreover, any X∈β(H) is a solution of eq.(1.4).

Example (1.2.12):

Consider eq.(1.4), where $A=\begin{bmatrix}0 & 0\\ 0 & 3\end{bmatrix}$ and $W=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}$. It is clear that $0\in\sigma(A)$. Therefore $\sigma(A)\cap\sigma(-A)\neq\phi$. Moreover, it is easy to check eq.(1.4) has no solution.

Now, we study the nature of the solution of eq.(1.4) for special types of operators.

Recall that an operator A is sAaid to be self–adjoint if $A^*=A$,[9,pp. 147]. The following remark is very usful here.

Remark (1.2.2):

If A and W are self–adjoint operators, then eq.(1.4) may or may not have a solution. Moreover, if it has a solution then it may be non self–adjoint. This remark can easily be observed in the matrices.

Next if A and W are self–adjoint operators, what conditions can one put on A (or W) to ensure the existence of self–adjoint solution for eq.(1.4)?

The following theorem gives one such conditions.

Theorem (1.2.2):

Let A and W be self-adjoint operators which are also positive. If $0\notin\sigma(A)$ then the solution X of eq.(1.4) is self – adjoint.

Proof:

Since $0\notin\sigma(A)$ then it is easy to see that $\sigma(A)\cap\sigma(-A)=\phi$ and hence eq.(1.4) has a unique solution X. Moreover, $AX^*+X^*A=W$

Therefore X^* is also a solution of eq.(1.4). By the uniqueness of the solution one gets $X = X^*$.♦

Recall that an operator A is said to be skew–adjoint in case $A^* = -A$, [9, pp.48].

Remark (1.2.3):

If A is a askew–adjoint operator, then $\sigma(A^*) \cap \sigma(-A) = \sigma(-A)$ and hence eq.(1.4) may have a solution or may not. If it has a solution then it may be askew–adjoint or may not.

To illustrate this remark, consider the following example

Example (1.2.4):

If A is a askew–adjoint operator and W is a zero operator then eq(1.4) has an infinite number of solutions. For example, X=A is a skew-adjoint solution of eq.(1.4). On the other hand X=I is not.

Now, the following proposition shows that if A and W are skew–adjoint and if eq. (1.4) has a unique solution then this solution is a skew–adjoint.

Proposition (1.2.1):

If A and W are skew–adjoint operators and eq.(1.4) has only one solution then this solution is also skew–adjoint.

Proof:

Since $A^* = -A$ and $W^* = -W$ then it is easy that to check $A^*(-X^*) + (-X^*)A = W$ and since the equation has only one solution then $X^* = -X$.♦

Next, recall that an operator A is said to be normal incase $A^*A = AA^*$, [9, ppP154].

The following remark is useful here.

Remark (1.2.4):

Consider eq.(1.4), where the solution of it exists. If A and W are normal operators then this solution is not necessarily normal.

This fact can easily be seen in the following example.

Example (1.2.5):

Let H=$\ell_2(C)$, consider eq.(1.4), where A=iI and W=0. Therefore, $iIX + XiI = 0$. It is easy to check the unilateral shift operator defined by

$U(x_1, x_2,\ldots)=(0,x_1, x_2,\ldots) \qquad \forall(x_1, x_2,\ldots)\in \ell_2(C)$

Is a solution of the above equation which is non normal operator.

There are cases in which the converse of corollary (1.2.2) holds but first we need some preliminaries.

Putnam–Fugled Theorem (1.2.3)

Assume that $M, N, T \in \beta(H)$, where M and N are normal. If $MT = TN$ then $M^*T = TN^*$.

Proof:

See [17, pp.300].

Next recall that an operator M is said to be dominant if $\|(T-z)^* x\| \le M_z \|(T-z)x\|$, for all $z \in \sigma(T)$ and $x \in H$. On the other hand, operator M is called M-hyponormal operator if $\|(T-z)^* x\| \le M\|(T-z)x\|$, for $z \in C$ and $x \in H$,[15].

In [15], the above theorem was generalized as follows.

Theorem (1.2.4):

Let M be a dominant operator and N^* is an M-hyponormal operator. Assume $MT = TN$ for some $T \in \beta(H)$ then $M^*T = TN^*$.

We also need the following lemma.

Lemma (1.2.1) [20]:

If $\begin{bmatrix} Q & R \\ S & T \end{bmatrix}$ is an invertible operator on the direct sum $H_1 \oplus H_2$ of Hilbert spaces H_1 and H_2 then $SS^* + TT^*$ is invertible on H_2.

Let us say that operators M and N satisfy Putnam- Fugled condition if $MT = TN$ for some $T \in \beta(H)$ implies that $M^*T = TN^*$.

The following theorem was proved in [21] if A and B are normal operators , we prove it for more general cases.

Theorem (1.2.5):

Let A and B be two operators that satisfy Putnam- Fugled condition. The operator equation $AX - XB = \overline{C}$ has a solution X iff $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ are similar operator on $H \oplus H$.

As a corollary, we have

Corollary (1.2.3):

If A is normal operator then eq.(1.4) has a solution if and only if $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$.

Proof:

Suppose that eq.(1.4) has a solution, then by the corollary (1.2.2) the operator $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$.

Conversely, assume similarity, then there exists an invertible operator $\begin{bmatrix} Q & R \\ S & T \end{bmatrix}$ such that

$$\begin{bmatrix} Q & R \\ S & T \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}=\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}\begin{bmatrix} Q & R \\ S & T \end{bmatrix},$$

So $QA^* - A^*Q = WS$, $-RA - A^*R = WT$, $SA^* = -AS$ and $-TA = -AT$. It follows that $-A$ commutes with both SS^* and TT^*.

Now

$$\begin{aligned} -W(SS^* + TT^*) &= (QA^* - A^*Q)S^* + (-RA - A^*R)T^* \\ &= (QA^*S^* - RAT^*) - (A^*QS^* + A^*RT^*) \\ &= (QS^* + RT^*)(-A) - A^*(QS^* + RT^*), \end{aligned}$$

by lemma (1.2.1), $SS^* + TT^*$ is invertible, moreover its inverse commutes with $(-A)$ so eq.(1.4) has the solution $X = -(QS^* + RT^*)(SS^* + TT^*)^{-1}$.♦

The following corollary follows directly from theorem (1.2.4)

Corollary (1.2.4):

If A is dominant or a M-hyponormal operator then the operator equation defined by eq.(1.4) has a solution iff $\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ and $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$ are similar operators on $H \oplus H$.

Recall that an operator T on a Hilbert space H is said to be binormal if T^*T commutes with TT^*, quasinormal if T commutes with T^*T, and $\theta-$operator if $[T^*T, T + T^*] = 0$, where

$[T^*T, T + T^*] = T^*T(T + T^*) - (T + T^*)T^*T$,[25].

In example (1.2.3), it is clear that A and W are binormal (quasinormal, $\theta-$operator).On the other hand, the unilateral shift operator is a solution which is not

Next, we recall the definition of another type of operators, namely the compact operators.

Definition (1.2.2), [9]:

An operator A is said to be compact in case, given any sequence of vectors $\{x_n\}$ such that $\|x_n\|$ is bounded, $\{Ax_n\}$ has a convergent subsequence.

Clearly, an operator A is compact iff $\|x\| \leq 1$ implies $\{Ax_n\}$ has a convergent subsequence.

Now, the following remark shows that the solution X of eq.(1.4) is not necessarily a compact operator in case A (or W) is compact.

Remark (1.2.5)

If A (or W) is compact and the solution of eq.(1.4) exists then it is not necessarily compact

As an illustration to this remark, consider the following examples.

Example (1.2.6):

Consider the equation $A^*X + XA = A^* + A$, where A is a compact operator on an infinite dimensional Hilbert space H. It is clear that $X = I$ is a solution of the above operator equation which is not compact.

Example (1.2.7):

Consider eq.(1.4) where W=0. It is clear that the zero operator is compact. Given A=iI, then X=I is a solution of eq.(1.4) which is not compact.

1.3 Invariant Subspaces

In this section, we recall the definition of an invariant subspace. Also give some propositions and remarks with some examples

Moreover, in this section, we study if the operator equation given by eq.(1.4) has a solution, when does this solution has a non-trivial invariant subspace. Also, we study the existence of non-trivial invariant subspaces for the operators X and W which are defined in the operator equation given by eq.(1.4) in case the operator A and A^* have reducing subspaces.

We start this section by recalling that a non-empty set $M \subseteq H$ is said to be a linear manifold of H if it is closed under addition and scalar multiplication which are defined on H and if this linear manifold is closed in the norm topology, then it is called a subspace,[3].

Definition (1.3.1), [3]:

Let $T: H \to H$ be a bounded linear operator on a Hilbert space H. A subspace $M \subseteq H$ is an invariant under T (or $T-$invariant) if $TM \subseteq M$.

The two trivial subspaces H and {0} are invariant subspaces under every bounded linear operator T. So, a non-trivial invariant subspace of a bounded linear operator T is a subspace M such that $0 \neq M \neq H$ and $TM \subseteq M$. The set of all invariant subspaces of a bounded linear operator T is called the Lattice of T and is denoted by Lat T. It is clear that the trivial subspaces are in Lat T for every bounded linear operator T.

Next, the definition of the common invariant subspace for a pair of bounded linear operators is given below

Definition (1.3.2), [3]:

Let $A, B \in \beta(H)$. A subspace $M \subseteq H$ is said to be a common invariant subspace of A and B if $AM \subseteq M$ and $BM \subseteq M$.

It is clear that H and $\{0\}$ are common invariant subspaces of any pair of bounded linear operators. So, by a common non–trivial invariant subspace, we mean a subspace M such that $0 \neq M \neq H$ and is an invariant subspace under both A and B.

Now, the next remark shows that if A and W have a common non-trivial invariant subspace M and the solution X of eq.(1.4) exists then it is not necessary that M is an invariant subspace under solution X.

Remark (1.3.1):

Consider eq.(1.4), if the operators A and W have a common non-trivial invariant subspace M then M is not a necessary invariant under the solution of this equation if it exists.

To see this, consider the following example.

Example (1.3.1):

Consider $A:\Re^2 \longrightarrow \Re^2$, where $A=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}$. W=I. It is easy to check that $M=\left\{\begin{pmatrix}x_1\\ 0\end{pmatrix} \middle|\ x_1\in\Re\right\}$ is a common non-trivial invariant subspace under the operators A and W. After simple computations, the solution of eq.(1.4) in this case takes the form $X=\begin{bmatrix}\frac{1}{4} & -\frac{1}{16}\\ \frac{-1}{16} & \frac{9}{32}\end{bmatrix}$. On the other hand,

$XM=\begin{bmatrix}\frac{1}{4} & -\frac{1}{16}\\ \frac{-1}{16} & \frac{9}{32}\end{bmatrix}\begin{bmatrix}x_1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{4}x_1\\ \frac{-1}{16}x_1\end{bmatrix}\not\subset M$, therefore M is a non invariant subspace under the solution X.

Moreover, the following remark is very useful here.

Remark (1.3.2):

Consider eq.(1.4). If the operators A and W have non-trivial invariant subspaces as M_1 and M_2 respectively then M_1 and M_2 may not be non-trivial invariant subspaces under the solution of eq.(1.4) if it exists.

To explain this remark, consider the following example.

Example (1.3.2):

Consider $A:\Re^2 \longrightarrow \Re^2$, where $A=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}$. $W=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$. It is easy to note that $M_1=\left\{\begin{pmatrix}x_1\\ 0\end{pmatrix}\middle|\, x_1\in\Re\right\}$ and $M_2=\left\{\begin{pmatrix}0\\ x_2\end{pmatrix}\middle|\, x_2\in\Re\right\}$ are non-trivial invariant subspaces under operators A and W respectively. After simple computations, the solution of eq.(1.4) in this case takes the form $X=\begin{bmatrix}\frac{1}{4} & -\frac{1}{16}\\ \frac{-1}{16} & \frac{34}{64}\end{bmatrix}$. On the other hand,

$XM_1=\begin{bmatrix}\frac{1}{4} & -\frac{1}{16}\\ \frac{-1}{16} & \frac{34}{64}\end{bmatrix}\begin{bmatrix}x_1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{4}x_1\\ \frac{-1}{16}x_1\end{bmatrix}\not\subset M_1$ if $x_1\neq 0$. Therefore, M_1 is a non invariant subspace under solution X.

Also, $XM_2=\begin{bmatrix}\frac{1}{4} & -\frac{1}{16}\\ \frac{-1}{16} & \frac{34}{64}\end{bmatrix}\begin{bmatrix}0\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{-1}{16}x_2\\ \frac{34}{64}x_2\end{bmatrix}\not\subset M_2$ if $x_2\neq 0$. Therefore, M_2 is a non invariant subspace under solution X.

The question now arises: if the operator equation given by eq.(1.4) has a solution, when does this solution have a non-trivial invariant subspace. The following proposition gives an answer.

Proposition (1.3.1):

Let $\sigma(A^*) \cap \sigma(-A) = \phi$. If A and A^* have a common non-trivial invariant subspace M, then for each $W \in \beta(H)$, W has M as a non-trivial invariant subspace iff X has M as a non-trivial invariant subspace where X is a solution of the operator equation $A^*X + XA = W$.

Proof:

Since A and A^* have a common non-trivial invariant subspace M, then the above operator equation can be written as

$$\begin{bmatrix} A_1^* & A_2^* \\ 0 & A_4^* \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}, \text{ where } A_1, A_2, A_4, X_1, X_2, X_3, X_4, W_1, W_2, W_3 \text{ and } W_4 \in \beta(H)$$

Therefore,

$$A_1^*X_1 + A_2^*X_3 + X_1A_1 = W_1$$
$$A_1^*X_2 + A_2^*X_4 + X_1A_2 + X_2A_4 = W_2$$
$$A_4^*X_3 + X_3A_1 = W_3$$

and

$$A_4^*X_4 + X_3A_2 + X_4A_4 = W_4$$

Since $\sigma(A^*) \cap \sigma(-A) = \phi$, thus $\sigma({A_4}^*) \cap \sigma(-A_1) = \phi$ and by using Sylvester–Rosenblum theorem one can get $W_3 = 0$ iff $X_3 = 0$. This completes the proof. ♦

Next recall that, a subspace M is said to reduce the operator T if M and $M^{\perp}$ are both invariant subspaces under T and in this case, the operator T can be represented as a diagonal matrix $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}$ where $T_1 : M \longrightarrow M$ and $T_4 : M^{\perp} \longrightarrow M^{\perp}$,[16].

Now, we study the existence of non-trivial invariant subspace for the operators X and W which are defined in the above operator equation in case the operators A and A^* have reducing subspaces.

Proposition (1.3.2):

Let $\sigma(A^*) \cap \sigma(-A) = \phi$. If A and A^* have a common a non-trivial reducing subspace M then for each $W \in \beta(H)$, W has M as a non-trivial invariant subspace iff X has M as a non-trivial invariant subspace where X is a solution of the operator equation $A^*X + XA = W$. Moreover M, is a reducing subspace of W iff it is a reducing subspace of X.

Proof:

Since A and A^* have a common non-trivial reducing subspace M then the above operator equation can be written as

$$\begin{bmatrix} A_1^* & 0 \\ 0 & A_4^* \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$$

Therefore

$$A_1^*X_1 + X_1A_1 = W_1$$
$$A_1^*X_2 + X_2A_4 = W_2$$
$$A_4^*X_3 + X_3A_1 = W_3$$

and

$$A_4^*X_4 + X_4A_4 = W_4$$

Since $\sigma(A^*) \cap \sigma(-A) = \phi$, thus $\sigma(A_1^*) \cap \sigma(-A_4) = \phi$ and $\sigma(A_4^*) \cap \sigma(-A_1) = \phi$. Therefore by using Sylvester–Rosenblum theorem one can gets $W_2 = 0$ iff $X_2 = 0$ and $W_3 = 0$ iff $X_3 = 0$. This completes the proof. ♦

Next, recall that an operator A on a Hilbert space is said to be isometry in case $A^{*}A = I$, an algebraic operator if there exists a non-zero polynomial P such that P(A)=0 and nilpotent if there exists a positive integer n such that $A^{n} = 0$; [9, pp.140]. We introduce the following corollary.

Corollary (1.3.1):

Let $\sigma(A^{*}) \cap \sigma(-A) = \phi$ and assume that A is normal. If one of the following conditions is satisfied, then for each $W \in \beta(H)$, W and X have a common non-trivial invariant subspace where X is a solution of eq.(1.4).

1. A^{*} is nilpotent operator.
2. A has an eignvalue .
3. A and A^{*} are algebraic operators.
4. A^{*} is a non-scalar isometry operator.

Proof:

In case 1,2,3 and 4, the operators A and A^{*} have a common non-trivial invariant subspace, see [3].

Recall that an operator $T \in \beta(H)$ is said to be hyponormal if $T^{*}T \geq T\,T^{*}$. [9, pp.161]

Now, consider the following proposition.

Proposition (1.3.3):

Let $\sigma(A^{*}) \cap \sigma(-A) = \phi$. If A is a hyponormal operator then for each $W \in \beta(H)$, W and X have a common invariant subspace , where X is a solution of eq.(1.4).

Proof:

Since A is a hyponormal operator then the null space of A is common invariant subspace of A and A^*, [16, pp.117] and the result follows from proposition (1.3.1). ♦

1.4 On the Range of τ_A

In this section, we discuss the injectivity of the map $\tau_A : \beta(H) \longrightarrow \beta(H)$ and show that in general, the map τ_A is not necessary one-to-one. Also, we study the range of τ_A when A is hyponormal.

Recall that, a linear mapping τ from a ring R to it self is called a derivation, if $\tau(ab) = a\tau(b) + \tau(a)b$, for all a, b in R ,[12].

Define the mapping $\tau : \beta(H) \longrightarrow \beta(H)$ by

$$\tau(X) = \tau_A(X) = A^*X + XA\,,\ X \in \beta(H).$$

where A is a fixed operator in β(H).

It is clear that the map τ_A is a linear map, in fact

$$\begin{aligned}\tau_A(\alpha X_1 + \beta X_2) &= A^*(\alpha X_1 + \beta X_2) + (\alpha X_1 + \beta X_2)A \\ &= \alpha\tau_A(X_1) + \beta\tau_A(X_2).\end{aligned}$$

Also, the map τ_A is bounded, since

$$\|\tau_A\| = \|A^*X + XA\| \le \|A^*X\| + \|XA\| \le \|X\|[\ \|A^*\| + \|A\|].$$

But $A \in \beta(H)$ and $\|A^*\| = \|A\|$, thus $\|\tau_A(X)\| \le M\|X\|$ where $M = 2\|A\|$, so τ_A is bounded.

The following remark shows that the mapping τ_A is not a derivation.

Remark (1.4.1):

Since $\tau_A(XY) = A^*(XY) + (XY)A$ for all $X, Y \in \beta(H)$ and $X\tau_A(Y) = XA^*Y + XYA$. Also $\tau_A(X)Y = A^*XY + XAY$. Then one can deduce that $\tau_A(XY) \neq X\tau_A(Y) + \tau_A(X)Y$.

To prove this, let $H = \ell_2(C)$ and A=U where U is a unilateral shift operator. Then $\tau_U(X) = BX + XU$, where B is the bilateral shift operator.

In this case τ_U is not derivation. To see this, consider

$\tau_U(IU) = \tau_U(U) = BU + U^2$ and

$$\begin{aligned} I\tau_U(U) + \tau_U(I)U &= \tau_U(U) + \tau_U(I)U \\ &= BU + U^2 + (B+U)U \\ &= 2BU + 2U^2 \end{aligned}$$

Now, let R be a ring. Recall that, a Jordan derivation $f : R \longrightarrow R$ is defined to be an additive mapping satisfying $f(a^2) = af(a) + f(a)a$. Now, let R be ∗-ring, i.e, a ring with involution ∗. A linear mapping $\tau : R \to R$ is called Jordan ∗-derivation, if for all $a, b \in R$ and $\tau(a^2) = a\tau(a) + \tau(a)a^*$,[2]. If R is a ring with the trivial involution, $a^* = a$, then the set of all Jordan ∗-derivations is equal to the set of all Jordan derivations.

It is easily seen that the mapping $\tau : \beta(H) \longrightarrow \beta(H)$ defined by $\tau(X) = \tau_A(X) = A^*X + XA$, $X \in \beta(H)$ is not Jordan ∗-derivation. To see this, see the above example.

Next, we discuss the injectivity of the map τ_A and show that, in general, the map $\tau_A : \beta(H) \longrightarrow \beta(H)$ is not necessary one-to-one.

Proposition (1.4.1):

Consider the map $\tau_A(X) = A^* X + XA$. If A is a skew-adjoint operator then τ_A is not one to one.

Proof:

Since A is a skew-adjoint operator, then $\ker \tau = \{X \in \beta(H) : AX = XA\}$. Therefore $I \in \ker \tau_A$ and thus τ_A is not one to one. ♦

Now, we have the following simple proposition.

Proposition (1.4.2):

1. $\text{Rang }(\tau_A)^* = \text{Rang}(\tau_A)$
2. $\alpha \text{ Rang }(\tau_A) = \text{Rang }(\tau_A)$

Proof:

1. Since $Rang(\tau_A)^* = \{X^* A + A^* X^*, X \in \beta(H)\}$. Then ,

$Rang(\tau_A)^* = \{A^* X_1 + X_1 A,\ X_1 \in \beta(H)\}$ where $X_1 = X^*$. Therefore

$\text{Rang}(\tau_A)^* = Rang(\tau_A)$.

2. $\alpha \text{ Rang}(\tau_A) = \{\alpha\,(A^* X + XA), X \in \beta(H)\}$

$$= \{A^* \alpha X + \alpha XA,\ X \in \beta(H)\}$$

Let $X_1 = \alpha X$, then

$\alpha \text{ Rang}(\tau_A) = \{\, A^* X_1 + X_1 A,\ X_1 \in \beta(H)\} = \text{Rang}(\tau_A)$.♦

Next we study the range of τ_A when A is hyponormal.

Now, we prove the following theorem.

Corollary (1.4.5):

Let $A \in \beta(H)$ such that A^* is a hyponormal operator and let $p(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$. If $p(A^*) \in \text{Rang}(\tau_A)$ then for each

$\lambda \in \sigma(A^*) \cap \sigma(-A)$ such that λ is an approximate eignvalue of -A and A^* with the same corresponding approximate eigenvector then $p(\lambda) = 0$.

Proof:

Let $\in > 0$ and let $\lambda \in \sigma(A^*) \cap \sigma(-A)$, such that λ is an approximate eigenvalue with the same approximate eigenvector. Then there exists a unit vector $f \in H$ such that

$$\|A^* f - \lambda f\| < \frac{\in}{4} \qquad (1.10)$$

and

$$\|Af - \lambda f\| < \frac{\in}{4} \qquad (1.11)$$

since λ is an approximate eignvalue for A^* then $p(\lambda)$ is an approximate eignvalue for $p(A^*)$ with the same corresponding eignvector, [18].

Thus $\|p(A^*)f - p(\lambda)f\| < \frac{\in}{4}$. Hence, $\|p(A^*)f - p(\lambda)f\|\|f\| < \frac{\in}{4}$.

So by Schwartz inequality

$$|< p(A^*)f, f > -p(\lambda)| < \frac{\in}{4} \qquad (1.12)$$

And since $p(A^*) \in \text{Range}(\tau_A)$, then there is $X \in \beta(H)$ such that $p(A^*) - (A^*X + XA) = 0$. So $< (p(A^*) - (A^*X + XA))g, g >= 0 \ \forall g \in H$.

In particular $< (p(A^*) - (A^*X + XA))f, f >= 0$. Thus

$$< p(A^*)f, f > - < A^*Xf, f > - < XAf, f >= 0 \qquad (1.13)$$

Since $\in$ is arbitrary, then we can assume in eq.(1.11) that

$$\|Af - \lambda f\|\|X^*\| < \frac{\in}{4}$$

By Schwartz inequality we get $|< Af, X^*f > -\lambda < f, X^*f >| < \frac{\in}{4}$

or

$$\left|\lambda < f, X^*f > - < Af, X^*f >\right| < \frac{\in}{4} \qquad (1.14)$$

Now the hyponormality of A^* implies that $\|Ay\| \le \|A^*y\|$ for each $y \in H$.

Hence, $\|X\|\|Af - \lambda f\| < \frac{\in}{4}$. Again by Schwartz inequality we get

$$\left|< Xf, Af > - < Xf, \lambda f >\right| < \frac{\in}{4}$$

or

$$\left|< A^*Xf,, f > -\lambda < f, X^*f >\right| < \frac{\in}{4} \qquad (1.15)$$

By adding the equation (1.13) and the inqualities (1.12), (1.14) and (1.15) we get $|p(\lambda)| < \frac{3}{4}\in \quad \forall \in > 0$. So $p(\lambda) = 0$.♦

Remark (1.4.2):

We can show that under the same conditions and the same argument if $p(A) \in Rang(\tau_A)$ then $p(\lambda) = 0$.

Corollary (1.4.1):

Let $A \in \beta(H)$ such that A^* is a hyponormal operator and let $p(X) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 X + a_0$. If $\sigma(A^*) \cap \sigma(-A)$ contains a number of approximate eignvalues with the same corresponding approximate eigenvectors, greater than the degree of p, then if $p(A^*) \in Rang(\tau_A)$, $p(X) = 0 \quad \forall\ X \in \beta(H)$..

Proof:

Suppose that $p(A^*) \in Range(\tau_A)$ then using theorem (1.4.1) for each $\lambda \in \sigma(A^*) \cap \sigma(-A)$ satisfying the hypothesis in the theorem, $p(\lambda) = 0$ which means that the number of zeros of $p(X)$ is greater than its degree so $p(X) = 0$.♦

Let f be a complex analytic function defined on a ball B_r of center zero and radius r , so by Taylors theorem $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and this series converges uniformally in $|z|<r$. Let A be an element in $\beta(H)$ such that $\|A\|<r$, the series $\sum_{n=0}^{\infty}a_n(A)^n$ converges in $\beta(H)$, [16]. Define $f(A^*)=\sum_{n=0}^{\infty}a_n(A^*)^n$. Since $|\sigma(A^*)|\leq\|A^*\|$, then $\sigma(A^*)\subseteq B_r$.

Now, we prove the following corollary.

Corollary (1.4.2):

Let $A\in\beta(H)$ such that A^* is a hyponormal operator and let f be a complex analytic function defined on a ball B_r of center zero and radius r such that $\|A^*\|<r$. If $f(A^*)\in \text{Rang}(\tau_A)$ then if λ is an approximate eignvalue of $-A$ and A^* with the same corresponding approximate eignvector, then $f(\lambda)=0$.

The proof of this corollary is similar to theorem(1.4.1).

Remark (1.4.3):

One can show that under the same conditions in corollary (1.4.2) and by the same way, if $f(A^*)\in \text{Rang}(\tau_A)$ then $f(\lambda)=0$.

Proposition (1.4.3):

Let $A\in\beta(H)$ such that A^* is a hyponormal operator and let f be a complex analytic function defined on a ball B_r of center zero and radius r such that $\|A^*\|<r$. If $\sigma(A^*)\cap\sigma(-A)$ contains a sequence of approximate eignvalues $\{\lambda_n\}_{n=1}^{\infty}$ with the same corresponding eignvectors, for both

$-A$ and A^* such that $\lambda_n \longrightarrow \lambda$ and $|\lambda| < r$ then if $f(A^*) \in Rang(\tau_A)$, f(z)=0, $\forall$ $z \in Br$.

Proof:

Let $f(A^*) \in \text{Rang}(\tau_A)$ then by corollary (1.4.2) if $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence approximate of eigenvalues with the same corresponding eigenvectors for both A and A^* then $f(\lambda_n) = 0$ for each n and since $\lambda_n \longrightarrow \lambda$ and $|\lambda| < r$ so by Taylor's theorem, f(z)=0 $\forall z \in \text{Br}$.

Following [4], we say that an operator T is pseudonormal if $Tx = \lambda x$ for some $x \in H$, $\lambda \in C$ then $T^* x = \bar{\lambda} x$, and following [21] we say that an operator T is *-paranormal if $\|T^* x\|^2 \leq \|T^2 x\|$ for every unit vector x in H.

Remarks (1.4.4):

(1) Every dominant operator, in particular, every M hyponormal, hyponormal, normal operator is a pseudonormal operator [14].

(2) Every *-paranormal is a pseudonormal operator [14].

Now, we can prove by the same way and the same condition that if $p(A) \in Rang(\tau_A)$ then $p(\lambda) = \dfrac{2\lambda < Xf, f >}{\|f\|^2}$.

Theorem (1.4.2):

Let $A \in \beta(H)$ such that A^* is a pseudonormal operator in $\beta(H)$ and let $p(X) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 X + a_0$. If $p(A^*) \in \text{Rang}(\tau_A)$ then if λ is an eigenvalue of $-A$ and A^* with the same eigenvector f, then $p(\lambda) = \dfrac{2\lambda < Xf, f >}{\|f\|^2}$, where $x \in \beta(H)$.

Proof:

Let λ is an eigenvalue of A and A^* with the same eigenvector f, then $A^*f - \lambda f = 0$ and $Af - \lambda f = 0$ and since A^* is pseudonormal then $Af - \bar{\lambda} f = 0$.

Since $p(A^*) \in Rang(\tau_A)$ then there exists $X \in \beta(H)$ such that $p(A^*) = A^*X + XA$ so $<(p(A^*) - (A^*X + XA)\, g, g >= 0, \forall g \in H$. In particular

$$<(p(A^*) - (A^*X + XA)f, f >= 0 \qquad (1.16)$$

But λ is an eigenvalue of A^* then $p(\lambda)$ is eigenvalue of $p(A^*)$ with the same eigenvector $p(A^*)f = p(\lambda)f$. So eq.(1.16) becomes

$$< p(A^*)f, f > - < A^*Xf, f > - < XAf, f >= 0$$

That is

$$< p(\lambda) f, f > - < Xf, \lambda f > - \lambda < Xf, f >= 0$$

Thus

$$p(\lambda) < f, f > - \lambda < Xf, f > - \lambda < Xf, f >= 0$$

Therefore, $p(\lambda) = \dfrac{2\lambda < Xf, f >}{\|f\|^2}$.♦

Finally, the following remark is essential here.

Remark (1.4.5):

The pervious results given in section(1.2),(1.3) and (1.4) of this chapter can be easily generalized to includes the generalization of the continuous – time Lyapunou equation given by eq.(1.6).

CHAPTER THREE

On the Discrete-Time Lyapunov Equation

Introduction

Recall that, the operator equation of the form

$$X - F^{*}XF = Q\,,$$

is called the discrete-time Lyapunov equation, or the Stein equation, [5].

In general this equation may have one solution, infinite set of solutions or no solution.

This chapter consists of three sections:

In section one, we study the nature of the solution of this equation has a unique solution and we study the nature of it for special type of operators.

In section two, we study some of the properties of $\mu_F(X)=X-F^*XF$, $X\in\beta(X)$.

Also, we prove that the range of μ_F is linear manifold of operators in $\beta(H)$ and show that the inverse of Q isn't necessary in Range (μ_F), in case $Q\in$ Range (μ_F) and Q is invertible.

In section three, we introduce some notation, theorems, corollaries and remarks about the spectrum of μ_F. Also we study the relation of the spectra of L_F and R_{F^*} with the spectra F and F* respectively.

3.1 The Discrete-Time Lyapunov Equation

Recall that, the operator equation

$$X-F^*XF=Q \qquad (3.1)$$

is said to be the discrete-time Lyapunov equation, where F and Q are known operators in β(H) and X is an unknown operator that must be determined,[5].

In general this equation may have one solution, infinite set of solutions, or no solution .

To see this, consider the following examples.

Example (3.1.1):

Let H= $\ell_2(C)$. The unilateral shift operator U on $\ell_2(C)$ is defined by

$$U(x_1, x_2,\ldots)=(0,x_1, x_2,\ldots)$$

Consider eq.(3.1) where F=U and Q=0. Therefore eq.(3.1) reduces to the operator equation

$$X-BXU=0$$

Where B is the bilateral shift operator which defined by

$$B (x_1,x_2,\ldots)=(x_2, x_3, x_4,\ldots)$$

and U*=B. The above discrete-time Lyapunov equation has infinite solutions, say X=0, X=U and X=cI where c is an arbitrary constant.

The following example shows that eq.(3.1) may have no solution.

Example (3.1.2):

Consider eq.(3.1), where F= $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and Q= $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$. It is easy to check that eq.(3.1) has no solution.

The following example shows that the solution of eq.(3.1) may be unique.

Example (3.1.3):

Consider eq.(3.1), take $F=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q=\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. After simple computations, one gets $X=\begin{bmatrix} -\frac{1}{3} & 2 \\ 3 & -1 \end{bmatrix}$, therefore, in this case eq.(3.1) has only one solution.

Now, we study the nature of the solution of eq.(3.1) for special types of operators.

Proposition (3.1.1):

Let Q and F be self adjoint operators. Assume that eq.(3.1) has a unique solution X then this solution is self adjoint.

Proof:

Consider eq.(3.1). Therefore $X^*-F^*X^*F=Q^*$. Since Q and F are self adjoint operator, thus $X^*-F^*X^*F=Q$. Also $X-F^*XF=Q$. but eq.(3.1) has a unique solution , thus, $X=X^*$, hence X is a self-adjoint operator.♦

Next, we study the nature of the solution of eq.(3.1) for normal (binormal, quasi normal, θ-operator).

The following example shows that the solution of eq.(3.1) is not necessarily normal (binormal, quasi normal, θ-operator) in case the known operators F and Q are normal (binormal, quasi normal, θ-operators).

Example (3.1.4):

Consider eq.(3.1).Take F=iI, Q=0. It is clear that this equation has the following solutions X=0, X=cI, X=U , where c is an arbitrary number and U is non normal (nonbinormal, nonquasi normal, non θ-operator).

Next, we study the nature of the solution of eq.(3.1) when it exists for another type of operators, namely the compact operators.

The following example shows that the solution of eq.(3.1), if it exists, is not necessarily compact operator incase the known operator Q is compact.

Example (3.1.5):

Consider eq.(3.1). Take F=U and Q=0. Clearly, the solution of this equation is X=cI, where c is an arbitrary number and I is not compact operator in infinite dimensional space.

Remark (3.1.1):

The following example shows that the subspace M is not necessarily invariant subspace under a solution of eq. (3.1) in case the operator, F and Q have common nontrivial invariant subspace M.

Example (3.1.6):

Consider eq.(3.1), and put Q=0, F=U. we get X-BXU=0. we have the following solutions X=0, X=cI, X=0 where c is an arbitrary numbers.

Remark (3.1.2):

Consider eq.(3.1). If the operator F has nontrivial invariant subspace M_1 and the operator Q has also nontrivial invariant subspace M_2 then a solution of this equation may have no invariant subspace.

To explain this remark, see example (1.3.2).

3.2 The Map μ_F

Let β(H) denote the algebra of all bounded operators on an infinite dimensional separable complex Hilbert space H. For F in β(H), let $\mu_F:\beta(H) \longrightarrow \beta(H)$ be a mapping defined by $\mu_F(X)=X-F^*XF$, $X\in\beta(H)$ then Rang $(\mu_F)=\{X-F^*XF: X\in\beta(H)\}$.

In this section, we study some of the properties of μ_F. It is clear that μ is a linear map, in fact $\mu(\alpha X_1+\beta X_2)=(\alpha X_1+\beta X_2)-F^*(\alpha X_1+\beta X_2)F$

$$=\alpha X_1+\beta X_2-\alpha F^*X_1F-\beta F^*X_2 F$$

$$=\alpha\mu(X_1)+\beta\mu(X_2).$$

Now, we have the following simple proposition.

Proposition (3.2.1):

(1) Rang $(\mu_F)^*$=Rang(μ_F)

(2) αRang (μ_F)=Rang(μ_F) $\forall\ \alpha\in C$

(3) The operator μ_F is bounded.

Proof:

(1) Rang$(\mu_F)^*=\{(X-F^*XF)^*, X\in\beta(H)\}$

$$=\{X^*-F^*X^*F, X\in\beta(H)\}.$$

Let $X_1=X^*$, then $(\text{Rang}(\mu_F))^* =\{ X_1-F^* X_1F, X_1\in\beta(H)\}$=Rang$(\mu_F)$.

(2) α Rang$(\mu_F)=\{\alpha(X-F^*XF), X\in\beta(H)\}$

$$=\{\alpha X-F^*\alpha XF, X\in\beta(H)\}.$$

Let $X_1=\alpha X$, then α Rang$(\mu_F)=\{X_1-F^*X_1 F, X_1\in\beta(H)\}$=Rang (μ_F).

(3) $\|\mu_F(X)\| = \|X - F^*XF\| \leq \|X\| + \|F^*XF\| \leq \|X\|(1 + \|F\|^2)$. But $F \in \beta(H)$, thus $\|\mu_F(X)\| \leq M\|X\|$, where $M = (1 + \|F\|^2)$, so μ_F is bounded.♦

Now, we prove that the range of μ_F is linear manifold of the operators in $\beta(H)$.

Proposition (3.2.2):

Rang(μ_F) is a linear mainfold of operators in $\beta(H)$.

Proof:

It is known that Rang($\mu_F(X)$)={Q | $\mu_F(X)=Q$, $X\in\beta(H)$}

(1) $0\in$ Rang(μ_F) since $X = 0 \in \beta(H)$ and $\mu_F(0) = 0$.

(2) Let Q_1, $Q_2 \in$ Rang(μ_F) we must prove $Q_1 - Q_2 \in$ Rang(μ_F)

Therefore, $\exists X_1 \in \beta(H)$ such that $\mu_F X_1 = Q_1$ and $\exists X_2 \in \beta(H)$ such that $\mu_F X_2 = Q_2$. Thus, $\mu_F(X_1 - X_2) = (X_1 - X_2) - F^*(X_1 - X_2)F$

But, $= X_1 - X_2 - F^*X_1F + F^*X_2F$

$= (X_1 - F^*X_1F) - (X_2 - F^*X_2F) = Q_1 - Q_2$.

Then $X_1 - X_2 \in \beta(H)$ such that $\mu_F(X_1 - X_2) = Q_1 - Q_2$. So $Q_1 - Q_2 \in$ Rang(μ_F).

Therefore, Rang (μ_F) is a linear manifold of operators.♦

Now, we state the following proposition.

Proposition (3.2.3):

If $Q \in \text{Rang}(\mu_F)$ then so does Q^*.

Proof:

Let $Q \in \text{Rang}(\mu_F)$ so $\exists X \in \beta(H)$ such that $\mu_F(X)=Q$. To prove $Q^* \in \text{Rang}(\mu_F)$ so $\exists\ y \in \beta(H)$ such that $\mu_F(y)=Q^*$. But $\mu_F(X)=Q$, thus $(X-F^*XF)^*=Q^*$. Hence $X^*-F^*X^*F=Q^*$. Since $X \in \beta(H)$ then $X^* \in \beta(H)$ so $Q^* \in \text{Rang}(\mu_F)$. ♦

The following example shows that the inverse of Q (denoted by Q^{-1}) is not necessarily in Rang (μ_F) in case $Q \in$ Rang (μ_F) and Q is invertible.

Example (3.2.1):

Consider eq.(3.1) $Q=\begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ and $F=\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Therefore $Q^{-1} = -\frac{1}{4}\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$. After simple computations one can get $X=\begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ is solution of eq.(3.1). In fact every matrix of the form $X=\begin{bmatrix} a & b \\ b & 1+4a \end{bmatrix}$ is a solution, where a and b are any numbers.

Now, if $Q \in \text{Rang}(\mu_F)$ then Q^n does not necessarily belong to $\text{Rang}(\mu_F)$, for all $n \in Z_+$. This fact can easily be shown in matrices.

Remark (3.2.1):

If $Q_1, Q_2 \in \text{Rang}(\mu_F)$ then Q_1Q_2 is not necessarily in $\text{Rang}(\mu_F)$, this can be seen in the following example.

Example (3.2.2):

Take $Q_1=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, therefore $Q_1 \in Rang(\mu_F)$ where $F=\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $Q_2=\begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$, therefore $Q_2 \in Rang(\mu_F)$. After simple computations one gets $Q_1Q_2=\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, but $Q_1Q_2 \notin Rang(\mu_F)$.

Remark (3.2.2):

$Rang(\mu_F)$ is not a subalgebra of $\beta(H)$.

Recall that $\mu_F(X)$ is one-to-one iff $\ker(\mu_F)=\{0\}$. Let $X \in \ker(\mu_F)$ so $\mu_F(X)=0$, then $X=F^*XF$ also if F is an isometric operator so X=I then $\ker(\mu_F) \neq \{0\}$. Therefore, μ_F is not one-to-one and thus it is non invertible.

To illustrate this fact, consider the following example.

Example (3.2.3):

Consider eq.(3.1) so $\mu_F=I-R_{F^*}L_F$, where $R_{F^*}=F^*X$ and $L_F=XF$ $\mu_F(X)=0$ therefore X=0.

$X-F^*XF=0$so $X=F^*XF$. Let F=U so I=BIU=I and 0=BOU=0. $F^*F=BU=I$. then U is an isometric operator, therefore F is an isometric operator and μ_F is not one-to-one and so is non invertible.

Recall that, an operator T on a Hilbert space H is said to be projection operator in case $T^2=T$ and $T^*=T$, [9,pp. 147].

Proposition (3.2.1):

If F is a non zero projection operator then μ_F is not one to one and so is not invertible.

Proof:

The prove μ_F is not one to one, we must prove that if x_1-x_2=$F^*(x_1$-$x_2)F$ then x_1-x_2=0 take x_1-x_2=F therefore F=F.

The following example shows that μ_F is not onto in general

Example (3.2.4):

Take $F=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ so $F^*F\neq I$. Then F is not isometric. Assume $X=F^*XF$, Let $X=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Then $\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ After simple computation one gets $X=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in\ker(\mu_F)$. Therefore μ_F is not one-to-one and is not onto.

We state the following remark.

Remark (3.2.3):

If F is a non isometric operator then μ_F may not be one-to-one.

To see this fact, consider the above example.

Remark (3.2.4):

Recall that, $\overline{\text{Rang}(\mu_F)}$ is self-adjoint where the closure is in the norm topology, if $\overline{\text{Rang}(\mu_F)}$=Rang($\mu_{F^*}$)

But we observed, in proposition (3.2.1), that Rang(μ_F)*= Rang(μ_F) so $\overline{\text{Rang}(\mu_F)}$ is self-adjoint if $\overline{\text{Rang}(\mu_F)}=\overline{\text{Rang}(\mu_F)}^*$

3.3 The Spectrum of μ_F:

In this section, we study the relation between the sepectra of L_F and R_{F*} with the sepactra of F and F* respectively.

Notation (3.3.1):

For A, B$\in\beta$(H). X is any Banach space. Let

$\sigma(A)+\sigma(B)=\{\alpha+\beta : \alpha\in\sigma(A), \beta\in\sigma(B)\}$

$\sigma_\pi(A)+\sigma_\pi(A)=\{\alpha+\beta: \alpha\in\sigma_\pi(A), \beta\in\sigma_\pi(B)\}$

$\sigma_\pi(A)\sigma_\pi(B)=\{\alpha\beta: \alpha\in\sigma_\pi(A), \beta\in\sigma_\pi(B)\}$

$\sigma_\delta(A)+\sigma_\delta(B)=\{\alpha+\beta:\alpha\in\sigma_\delta(A), \beta\in\sigma_\delta(B)\}$

$\sigma_\delta(A)\sigma_\delta(B)=\{\alpha\beta: \alpha\in\sigma_\delta(A), \beta\in\sigma_\delta(B)\}$

In the following lemma we give the relation between the parts of spectrum of the sum of two operators A and B defined on a Banach space X and the sum of the spectrum.

Lemma (3.3.2), [10]:

If A, B $\in\beta$(H), and AB=BA, then

(i) $\sigma_\pi(A+B)\subseteq\sigma_\pi(A)+\sigma_\pi(B)$.

(ii) $\sigma_\pi(AB)\subseteq\sigma_\pi(A)\sigma_\pi(B)$.

We have this immediate corollary.

Corollary (3.3.1), [10]:

If A, B $\in\beta$(H) and AB=BA then

(i) $\sigma_\delta(A+B)\subseteq\sigma_\delta(A)+\sigma_\delta(B)$.

(ii) $\sigma_\delta(AB)\subseteq\sigma_\delta(A)\sigma_\delta(B)$.

In [26] Herro,D.A. proved that if X is a Hilbert space then corollary (3.3.1) becomes

Remark (3.3.1)

Let $A,B \in \beta(H)$ and $AB=BA$ then

(1) $\sigma_\pi(A+B)=\sigma_\pi(A)+\sigma_\pi(B)$, $\sigma_\pi(AB)=\sigma_\pi(A)\sigma_\pi(B)$.

(2) $\sigma_\delta(A+B)=\sigma_\delta(A)+\sigma_\delta(B)$, $\sigma_\delta(AB)\subseteq\sigma_\delta(A)\sigma_\delta(B)$.

Let $\beta(\beta(H))$ be the Banach algebra of operators on $\beta(H)$ considerd as a Banach space. Define the mappings L_F and R_F from $\beta(H)$ into $\beta(\beta(H))$ such that $L_F(X)=XF$, $R_{F^*}(X)=F^*X$. It is clear that for all $F\in\beta(H)$, $L_F R_{F^*}=L_F R_{F^*}$.

Now, we return to our problem, we want to relate the spectra of L_F and $R_{F^*}$with the spectra of F and F* respectively.

Theorem (3.3.1):

Let $F\in\beta(H)$, then

(1) $\sigma_\pi(R_{F^*})=\sigma_\pi(F^*)$ and $\sigma_\pi(L_F)=\sigma_\delta(F)$

(2) $\sigma_\delta(R_{F^*})=\sigma_\delta(F^*)$ and $\sigma_\delta(L_F)=\sigma_\pi(F)$.

Proof:

To prove this theorem, we have to look several possibilities, but we prove some of them and the rest can be proved by similar ways. To prove $\sigma_\pi(R_{F^*})=\sigma_\pi(F^*)$ we have to show that $\sigma_\pi(F^*)\subseteq\sigma_\pi(R_{F^*})$ and $\sigma_\pi(R_{F^*})\subseteq\sigma_\pi(F^*)$.

Let $\lambda\in\sigma_\pi(F^*)$, without loss of generality, we may take $\lambda=0$. Suppose that F* is not bounded below on H, but R_{F^*} is bounded below on $\beta(H)$. This means that there exists m>0 such that $\|R_{F^*}(X)\| \geq m\|X\|$ for all $X\in\beta(H)$.

Since F* is not bounded below, one can choose a unit vector $f \in H$ such that $\|F^* f\| < m$. Now we set $X = f \otimes g$ where g is a unit vector and $f \otimes g$ represents the operator on H which is defined by $(f \otimes g)h = \langle h,g \rangle f$ for all h in H. It is easily checked that $\|f \otimes g\| = \|f\| \|g\|$. Thus $\|X\| = \|f \otimes g\| = \|f\| \|g\| = 1$. On the other hand, $\|F^* X\| \|F^*(f \otimes g)\| = \|F^* f \otimes g\| = \|F^* f\| \|g\| < m = m\|X\|$, a contradiction.

To prove that $o \in \sigma_\delta(L_F)$ implies $o \in \sigma_\pi(F)$. If $o \notin \sigma_\pi(F)$, then F maps H, 1-1 and invertibly onto rang(F). Then the equation XF=C has a unique solution in $\beta(H)$ for any $C \in \beta(H)$, that is $X = CF^{-1}$, where F^{-1} is the operator whose restriction to rang(F) is the inverse of F and whose restriction to $\text{Rang}(F)^\perp$ is zero. This means that $\text{Rang}(L_F) = \beta(H)$ so $0 \notin \sigma_\delta(L_F)$.

Now, let $F \in \beta(H)$, $R_{F^*}L_F = L_F R_{F^*}$ and R_{F^*} and $L_F \in \beta(H)$. So

$\sigma_\pi(\mu_F) = \sigma_\pi(I - R_{F^*}L_F) = \sigma_\pi(I) - \sigma_\pi(R_{F^*}L_F)$ (by spectral mapping theorem)

$= \sigma_\pi(I) - \sigma_\pi(R_{F^*})\sigma_\pi(L_F)$ by remark (3.3.1)

$= 1 - \sigma_\delta(F^*)\sigma_\pi(F)$.

Also $\sigma_\delta(\mu_F) = \sigma_\delta(I - R_{F^*}L_F) = \sigma_\delta(I) - \sigma_\delta(R_{F^*}L_F)$ by remark (3.3.1)

$= 1 - \sigma_\pi(F^*)\sigma_\delta(F)$.

and this completes the proof.♦

Corollary (3.3.2):

(1) $\sigma_\pi(\mu_F) = 1 - \sigma_\delta(F^*)\sigma_\pi(F)$.

(2) $\sigma_\delta(\mu_F) = 1 - \sigma_\pi(F^*)\sigma_\delta(F)$.

Now, we state the main theorem

Theorem (3.3.2):

$\sigma(\mu_F)=1-\sigma(F^*)\sigma(F)$

Proof:

Since $\beta(\beta(H))$ is the Banach algebra of endmorphisms on a Banach space $\beta(H)$ then $\sigma(\mu_F)=\sigma(I-R_{F^*}L_F)=1-\sigma(R_{F^*})\sigma(L_F)$. Since $\sigma(R_{F^*})=\sigma(F^*)$ and $\sigma(L_F)=\sigma(F)$ so $\sigma(\mu_F)=1-\sigma(F^*)\sigma(F)$. ♦

CHAPTER TWO

On The Range Of ρ_A

Introduction

Assume that $\rho_A : \beta(H) \longrightarrow \beta(H)$ is amapping which is defined by $\rho_A(X) = X^*A + AX$, $X \in \beta(H)$. The range of ρ_A is denoted by R_A and defined by $R_A = \{X^*A + AX : X \in \beta(H)\}$.This mapping is said to be quasi-Jordan $*$-derivation.

In this chapter, we study the range of the quasi-Jordan *-derivation ρ_A, when A is either normal or compact operator. Furthermore, we study some properties of ρ_A, like, surjectivity and density .

This chapter consists of four sections.

In section one, a study of the surjectivity of the map ρ_A is introduced. We prove that ρ_A is not surjective in general. In fact if $A - A^*$ or $A + A^*$ is non invertible then $R_A \neq \beta(H)$.

In section two, a study of the range of ρ_A when A is either a normal or compact operator. We prove that, if A is a self-adjoint operator, (skew-adjoint) which is an invertable operator then the range of ρ_A is equal to the set of all skew-adjoint (self-adjoint) operators. Moreover, if A is a skew-adjoint operator, we prove that the range of ρ_A is equal to the set of all self-adjoint operators.

In section three, we define the quasi-commutator of two operators in $\beta(H)$. Also, if H is a complex Hilbert space, then $\vartheta^*(H) = \beta(H)$. And in case infinite dimensional real Hilbert space, we show that every operator in $\beta(H)$ can be written as a sum of two quasi-commutators.

In section four, we study the density of the range of ρ_A with respect to the norm topology.

In fact, we show that $R_A = \{XA + AX^* : X \text{ is a skew - adjoint}\}$ is not a norm dense in $\beta(H)$.

We start this chapter by studying the surjectivity of the map ρ_A.

2.1 Surjectivity of the map ρ_A

For each $A \in \beta(H)$, let $\rho_A : \beta(H) \longrightarrow \beta(H)$ be a mapping which is defined by $\rho_A(X) = X^*A + AX$,. For every $A \in \beta(H)$, we call ρ_A a quasi-Jordan$_*$-derivation. Let R_A denote the range of the quasi–Jordan $_*$-derivation ρ_A, that is, $R_A = \{X^*A + AX : X \in \beta(H)\}$.

In this section we study the surjectivity of the map ρ_A.

First the following simple proposition is given.

Proposition (2.1.1):

(1) $(R_A)^* = R_{A^*}$.

(2) $i R_A = \{AX - X^*A : X \in \beta(H)\}$

Proof

(1) $(R_A)^* = \{(X^*A + AX)^* : X \in \beta(H)\}$

$= \{A^*X + X^*A^* : X \in \beta(H)\} = \{X^*A^* + A^*X : X \in \beta(H)\} = R_{A^*}$

(2) $i R_A = \{i(X^*A + AX) : X \in \beta(H)\}$

$= \{(iX^*)A + A(iX) : X \in \beta(H)\}$

$= \{-(iX)^*A + A(iX) : X \in \beta(H)\} = \{-X_1^*A + AX_1 : X_1 \in \beta(H)\}$

$= \{AX_1 - X_1^* A : X_1 \in \beta(H)\}$ where $X_1 = iX$. ♦

The question that may be asked is, whether the map ρ_A is 1-1 for any fixed operator A in $\beta(H)$.

The following proposition gives an answer.

Proposition (2.1.2): ρ_A is not 1-1.

Proof:

It is known that $Ker\ \rho_A = \{X \in \beta(H) : XA^* + AX = 0\}$. If $X^* = -X$, then $\text{Ker}\ \rho_A = \{X \in \beta(H) : -XA + AX = 0\}$. It is clear that $I \in \text{Ker}\ \rho_A$. ♦

The following theorems shows that ρ_A is not generally surjective.

Theorem (2.1.1):

Let $A \in \beta(H)$ such that $A - A^*$ is non invertible, then $iI \notin R_A$ and hence $R_A \neq \beta(H)$.

Proof:

Assume that there exists an operator $X \in \beta(H)$ such that

$$X^* A + AX = iI \qquad (2.1)$$

Hence

$$A^* X + X^* A^* = -iI \qquad (2.2)$$

By subtracting eq.(2.2) from eq.(2.1), one can get

$$X^* A + AX - A^* X - X^* A^* = 2iI .$$

It follows that

$$X^*(A - A^*) + (A - A^*)X = 2iI .$$

Thus

$$X^*(A - A^*) - (X^*(A - A^*))^* = 2iI .$$

By assuming that $B = A - A^*$, then it is easily seen that B is skew-adjoint and $X^*B - (X^*B)^* = 2iI$. Therefore $X^*B = [iI + (X^*B)^*] + iI$.Let $C = iI + (X^*B)^*$ then $C = X^*B - iI$. Also, $C^* = [iI + (X^*B)^*]^* = -iI + X^*B = C$.

Thus the spectrum of C, shown by $\sigma(C)$, consists of real numbers only. On the other hand, since B is non invertible, so is X^*B, and hence $0 \in \sigma(X^*B)$. But $C = X^*B - iI$, hence $-i \in \sigma(C)$ which is a contradiction.♦

<u>Theorem (2.1.4):</u>

Let $A \in \beta(H)$ such that $A + A^*$ is non invertible, then $I \notin R_A$, and hence $R_A \neq \beta(H)$.

<u>Proof:</u>

Assume that there exists an operator $X \in \beta(H)$ such that

$$X^*A + AX = I \qquad (2.3)$$

Hence

$$A^*X + X^*A^* = I \qquad (2.4)$$

By adding eq.(2.3) and eq.(2.4), one can get

$$X^*(A + A^*) + (X^*(A + A^*))^* = 2I.$$

By assuming that $B = A + A^*$, then it is clear that B is self-adjoint and $X^*B = [I - (X^*B)^*] + I$. Let $C = I - (X^*B)^*$ then $C = X^*B - I$. Also,

$C^* = [I - (X^*B)^*]^* = I - X^*B = -C$.

Thus $\sigma(C)$ consists of the pure imaginary numbers only. On the other hand, since B is non invertible, so is X^*B, and hence $0 \in \sigma(X^*B)$. But

$C = X^* B - I$, hence $-1 \in \sigma(C)$ which is a contradiction. Therefore $R_A \neq \beta(H)$. ♦

2.2 On the Range of ρ_A

In this section, we study the range of ρ_A for special types of operator A. This study includes the compact operators and the normal operators.

2.2.1 On The Range Of ρ_A In Case A Is A Normal Operator

This section given a study of the range of ρ_A in case A is a special type of operators, namely the normal operator.

We start this section by the following proposition.

Proposition (2.2.1): $R_I = \zeta(H)$ where $\zeta(H)$ is the set of all self-adjoint operators defined on H.

Proof:

It is clear that $R_I = \{X^* + X \mid X \in \beta(H)\}$. Since X^*+X is self-adjoint operator then $R_I \subseteq \zeta(H)$.

Conversely let $Y \in \zeta(H)$, by assuming $X=\frac{1}{2}Y$ then $X^*+X=Y \in R_I$. Therefore $R_I=\zeta(H)$.

Next, we give a generalization of the above proposition to include another special type of normal operators, namely the self-adjoint operator.

Proposition (2.2.2):

Let A be self-adjoint operator, then $R_A \subseteq \zeta(H)$.

Proof:

Let $Y \in R_A$, then there exists an operator $X \in \beta(H)$ such that $Y = X^*A + AX$. Hence $Y^* = A^*X + X^*A^*$. Since A is self-adjoint, one can get $Y^* = AX + X^*A = Y$. Thus Y is self-adjoint. Therefore $R_A \subseteq \zeta(H)$.♦

The question now is pertinent, does $\zeta(H) \subseteq R_A$ for fixed operator $A \in \beta(H)$?

The answer is negative, in fact, if A=0 where 0 is the zero operator then $R_o=\{0\}$ and hence $\xi(H) \neq \{0\}$.

On the other hand the following proposition shows that if A is any self-adjoint operator, then one can get the same result that is $\zeta(H) \not\subset R_A$.

Proposition (2.2.3):

If A is a self-adjoint operator which is non–invertible, then $\xi(H) \not\subset R_A$.

Proof:

Since A is self-adjoint, then $A + A^* = 2A$, and since A is non-invertible, then so does $A + A^*$. Thus by theorem (2.1.4), $I \notin R_A$.But $I \in \xi(H)$ therefore $\xi(H) \not\subset R_A$.♦

Next, what conditions can one put it on the self-adjoint operator A that gives the relation $\zeta(H)=R_A$?

The following proposition gives one of the such conditions.

Proposition (2.2.4):

If A is a self-adjoint operator which is invertible, then $R_A = \zeta(H)$.

Proof:

Let $Y \in \zeta(H)$. To prove $Y \in R_A$, take $X = \frac{1}{2}A^{-1}Y$, then $X^*A + AX = \frac{1}{2}Y^*\left(A^{-1}\right)^*A + \frac{1}{2}AA^{-1}Y$. Since Y is self-adjoint operator, then $X^*A + AX = \frac{1}{2}Y(A^{-1})^*A + \frac{1}{2}AA^{-1}Y$. But A is self-adjoint, then $X^*A + AX = \frac{1}{2}YA^{-1}A + \frac{1}{2}AA^{-1}Y = Y$. Thus $\zeta(H) \subseteq R_A$ and by using proposition (2.2.2), one can get $R_A = \zeta(H)$. ♦

Now, we study the nature of R_A in case A is another special type of normal operators, namely the skew-adjoint operator.

Proposition (2.2.5):

If A is a skew-adjoint operator, then $R_A \subseteq \gamma(H)$. Where $\gamma(H)$ denoted the set of all skew-adjoint operator.

Proof:

Let $Y = R_A$. Then there exist an operator $X \in \beta(H)$ such that $Y = X^*A + AX$. Hence $Y^* = A^*X + X^*A^*$.Since A is a skew-adjoint , one can get $Y^* = (-A)X + X^*(-A)$. Therefore , $-Y^* = X^*A + AX$. Thus $-Y^* = Y$ and hence is a skew-adjoint operator. Therefore, $Y \in \gamma(H)$.♦

Now, we study R_A in case A is a normal operator. But before that, we need the following definition and remarks.

Definition (2.2.1), [2]: A nonempty subset $S \subseteq \beta(H)$ is said to be self-adjoint set if for each $A \in S$ implies $A^* \in S$.

Remarks (2.2.1), [2]:

(1) A nonempty set $S \subseteq \beta(H)$ is self-adjoint iff $S = S^*$.

(2) If S is a self-adjoint subset of $\beta(H)$ then so does $\bar{S}$.

The next theorem is a modification of a theorem proved by Molnar L. in [13]. It shows that if A is a normal operator and $\overline{R_A}$ is a self-adjoint subset of β(H), then so does R_A.

Theorem (2.2.1):

If $A \in \beta(H)$ is a normal operator, then the following statements are equivalent :

(1) R_A is a self-adjoint subset of β(H).

(2) $\overline{R_A}$ is a self-adjoint subset of β(H).

(3) $\sigma(A) \subseteq \Re \cup i\Re$

Proof:

(1) $\Rightarrow$ (2): follows from remarks (2.2.7).

(2) $\Rightarrow$ (3): it is clear that $\left(\frac{1}{2}I + \frac{1}{n}\right)^* A + A\left(\frac{1}{2}I + \frac{1}{n}\right) \longrightarrow A$ as $n \longrightarrow \infty$.

Thus by proposition (2.1.1), $A \in i\overline{R_A}$ and hence $-iA \in \overline{R_A}$ .On the other hand, since $\overline{R_A}$ is self-adjoint, then $\overline{R_A} = (\overline{R_A})^*$. Hence $\overline{R_A} = (\overline{(R_A)^*}) = \overline{R}_{A^*}$, thus $\overline{R_A} = \overline{R}_{A^*}$ and $-iA \in \overline{R}_{A^*}$, hence $A \in i\overline{R}_{A^*}$.

Next, let $\varepsilon > 0$ and let $\lambda \in \sigma(A)$. Then λ is an approximate eigenvalue for A, thus there exists a unit vector $y \in H$ such that

$$\|Ay - \lambda y\| < \frac{\varepsilon}{4} \quad , \quad \|y\| = 1 \qquad (2.5)$$

This implies that $\|Ay - \lambda y\|\|y\| < \frac{\varepsilon}{4}$. Thus by Schwarz inequality

$$\left|< Ay-\lambda y, y >\right| < \frac{\varepsilon}{4}.$$

Hence

$$\left|< Ay, y > -\lambda < y, y >\right| < \frac{\varepsilon}{4}.$$

Thus

$$\left|< Ay, y > -\lambda\right| < \frac{\varepsilon}{4} \tag{2.6}$$

Moreover, since $A \in i\overline{R}_{A^*}$,then there exists X in $\beta(H)$ such that

$$\left\|A-(X^*A^*+A^*X)\right\| < \frac{\varepsilon}{4}.$$

Hence

$$\left\|A-X^*A^*-A^*X\right\|\left\|y\right\| < \frac{\varepsilon}{4}$$

So by Schwarz inequality,

$$\left|< (A-X^*A^*-A^*X)y, y >\right| < \frac{\varepsilon}{4}.$$

Thus

$$\left|< Ay, y > - < A^*y, Xy > - < Xy, Ay >\right| < \frac{\varepsilon}{4} \tag{2.7}$$

Note that, since ε is arbitray, then we may assume in eq.(2.5) that

$$\left\|X^*\right\| \left\|Ay-\lambda y\right\| < \frac{\varepsilon}{4} \tag{2.8}$$

Since A is normal, then $\left\|Az\right\| = \left\|A^*z\right\|$ for all $z \in H$.

It follows that

$$\left\|A^*y-\overline{\lambda}y\right\|\left\|X^*\right\| < \frac{\varepsilon}{4} \tag{2.9}$$

From ineq.(2.8) and ineq.(2.9) one can get

$$\left|< X^*y, Ay-\lambda y >\right| < \frac{\varepsilon}{4}$$

and $\left| < A^* y - \bar{\lambda} y, X^* y > \right| < \frac{\varepsilon}{4}$

Hence

$$\left| < Xy, Ay > + \bar{\lambda} < y, X^* y > \right| < \frac{\varepsilon}{4} \qquad (2.10)$$

and

$$\left| < A^* y, Xy > + \bar{\lambda} < y, Xy > \right| < \frac{\varepsilon}{4} \qquad (2.11)$$

By adding the inequalities (2.6), (2.7), (2.10) and (2.11) and after simple computations one can have

$$\left| 2 < Ay, y > - \lambda + \bar{\lambda}(< Xy, y > + < y, Xy >) \right| < \varepsilon .$$

Note that $< Xy, y > + < y, Xy > =$r and $2 < Ay, y > = r_1$. Hence

$$\left| r_1 - \lambda + \overline{\lambda r} \right| < \varepsilon \text{ for each } \varepsilon > 0 .$$

Now, we check that $\lambda \in \Re \cup i\Re$. In fact, let $\lambda = a + ib$ .Thus

$$\left| r_1 - (a + ib) + (a - ib) r \right| < \varepsilon$$

It follows that,

$$\left| r_1 - a - ib + ar - ibr \right| < \varepsilon ,$$

$$\left| r_1 + (r - 1)a - i(1 + r)b \right| < \varepsilon$$

since ε arbitrary, then

$$\left| r_1 + (r - 1)a - i(1 + r)b \right| = 0$$

Hence

$$r_1 + (r - 1)a = 0$$

and

$$-(1 + r)b = 0 .$$

It is clear that if $b \neq 0$ then then (1+r)=0 which implies that $r = -1$ which is absurd. Thus either $a = 0$ or $b = 0$ and hence $\lambda \in \Re \cup i\Re$.♦

Now, we study the range of ρ_A in case A is compact.

2.2.3 On The Range Of ρ_A In Case A Is A Compact Operator

This section gives a study of R_A in case A is a compact operator.

We start this section by the following proposition.

Proposition (2.2.6),[13]:

R_A cannot contains a non trivial ideal of $\beta(H)$.

From this result we can deduce that R_A does not contain the ideal of compact operators and hence R_A does not contain the ideal of finite rank operators, But there exists a compact operator A such that $\wp(H) \subseteq \overline{R}_A$ and thus $\overline{R}_A = \wp(H)$, where $\wp(H)$ denote the set of all compact operators defined on H.

Before giving the definition of this operator we need the following definition and lemma,[13].

Lemma (2.2.1),[13]:

Let $\{x_i\}$ and $\{y_i\}$ be two orthonormal sequence of vectors, and $\{\lambda_i\}$ be a sequence of complex numbers that converges to 0. Then operator $T = \sum_i \lambda_i x_i \otimes y_i$ is compact.

Proof:

Let $T_n = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$, $n = 1,2,\ldots$. Since T_n is a finite rank operator $\forall n$, then T_n is compact $\forall n$. Also,

$$\|T - T_n\| = \left\| \sum_i \lambda_i x_i \otimes y_i - \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \right\|$$

$$= \left\| \sum_{i>n} \lambda_i x_i \otimes y_i \right\| = \sup_{i>n} |\lambda_i|$$

But $\{\lambda_i\}$ converges to 0, thus $\|T - T_n\| \longrightarrow 0$ as $n \to \infty$. Hence T is compact being a limit of a sequence of finite rank operators. ♦

Next, we give the definition of the operator A such that $\wp(H) \subseteq \overline{R_A}$.

Definition (2.2.3),[13]:

Let $\{e_n\}$, $n \in N$, be a complete orthonormal set in H. For a real number q, $|q| < 1$, define the operator A as follows

$$A = \sum_n q^n e_{n+1} \otimes e_n$$

The following result gives some properties of operator A.

Proposition (2.2.7),[13]:

(1) A is compact.

(2) $\gamma(H) \subseteq \overline{R}_A$.

The following proposition shows that if operator A is compact then I does not belong to the range of ρ_A.

Proposition (2.2.8):

If A is a compact operator. In $\beta(H)$, then $I \notin R_A$.

Proof:

Since A is compact, this implies that $X^*A + AX$ is compact for each $X \in \beta(H)$. Therefore, if $I \notin R_A$, then this implies that I is compact which is contradiction. Thus $I \notin R_A$. ♦

2.3 Quasi-Commutator Operators

In this section, we give a definition of the quasi-commutator operator. Also we prove that $\beta(H) = \vartheta^*(H)$ where ϑ^* is the set of all quasi-commutator operators on H.

We start this section by the following definitions

Definition (2.3.1):

Let A, X∈β(H). The quasi-commutator of A and X is defined to be the operator $X^*A + AX$.

Definition (2.3.2):

An operator Y∈β(H) is said to be quasi-commutator on H if there exists two operators X and A in β(H) such that $Y = Y^*A + AX$.

The following remarks are useful.

Remarks (2.3.3):

(1) If Y is a quasi-commutator, then αY and Y^* are also, where $\alpha \in C$.

(2) If Y is a quasi-commutator on H, then $Y \oplus Y$ is a quasi-commutator on $H \oplus H$. In fact, if $Y = X^*A + AX$

then

$$\begin{bmatrix} X^* & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

$$= \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} = Y \oplus Y .$$

Now, the following proposition shows that every operator in β(H) is quasi-commutator.

Proposition (2.3.4):

β(H)=ϑ^*(H) where ϑ^*(H) denoted the set of all quasi-commutator opertors on H.

Proof:

Let Y∈β(H). Define operators X and A as $X = -\frac{1}{2}I$ and $A = -Y$,

then

$$X^*A + AX = \left(-\frac{1}{2}I\right)(-Y) + (-Y)\left(-\frac{1}{2}I\right) = Y$$

Hence $\beta(H) = \vartheta^*(H)$. ♦

2.4 Density Of R_A In the Norm Topology

In this section we study the density of the range of $\rho_A\ (R_A)$ with respect to the norm topology.

We start this section by recalling the following definitions:

Definitions (2.4.1), [13]:

(1) A family S of operators in $\beta(H)$ is said to be dense in the operator norm topology, or uniformally dense, if for each operator $A \in \beta(H)$, there exists a sequence of operators $A_n \in S$ such that $A_n \longrightarrow A$ in the norm of $\beta(H)$, that is $\|A_n - A\| \longrightarrow 0 \quad \text{as} \quad n \to \infty$.

(2) A family S of operators in $\beta(H)$ is said to be dense in the strong topology, or strongly dense, if for each operator $A \in \beta(H)$, there exists a sequence of operators $A_n \in S$ such that, $A_n X \to AX$ strongly for all $X \in H$, that is $\|A_n X - AX\| \longrightarrow 0$ as $n \to \infty$ for all $X \in H$.

(3) A family S of operators in $\beta(H)$ is said to be dense in the weak topology, or weakly dense, if for each operator $A \in \beta(H)$, there exists a sequence of operators $A_n \in S$ such that, $A_n X \to AX$ weakly for all $X \in H$, that is $|< A_n x, y > - < Ax, y >| \to 0 \quad as \quad n \to \infty$ for all $x, y \in H$.

Remark (2.4.1):

It is obvious that the density in the norm topology $\Rightarrow$ density in the strong topology $\Rightarrow$ density in the weak topology.

Now, we show that the range $R_A = \{XA + AX^* : X \in \beta(H) |\ X \text{ is a skew-adjoint operator}\}$ is never dense in the norm topology.

Theorem (2.4.1):

Let $A \in \beta(H)$, and $R_A = \{XA + AX^* : X \in \beta(H) \mid X$ is a skew-adjoint operator$\}$, then R_A, can not be dense in $\beta(H)$ with respect to the operator norm topology.

Proof:

Assume the contrary, that is assume R_A is dense in $\beta(H)$. Let A_1 and A_2 be the real and imaginary parts of A respectively in the Cartesian decomposition of A, thus $A_1 = \frac{1}{2}\left(A + A^*\right)$ and $A_2 = \frac{1}{2i}\left(A + A^*\right)$. It is known that decomposition $A = A_1 + iA_2$ is unique,[13].

Now, let B_1, B_2 be any pair of skew-adjoint operators on H, and let $B = B_1 + iB_2$. Since R_A is dense in $\beta(H)$, then there exists a sequence of operators $\{X_n\}$ in $\beta(H)$ such that $X_n A + AX_n^* \longrightarrow B = B_1 + iB_2$. Since X is a skew-adjoint therefore:

$$X_n A - AX_n^* \longrightarrow B = B_1 + iB_2$$

Thus,

$$X_n(A + iA_2) - (A_1 + iA_2)X_n \longrightarrow B_1 + iB_2$$

since each of A_1 and A_2 is self-adjoint then by proposition (2.2.2) each of $X_n A_1 - A_1 X_n$ and $X_n A_2 - A_2 X_n$ is askew-adjoint for each $n \in N$. It is clear that each of the sequences $\{X_n A_1 - A_1 X_n\}$ and $\{X_n A_2 - A_2 X_n\}$ converges and by the closeness of a askew-adjoint operator, there limits are skew-adjoints, call these limits by C_1 and C_2. Thus,

$$(X_n A_1 - A_1 X_n) + i(X_n A_2 - A_2 X_n) \longrightarrow C_1 + iC_2$$

Because of the uniqueness of converging points, one can get $C_1 + iC_2 = B_1 + iB_2$. The uniqueness of the Cartesian decomposition implies $C_1 = B_1$ and $C_2 = B_2$. It is easily seen from the Cartesian decomposition, that the space $\beta(H)$ is generated by skew-adjoint operators over C. Thus the complex linear subspaces generated by the ranges R_{A_1} and R_{A_2} are dense in $\beta(H)$. Hence each of A_1 and A_2 has a left or a right inverse in $\beta(H)$.

Next, let $B \in \beta(H)$ be an arbitrary skew-adjoint operator and apply the last paragraph to the pair of the operators B and θ, to get the existence of a sequence $\{X_n\}$ in $\beta(H)$ such that $X_nA_1 - A_1X_n \longrightarrow B$ and . $X_nA_2 - A_2X_n \to \theta$. If A_2 has a right inverse then

$$X_nA_1A_2^{-1} - A_1X_nA_2^{-1} \longrightarrow BA_2^{-1} \tag{2.12}$$

and

$$X_n - A_2X_nA_2^{-1} \longrightarrow \theta$$

Thus,

$$X_n^* - A_2^{-1}X_n^*A_2 \longrightarrow \theta$$

and

$$X_n^*A_2^{-1} - A_2^{-1}X_n^* \longrightarrow \theta$$

Hence,

$$A_1X_n^*A_2^{-1} - A_1A_2^{-1}X_n^* \longrightarrow 0 \tag{2.13}$$

Therefore,

$$X_n^*(A_1A_2^{-1}) + (A_1A_2^{-1})X_n^* \longrightarrow BA_2^{-1},$$

which shows that the range of the map $X \longrightarrow X(A_1A_2^{-1}) + (A_1A_2^{-1})X^*$ is dense in $\beta(H)$ and this is a contradiction. A similar argument applies if A_2 has a left inverse.♦

Contents

Introduction

The operator equations (linear and non-linear) play an important rule in differential equations, integral equations and control theory, [4], [11] and [19].

The operator equation is an equation of the form LX=C, where L and C are known operators defined on a Hilbert space H and X is an operator that must be determined. If L is linear then the above equation is said to be linear operator equation, [12]. Otherwise, it is non-linear operator equation,[12].

Many authors studied the operator equations for example Goldstein J. in 1978 studied the existence and uniqueness of the solution for the linear operator equation of the form AX+XB=Q, where A,B and Q are known operators defined on a Hilbert space H, and X is the operator that must be determined, Lin S. in 1988 discussed the nature of the solution for the linear operator equations of the forms AX=Q and AX - XQ=W, where A,Q and W are known operators defined on a Hilbert space H, and X is the operator that must be determined. Bahatia and Rosenthal. in 1997 illustrate the importance of the study of the previous linear equations. Also, in 2001 Bahatia studied a special type of linear operator equations of the form $A^*X+XA+tA^* A^{½}=W$, where A and W are known operators defined on a Hilbert space H, t is any scalar and X is the operator that must be determined.

This work concerns with special types of the linear operation equations namely, the Lyapunov equations.

These types of linear operator equations have many real life applications in physics, weather, and atmospheric models [4],[8] and [25].

The Lyapunov equations are of two types, the first type is the continuous-time Lyapunov equation which takes the form $A_{*}X+XA=W$; were A and W are known operators defined on a Hilbert space H, and X is the unknown operator that must be determined, [8] and [23].

The second type is the discrete-time Lyapunov equation which takes the form $X-A^{*}XA=W$, where A and W are known operator defined on a Hilbert space H, and X is the unknown operator that must be determined, [5].

This work is a study of the nature of the solution for the linear Lyapunov equation of two types with simple generalization.

This thesis consists of three chapters.

In chapter one, we modify some theorems to ensure the existence and uniqueness of the solution for the continuous-time Lyapunov equation. Also, some study is presented to include the more general continuous-time Lyapunov equation.

This chapter consists of four sections:

In section one; some types of linear operator equations are presented.

In section two, we give some modification for the Sylvester Rosenblum theorem to guarantee the existence and uniqueness for the solution of the continuous-time Lyapunov equation.

In section three the invariant subspace problem is studied of the continuous-time Lyapunov equation.

In section four, the range of the map $\tau_A:\beta(H)\to\beta(H)$ which is defined by $\tau_A(X)=A_*X+XA$ is discussed. This discussion includes the injectivity, surjectivity for some special types of operators A.

In chapter two, a study of the range of the quasi-Jordan *-derivation $\rho_A:\beta(H)\to\beta(H)$ such that $\rho_A(X)=X^*A+AX$ is presented.

This chapter consists of four sections:

In section one, we discuss the injectivity, surjectivity of ρ_A.

In section two, we study the range of ρ_A in case A is normal or compact operator.

In section three, we defined the quasi-commutator operator for any pair of operators. Also, we proved that any operator in $\beta(H)$ can be written as a sum of two quasi-commutator operators.

In section four, we study the density of the range ρ_A with respect to the norm topology.

Chapter three concerns with the study of the discrete-time Lyapunov equation. This chapter consists of three sections:

In section one, the nature of the solution for the discrete-time Lyapunov equation is discussed.

In section two, we study the map $\mu_A:\beta(H)\to\beta(H)$ defined by $\mu_A(X)=X-F^*XF$.

In section three, the nature of the spectrum of μ_F with its parts is studied.

To the best of our knowledge, theorem (1.4.1), theorem (2.1.1) theorem (2.1.2), theorem (2.4.1) and proposition (3.2.2) seem to be new.

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Table of Notation

$\Re$	The field of real numbers.
₵	The field of complex numbers.
H	Infinite dimensional complex separable Hilbert space.
β(H)	The Banach algebra of all bounded linear operators defined on H.
σ(T)	Spectrum of the operator T.
σ_π(T)	The approximate point spectrum of T.
σ_δ(T)	The defect spectrum of T.
Rang(T)	The range of the operator T.
$< \, , \, >$	Inner product.
$\\| \\|$	Norm.
T*	The adjoint of the operator T.
Ker(X)	The kernel of the operator X.

المستخلص

الغرض الرئيسي من هذا العمل يمكن تقسيمةالى ثلاثه اقسام

اولاَ دراسة وجود و وحدانية الحل لانواع خاصة من معادلات المؤثرات الخطية و التي هي معادلة ليبانوف.

ثانياَ دراسة و مناقشة المدى لشبة اشتقاق * جوردن.

ثالثاَ قدمت الدراسة لطبيعية حل نوع لمعادلة لبيانوفو التي هي معادلة "stein".

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الاهداء

الى من علمني ورباني وادخل السرور الى قلبي

الى من افكر فيه دائماً ولن انساه ابداً

(أمي)

الى من كنت اتمنى ان يكون معي في هذه اللحظات

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We certify that this thesis was predared under our supervision at the University of AL- Nahrian, College of Science as a partial fulfillment of the Requirements of Doctor of Philosophy of science in Mathematics

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