Ministry of Higher Education
And Scientific Research
Al-Nahrain University
College of Science

On

The Volume And Integral Points Of A polyhedron In R^n

A Thesis
Submitted to the Department of Mathematics and Computer Applications/College of Science/ Al- Nahrain University In Partial Fulfillment of the Requirements For the Degree of Doctor of Philosophy in Mathematics

By

Shatha Assaad Salman Al-Najjar

(B.Sc., 1991)
(M.Sc., 1995)

Supervised by

Dr. Adil G. Naoum and Dr.Ahlam J. Khaleel

July 2005

وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

حول الحجم والنقاط الصحيحة لمتعدد السطوح في IJ [illegible]

رسـالة
مقدمة الى قسم الرياضيات وتطبيقات الحاسوب/كلية العلوم/ جامعة النهرين كجزء من متطلبات نيل درجة دكتوراه فلسفة في الرياضيات

من قبل
شذى أسعد سلمان النجار

(بكلوريوس علوم/الجامعة التكنولوجية/١٩٩١)
(ماجستير علوم/الجامعة التكنولوجية/١٩٩٥)

باشـراف

د.عادل غسان نعوم و **د.أحلام جميل خليل**

تموز ٢٠٠٥ ***جمادي الثاني ١٤٢٦***

Abstract

Computing the volume and integral points of a polyhedron in $\Re^n$ is a very important subject in different areas of mathematics.

There are two representations for the polyhedron, namely the H-representation and the V-representation. For each representation we give a different method of finding the volume and number of integral points.

Moreover, the Ehrhart polynomial of a bounded polyhedron is discussed with some methods for finding it. One of these methods is modified and we prove two theorems for computing the coefficients of the Ehrhart polynomial.

Also, a modified method for counting the number of integral points of a bounded polyhedron is given, and it makes matrix operations on the matrix that represents the bounded polyhedron, and studies the effect of these operations on these numbers.

All of the used methods are demonstrated with different examples.

Introduction

A wide variety of pure and applied mathematics involve the problem of counting the number of integral points inside a region in space. Applications range from the very pure: number theory, toric Hilbert functions, Kostant's partition function in representation theory, Ehrhart polynomial in combinatorics to the very applied: cryptography, integer programming, statistical contingency, mass spectroscope analysis. Perhaps the most basic case is when the region is a convex bounded polyhedron. Convex polyhedra, i.e., the intersections of a finite number of half spaces of the space $\Re^d$, are important objects in various areas of mathematics and other disciplines as seen before. In particular, the compact ones among them (polytopes), which, can equivalently be defined as the convex hulls of finitely many points in $\Re^d$, have been studied since ancient times, for example, platonic solids, diamonds, the great pyramids in Egypt etc., [27]. polytopes appear as building blocks of more complicated structures, e.g. in combinatorial, topology, numerical mathematics and computed aided designs. Even in physics polytopes are relevant e.g., in crystallography or string theory, [31].

Probably the most important reason of the tremendous growth of interest in the theory of convex polyhedra in the second half of 20'th century was the fact that linear programming i.e., optimizing a linear function over the solutions of a system of linear inequalities became a wide spread tool to solve practical problems in industry and military. Dantzig's simplex algorithm, developed in the 40's of the last century, showed that geometric and combinatorial knowledge of polyhedra (as the domains of linear programming problems), is quite helpful for finding and analyzing solution procedures for linear programming problems, [31].

Since the interest in the theory of convex polyhedra to a large extent comes from algorithmic problems, it is not surprising that many algorithmic questions on polyhedra rose in the past, but also inherently, convex polyhedra (in particular: polytopes) give rise to algorithmic questions, because they can be treated as finite objects by definition; this makes it possible to investigate the smaller ones among them by computer programs like the polymake - system written by Gawilow and Jowing, [24].

Once chosen to exploit this possibility one immediately finds oneself confronted with many algorithmic challenges.

Also, the notion of the volume of a polytope is basic and intuitive; its computation has raised a lot of problems. In this thesis we attempt to answer some fundamental and practical question on volume computation of higher dimensional convex polytopes given by their vertices and / or facets. In particular, we study through extensive computational experiment typical behavior of the exact methods, including Delaunay and boundary triangulation, the triangulation scheme described by Cohen and Hickey and the methods presented by Lawrence, [13].

This thesis consists of three chapters.

In chapter one we try to give a short introduction, provide a sketch of what bounded polyhedron looks like and how they behave with many examples. Also we recall some methods for finding the number of integral points inside a convex polytope, [13], [25] and [4].

In chapter two we present some methods for computing the coefficients of Ehrhart polynomial that depend on the concepts of Dedekind sum and residue theorem in complex analysis. Also, a method for counting these coefficients is introduced. The polytope that we take are with V-representations, [5] and [60]. We give a method for computing the coefficients of the Ehrhart polynomial, c_{d-3}, c_{d-4} until c_{d-9} also we give a formula for the differentiation of the given method.

In chapter three, a method for finding the volume of H-representation of a polytope using Laplace transform is presented and some basic concepts and remarks about the Birkhoff polytope and their volumes are discussed with their Ehrhart polynomials, [33], [11], [7], [8] and [9]. We make a change on the matrix, which represents the polytopes and finds a general formula for the number of integral points; also we make a change of matrix operation and study the effect of this change on the number of integral points of the polytopes. To the best of our knowledge, this result seems to be new.

List of symbols

$\Re$	the set of all real numbers.
Z	the set of all integers.
$\Re^d$	the vector space of d-dimension.
$\Re^{m\times d}$	an $m\times d$ real matrix.
Z^d	the standard integer lattice.
$\Re_+^d$	d-space of vectors with positive components.
Vol(P)	volume of P.
ext(P)	extreme points of P.
$\lfloor x \rfloor$	greatest integer ≤x.
$\lceil x \rceil$	least integer ≥ x.
νo(P)	Voronoi cell of p.
nb(S,v)	nearest neighbor set of v in S.
conv(nb(S,v))	Delaunay cell of v.
$\lvert P\cap Z^d \rvert$	number of integral points of a polyhedron.
$\lvert \partial P\cap Z^2 \rvert$	number of integral points on the boundary of the polyhedron.
$\delta(x,p)$	delta function of x and p.
$\lVert x \rVert$	the Euclidean norm of a vector x.
rank(A)	rank of the matrix A.
dim(P)	dimension of the polytope P.
$\Delta(v_0,v_1,...,v_d)$	simplex in $\Re^d$ with vertices $v_0,v_1,...,v_d\in\Re^d$.
$\det(v_1-v_0,...,v_d-v_0)$	*determinant of* $d\times d$ *matrix whose columns are* $v_1-v_0,...,v_d-v_0$.
$P=\bigcup_+ p_i$	signed union of p_i .
$\prod_{i=1}^{n} x_i$	multiplication symbol of $x_1,x_2,...,x_n$.
L(P,t)	the Ehrhart polynomial of a polytope P.
S(a,b)	the Dedekind sum of a and b.
Res(f(z), z=a)	residue of f(z) about z = a.

$S_1(m,n)$	Stirling number of the first kind.
$S_2(m,n)$	Stirling number of the second kind.
$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$	
P°	interior of a polytope P.
B_n	Birkhoff polytope.
$H_n(t)$	Ehrhart polynomial of the Birkhoff polytope.
int |Z| = r	interior of the circle |Z| = r.

Contents

Chapter One

Preliminaries

Introduction

Convex bounded polyhedrons are fundamental geometric objects that have been investigated since antiquity. The beauty of their theory is nowadays complemented by their importance for many other mathematical subjects, ranging from the integration theory, algebraic topology, and algebraic geometry (toric varieties) to the linear and combinatorial optimization.

In this chapter we try to give a short introduction and provide a sketch of what bounded polyhedron looks like and how they behave with many examples.

A convex polyhedron is an intersection of a finite number of half spaces of the space $\Re^d$, and a convex polytope is a bounded convex polyhedron. Every convex polyhedron has two natural representations, a half space representation (H-representation) and a vertex representation (V-representation). In recent years various techniques of geometric computations associated with convex polyhedron have been discovered, see [13], [31], [16] and [17].

Also, we recall some methods for finding the volume of a convex polytope and other methods for finding the number of integral points inside a convex polytope.

This chapter consists of three sections:

In section one, some basic definitions with some useful remarks about representation of the polyhedron are presented.

In section two, some methods for computing the volume of convex polytopes are given with some illustrative examples. These are classified into two groups: triangulation methods and signed decomposition methods.

The triangulation methods include boundary triangulation, Delaunay triangulation and Cohen & Hickey's triangulation, [13]. The signed decomposition methods include Lawerence's method, [25].

In section three, some methods for finding the number of integral points of a convex polytope are discussed; these methods are demonstrated with some examples.

1.1 Representation of a polyhedron

The volume of a convex bounded polyhedron is not easy to compute and the basic methods for exact computation of this volume can be classified according to whether a half space representation or a vertex representation of it, [33]. Therefore, in this section some basic definitions on a convex bounded polyhedron and its representations are given.

We start this section by the following definitions:

Definition (1.1.1), [35, p.85]:

Let $AX \leq b$ where $A \in \Re^{m \times d}$ is a given real matrix, and $b \in \Re^m$ is a known real vector. A set $P = \{X \in \Re^d : AX \leq b\}$ is said to be a polyhedron.

A polyhedron P is bounded if there exists $\omega \in \Re^1_+$ such that $\|X\| \leq \omega$ for every $X \in P$, [35, p.86].

Definition (1.1.2), [35, p.85]:

Every bounded polyhedron is said to be a polytope.

Definition (1.1.3), [35, p.84]:

Let $S = \{x_1, x_2, ..., x_k\}$ where $x_i \in \Re^d$, $1 \leq i \leq k$, then S is said to be affinely independent if the unique solution of $\sum_{i=1}^{k} a_i x_i = 0$ and $\sum_{i=1}^{k} a_i = 0$ is $a_i = 0$, for i = 1, …, k.

Recall that a polyhedron P is of dimension k, denoted by dim(P)=k, if the maximum number of affinely independent points in P is k+1. In this case a polyhedron (polytope) is said to be k-polyhedron (k-polytope). On the other hand, a polyhedron is of a full dimensional if dim (P) = d, [35, P.86].

Remark (1.1.1):

If the polyhedron P which is defined by, $P = \{X \in \Re^d : AX \leq b\}$ is not full dimensional, then at least one of the inequalities $a_i X \leq b_i$, i=1,2,…,k, is satisfied as equality by all points of P, where a_i is the i-*th* row of the matrix A and b_i are the values of the vector b, [35, p.86].

Proposition (1.1.1), [35, p.84]:

Let $P=\{X\in\Re^d : AX\le b\}$ then the following statements are equivalent:

(a) $\{X\in\Re^d : AX\le b\}\ne\phi$.

(b) rank(A)=rank(A|b),

where $A\in\Re^{m\times d}, b\in\Re^m$, A|b is the augmented matrix of the system AX=b and rank(A|b) is the maximum number of linearly independent rows (columns) of A|b.

Now, if P takes the form $P=\{X\in\Re^d : AX\le b\}$, the pair A|b is said to be a half space representation or simply H-representation of P, where $A\in\Re^{m\times d}, b\in\Re^m$, [13].

Proposition (1.1.2), [35, p.87]:

Let $P=\{X\in\Re^d : AX\le b\}$ be a polytope, then:

dim(P)+rank($A^*|b^*$)=d,

where $A^*|b^*$ denotes the corresponding rows of A|b, which represent the equality sets of the representation A|b of P, that is, $P=\{X\in\Re^d : A^*X=b^*\}$, [35,p.86].

Definition (1.1.4), [4]:

Let $P=\{X\in\Re^d : AX\le b\}$ be a polyhedron. If the entries of A and b have integer values then this polyhedron is said to be rational polyhedron.

Recall that for a given convex set S, a point $X\in S$ is said to be vertex (or sometimes extreme point) if it does not lie on a line segment joining two other points of this set. In this case the line joining any two vertices is said to be an edge [41, p.98].

It can be easily seen that any polyhedron is a convex set in $\Re^d$.

Definition (1.1.5), [38]:

A lattice polytope in $\Re^d$ (sometimes called integral polytope) is a polytope whose vertices are lattice points (integral points), that is, points in Z^d. If the lattice polytope is of dimension d then this polytope is said to be a d-dimensional lattice polytope

<u>Definition (1.1.6), [41, p.96]:</u>

Given $\sum_{i=1}^{d} a_i x_i = b$, where a_i and b are known real constants for $1 \le i \le d$. The set of points $X = \{x_i\}_{i=1}^{d}$, which satisfies the above equation, is said to be a hyperplane.

Moreover, the set of points $X = \{x_i\}_{i=1}^{d}$ is called a half- space if it satisfies the inequality $\sum_{i=1}^{d} a_i x_i \ge b$, [36, p.413].

<u>Definition (1.1.7), [10]:</u>

Let P be a polyhedron in $\Re^d$. For $c \in \Re^d$ and $b \in \Re$, the inequality $\sum_{i=1}^{d} c_i y_i \le b$ is called valid for P if it is satisfied by all points in P, where $c = \{c_i\}_{i=1}^{d}$. The faces of P are the sets of the form $P \cap \{Y = \{y_i\}_{i=1}^{d} : \sum_{i=1}^{d} c_i y_i = b\}$ for some valid inequality $\sum_{i=1}^{d} c_i y_i \le b$.

Recall that a face F is said to be proper if $\phi \ne F \ne P$. On the other hand the faces of dimension 0 and 1 are called vertices and edges respectively. However the faces of highest dimension are termed facets.

<u>Definition (1.1.8), [12]:</u>

A polytope in $\Re^d$ is said to be simple if there are exactly d edges through each vertex, and it is called simplicial if each facet contains exactly d vertices.

It is known that a simplex in $\Re^d$ is a d-dimensional polyhedron, which has exactly d+1 vertices, [23, p.37].

<u>Definition (1.1.9), [35, p.83]:</u>

Given a non empty set $S \subseteq \Re^d$, a point $X \in \Re^d$ is a convex combination of points of S if there exists a finite set of points $\{x_i\}_{i=1}^{t}$ in S and $\lambda \in \Re_+^t$ with $\sum_{i=1}^{t} \lambda_i = 1$ and $X = \sum_{i=1}^{t} \lambda_i x_i$.

It is known that the convex hull of S, denoted by conv(S) is the set of all points that are convex combinations of all points in S.

Now, if $V = \{v_0, v_1, ..., v_n\}$ is a finite set of points in $\Re^d$, the convex hull of V denoted by conv(V) is said to be convex polytope. In this case, V is called vertex representation or simply V-representation of P, [13].

1.2 Some methods for the volume computation of a polytope

As mentioned before, computing the volume of a polytope is very important in many real life applications, so in this section we give some methods for finding it. There is a comparative study of various volume computation algorithms for polytopes in [13]. However there is no single algorithm that works well for many different types of them, [22].

For simple polytopes, triangulation-based algorithms are more efficient and for simplicial polytopes sign-decomposition based algorithms are better, [13].

In this section, some methods for volume computation are given with different examples.

We start this section by the following remark.

Remark (1.2.1):

All known algorithms for exact volume computation decompose a given polytope into simplices, and thus they all rely on the volume formula of a simplex which is given by the following proposition, [13]:

Proposition (1.2.1), [13]:

For a polytope represented by its vertices $v_0, v_1, ..., v_d \in \Re^d$*, the volume of it is given by*

$$Vol(\Delta(v_0, v_1, ..., v_d)) = \frac{1}{d!}\left|\det(v_1 - v_0, ..., v_d - v_0)\right|$$

Where $\Delta(v_0, v_1, ..., v_d)$ *denotes the simplex in* $\Re^d$ *with vertices* $v_0, v_1, ..., v_d \in \Re^d$ *and* $(v_1 - v_0, ..., v_d - v_0)$ *is* $d \times d$ *matrix whose columns are* $v_1 - v_0, ..., v_d - v_0$.

Next, there are two types of methods for exact volume computation of the simple polytopes, which are discussed below:

I.Triangulation methods:

In these methods one has a simple polytope P in $\Re^d$*. P is triangulated into simplices* $\Delta_i (i = 1,2,...,s)$ $P = \bigcup_{i=1}^{s} \Delta_i$ *. The volume of P is simply the sum of the volumes of the simplices.*

$$Vol(P) = \sum_{i=1}^{s} Vol(\Delta_i) \qquad (1.1)$$

The following: boundary triangulation, Delaunay triangulation and Cohen & Hickey's combinatorial triangulation by dimensional recursion named, as triangulations method, [13].

An important difference between these methods is that the former two methods need only a V–representation while the last method requires both V- and H-representations, [13].

Before giving the signed decomposition methods, we need the following definition.

Definition (1.2.1):

Let $\mathrm{P} \in \Re^d$ *be a polytope, a signed union of P means, a collection of polytopes* $\mathrm{P}_1, \mathrm{P}_2, ..., \mathrm{P}_k \subseteq \Re^d$ *such that* $\mathrm{P} = \bigcup_{i=1}^{k} \mathrm{P}_i$*, and* $\mathrm{P}_i \cap \mathrm{P}_j$ *is a proper face of* P_i *and* P_j*, for* $i \neq j$*.In this case we write* $P = \bigcup_{+} \mathrm{P}_i$*, [30].*

II. Signed decomposition methods:

Instead of triangulating a polytope P, one can decompose P into signed simplices whose signed union is exactly P. More specifically, P is represented as a signed union of simplices Δ_i*, i =1, 2, …, s. This means,*

$$\mathrm{P} = \bigcup_{i=1}^{s} \sigma_i \Delta_i \qquad (1.2)$$

Where σ_i *is either +1 or –1. The volume of P is, [13].*

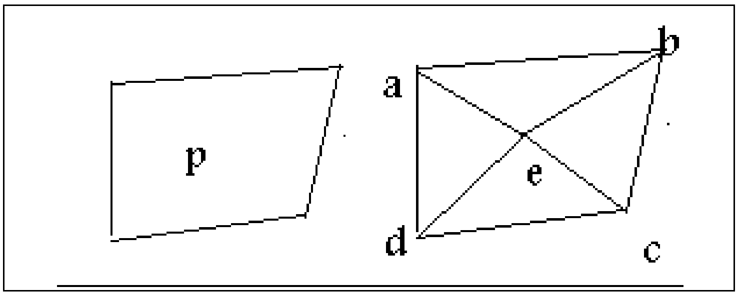
$$Vol(P) = \sum_{i=1}^{s} \sigma_i Vol(\Delta_i)$$

1.2.1 Triangulation methods

In this subsection we discuss briefly some of the known triangulation methods that compute the volume of the polytope.

(i) Boundary triangulation, [13]:

In boundary triangulation, one computes the convex hull of the perturbed points, interpreting the result in terms of the original vertices leads to a triangulation of the boundary, which by linking with a fixed interior point yields, a triangulation of P. For the convex hull computation the reverse search algorithm is chosen, [3], where only the V-representation of a polytope is required. To illustrate this method, consider the polytope which is represented by a set of vertices named {a, b, c , d} as given in figure (1). Using an interior point e where the boundary of a polytope is easily triangulated or already triangulated as in the case of simplicial polytopes. By linking a point e with the vertices a, b, c and d yields four triangles then, volumes of these triangles are found, summing all of these volumes the volume of this polytope is obtained.



(a) (b)

Figure (1):
(a) represents a polytope P.
(b) represents a partition of the given polytope P by using the boundary triangulation method.

(ii) Delaunay triangulation, [13]:

Before we discuss this method, some basic definitions concerning the Delaunay triangulation are needed.

Definition (1.2.2), [22]:

Given a set S of n distinct points in $\Re^d$*, Voronoi diagram is the partition of* $\Re^d$ *into n polyhedron regions (denoted by* $vo(\rho), \rho \in S$*). Each region* $vo(\rho)$ *is called Voronoi cell of* ρ*, which is defined as set of points in* $\Re^d$ *that are closer to* ρ *than other points in S, or more precisely*

$$vo(\rho) = \{X \in \Re^d : \|X - \rho\| \leq \|X - q\|, \forall q \in S - \rho\}.$$

Definition (1.2.3), [22]:

Let S be a set of n points in $\Re^d$*. For each point* $v \in \Re^d$*, the nearest neighbor set denoted by* $(nb(S, v))$ *of* v *in* S *is the set of points* $\rho \in S - v$*, which are closest to* v *in Euclidean distance.*

Definition (1.2.4), [22]:

Let S be a set of n points in $\Re^d$*. A point* $v \in \Re^d$ *is said to be a Voronoi vertex of S if* $nb(S, v)$ *is maximal over all nearest neighbor sets.*

Definition (1.2.5), [22]:

Let S be a set of n points in $\Re^d$*. The convex hull of the nearest neighbor set of Voronoi vertex* v *denoted by* $conv(nb(S, v))$ *is said to be a Delaunay cell of* v*.*

The Delaunay triangulation of S is a partition of the convex hull conv(S) into the Delaunay cells of Voronoi vertices together with their faces, [22].

Now we discuss the method of Delaunay triangulation method that requires only the V–representation of the polytope.

The geometric idea behind a Delaunay triangulation of a d–polytope is to ' lift ' it on a paraboloid in dimension d+1. The following construction is very important to compute the Voronoi diagram, [22].

Let S be a set of n points in $\Re^d$*. For each point* $\rho \in S \subseteq \Re^d$*, consider the hyperplane tangent to the paraboloid* $x_{d+1} = x_1^2 + \ldots + x_d^2$ *in* $\Re^{d+1}$ *at* ρ*:*

This hyperplane is represented by h (ρ) as:

$$\sum_{j=1}^{d}\rho_j^2 - \sum_{j=1}^{d}2\rho_j x_j + x_{d+1} = 0$$

where $\rho_j \ (j=1,2,...,d)$ *are the coordinates of* ρ*, for each point* ρ*, the equality in the above equation is replaced by the inequalities* $(\geq)$*, which yields a system of n inequalities that is denoted by* $b - AX \geq 0$*. The polyhedron P in* $\Re^{d+1}$ *of all solutions X to the system of inequality is a lifting of the Voronoi diagram to one higher dimensional space. [13], shows that the underlying convex hull algorithm uses the ' beneath – beyond ' method.*

Example (1.2.1):

Consider the set of vertices:

$$S = \{\rho_1 = (0,0), \rho_2 = (2,1), \rho_3 = (1,2), \rho_4 = (4,0), \rho_5 = (0,4), \rho_6 = (4,4)\}.$$

Here the volume of the polytope given by these vertices is to be determined. To do so, Delaunay triangulation is used to compute the volume of this polytope.

First, write down the system of linear inequalities in three variables as explained before. That is for each $\rho_j \in S, j = 1,2,...,6$*, apply the inequalities:*

$$\sum_{j=1}^{2}\rho_j^2 - \sum_{j=1}^{2}2\rho_j x_j + x_3 \geq 0$$

we get a system of six inequalities

$$x_3 \geq 0$$
$$5 - 4x_1 - 2x_2 + x_3 \geq 0$$
$$5 - 2x_1 - 4x_2 + x_3 \geq 0$$
$$16 - 8x_1 + x_3 \geq 0$$
$$16 - 8x_2 + x_3 \geq 0$$
$$32 - 8x_1 - 8x_2 + x_3 \geq 0.$$

The set of solutions $X \in \Re^3$ of the above inequalities represents a polyhedron P. By applying the cdd+ program [22], the Delaunay cells, (ρ_1,ρ_3,ρ_5), (ρ_1,ρ_2,ρ_3), (ρ_1,ρ_2,ρ_4), (ρ_2,ρ_3,ρ_6), (ρ_2,ρ_4,ρ_6) *and* (ρ_3,ρ_5,ρ_6) *are obtained. The cell* (ρ_1,ρ_3,ρ_5) *means the triangle which is represented by three vertices* ρ_1,ρ_3 *and* ρ_5*, and similarly for the other cells. Therefore six triangles are obtained, summing the volumes of these triangles yields the volume of the polyhedron P is equal to 16.*

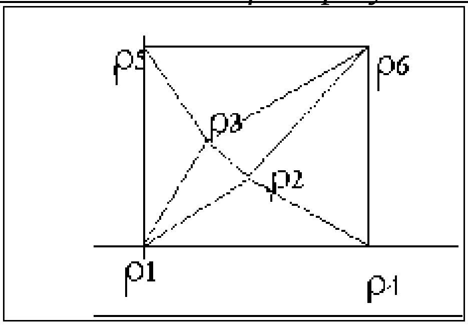


Figure (2): represents the polyhedron P with the set of vertices $\{\rho_1,\rho_2,...,\rho_6\}$ *and Delaunay cells.*

(iii) Triangulation by Cohen & Hickey, [13]:

This recursive scheme triangulates a d- polytope P by choosing any vertex $v \in P$ *as an apex and connecting it with the (d-1)–dimensional simplices resulting from a triangulation of all facets of P not containing* v*. To be precise, denote by* θ^k, $0 \le k \le d$*, the k-dimensional faces of P, and let* η *be a ' map ' which associates to each face one of its vertices. Then the pyramids with apex* $\eta(\theta^d)$ *and bases among the facets* θ^{d-1} *with* $\eta(\theta^d) \notin \theta^{d-1}$ *form a dissection of the polytope.*

Applying the scheme recursively to all θ^{d-1} *results in a set of decreasing chains of faces* $\theta^0 \subset \theta^1 \subset ... \subset \theta^{d-1} \subset \theta^d$ *such that* $\eta(\theta^k) \notin \theta^{k-1}$ *for* $1 \le k \le d$*. Then the set of corresponding simplices* $\Delta(\eta(\theta^0),\eta(\theta^1),...,\eta(\theta^d))$ *is a triangulation of P.*

To implement this recursive method, an extensive use of the double description as V-representation and H-representation is made by representing all faces as sets of vertices.

Note that in the case of Cohen & Hickey compared to a boundary triangulation all simplices in the facets containing the apex v are eliminated and therefore the number of simplices is usually reduced.

Example (1.2.2):

Consider the polytope which is represented by set of vertices $\{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}$ *as illustrated in figure (3), let* η *be the 'map' which assigns to each face of the polytope its vertex with the lowest number, so* $\eta(P) = \rho_0$, *all facets which do not contain the vertex* ρ_0 *are examined, that is, II, III and IV. The scheme of the Cohen and Hickey is applied to facet II with* $\eta(II) = \rho_1$. *II is intersected with all facets not containing the vertex* ρ_1, *these are III, IV and V. The intersections with IV and V are empty, so this recursion is unsuccessful. The intersection with III yields the vertex* ρ_2, *and the fixed vertices* ρ_0, ρ_1, ρ_2 *forms a first simplex. The other simplices obtained from III and IV is also marked in the figure (3). Therefore we have three triangles. Summing the volumes of these triangles yields the volume of the polytope.*

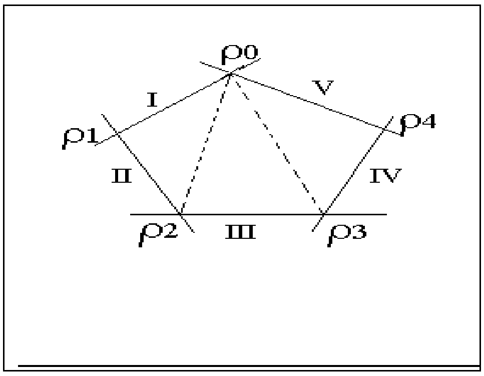


Figure (3): represents the partition of the polytope by Cohen & Hickey's triangulation method.

1.2.2 Signed decomposition method

In this subsection Lawrence's volume formula, which is one of the signed decomposition methods, is discussed.

(i) Lawrence's volume formula, [13]:

Assume the polytope P is simple and choose a vector $C \in \Re^d$ *and* $q \in \Re$ *such that the function* $f : \Re^d \to \Re$ *which is defined by f(X)=*$C^T X + q$ *is not constant along any edge that connected the vertices of the polytope P and* C^T *is the transpose of* C. *Let V be the set* *of vertices defining the polytope P. For each vertex* $v \in V$ *, let* A_v *be the* $d \times d$ *– matrix composed by the rows of A which are binding at v. Then by using [13],* A_v *is invertible and* $\gamma^v = \left[A_v^T\right]^{-1} C$ *. The assumption imposed on* C *assures that none of the entries of* γ^v *is zero. It is shown that*

$$Vol(P) = \sum_{v \in V} \frac{(C^T v + q)^d}{d! \ \left|\det A_v\right| \prod_{i=1}^{d} \gamma_i^v}$$

To illustrate this method, consider the following example.

Example (1.2.4) :

Consider the polytope P which is described by the following constraints

$$-x_1 \le 0$$
$$-x_2 \le 0$$
$$x_1 \le 2$$
$$x_2 \le 2$$
$$x_1 + x_2 \le 3$$

then

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

It is easy to check that the polytope of this example is simple, therefore the Lawerence volume formula can be applied. Define a function f by $f(X) = x_1 - x_2$ *where* $C^T = (1,-1)$ *and q = 0. Note that f (X) is non – constant on each edge of the polytope in the figure (4), for example, on edge (1) which connect* v_1 and v_2 *,* $x_2 = 0$ *and* x_1 *varies from 0 to 2. Therefore f (X) =* x_1 *which varies from 0 to 2 which means that it is nonconstant on edge (1) and similarly on each edge of P. According to figure (4), it is seen that the set of vertices, which represents the polytope, is*

$$\{v_1 = (0,0), v_2 = (2,0), v_3 = (2,1), v_4 = (1,2), v_5 = (0,2)\}.$$

Now, consider $v_1=(0,0)$, *this vertex satisfies the first two constraints, and this implies that*

$$A_{v_1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \quad \textit{hence} \quad |\det A_{v_1}|=1 \quad \textit{and} \quad \gamma^{v_1}=[A_{v_1}^T]^{-1}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$\gamma^{v_1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

and for $v_2=(2,0)$, *this vertex satisfies the second and third constraints, that is,*

$$A_{v_2}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad |\det A_{v_2}|=1 \quad \textit{And} \quad \gamma^{v_2}=[A_{v_2}^T]^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

And similarly for the other vertices we get

$$\gamma^{v_3}=\begin{bmatrix}2\\ -1\end{bmatrix}, \quad \gamma^{v_4}=\begin{bmatrix}-2\\ 1\end{bmatrix} \quad \textit{And} \quad \gamma^{v_5}=\begin{bmatrix}-1\\ -1\end{bmatrix}$$

Then Lawrence's volume formula is applied to get the volume of P.

$$Vol(P)=\sum_{v\in V}\frac{(C^Tv)^2}{2!\,|\det A_v|\prod_{i=1}^{2}\gamma_i^v}$$

$$=\frac{0^2}{2!(1)(-1)(1)}+\frac{2^2}{2!(1)(1)(1)}+\frac{1^2}{2!(1)(2)(-1)}+\frac{(-1)^2}{2!(1)(-2)(1)}+\frac{(-2)^2}{2!(1)(-1)(-1)}$$

$$=0+2+(-1/4)+(-1/4)+2=3\frac{1}{2}.$$

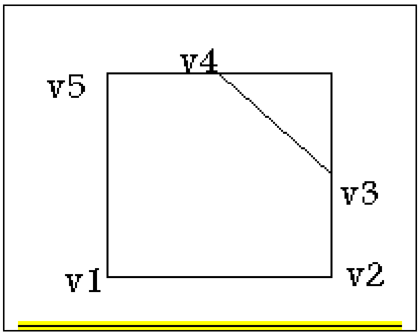


Figure (4): represents a polytope with the vertices v_1, v_2, v_3, v_4 and v_5

1.3 Methods for computing integral points

The main objective of this section is to recall some methods for finding integral points of a polyhedron. In this work, we use the symbol $\left|P \cap \mathrm{Z}^d\right|$ to denote the number of integral points in the polyhedron P, where Z^d is the integer lattice and P is a rational polyhedron. These methods are:

Method (1), [4]:

For d = 2, $P \subset \Re^2$ and P is an integral polyhedron. The famous formula, [42, p.240] states that

$$\left|P \cap \mathrm{Z}^2\right| = area(P) + \frac{\left|\partial P \cap \mathrm{Z}^2\right|}{2} + 1$$

That is, the number of integral points in an integral polyhedron is equal to the area of the polyhedron plus half the number of integral points on the boundary of the polyhedron plus one. This formula is useful because it is much more efficient than the direct enumeration of integral points in a polyhedron. The area of P is computed by triangulating the polyhedron. Furthermore, the boundary ∂P is a union of finitely many straight-line intervals, and counting integral points in intervals is easy.

Method (2), [4]:

Let $P \subset \Re^d$ be a polytope, then one can write the number of integral points in P as

$$|P \cap Z^d| = \sum_{X \in Z^d} \delta(X, P)$$

where $$\delta(X, P) = \begin{cases} 1 & \text{if } X \in P \\ 0 & \text{if } X \notin P \end{cases}$$

Before we give the next method, we need the following definition.

Definition (1.3.1), [2, p.61]:

The Dedekind sum of two relatively prime positive integers a and b denoted by $S(a,b)$ *can be defined as follows,*

$$S(a,b) = \sum_{i=1}^{b} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{ai}{b}\right)\right)$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin Z \\ 0 & \text{if } x \in Z \end{cases}$$

and $\lfloor x \rfloor$ *is the greatest integer* $\leq x$.

Remarks (1.3.1):

Dedekind sums appear in various branches of mathematics: the number theory, algebraic geometry and topology. These include the quadratic reciprocity law, random number generators [32], and lattice point problems [19]. More details about Dedekind sums are given in chapter two

Now, we are in a position that we can explain the following method.

Method (3), [4]:

Let $\Delta \subset \Re^3$ *be the tetrahedron with vertices (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c) where a, b and c are pairwise coprime positive integers, then the number of integral points in* Δ *can be expressed as:*

$$|P \cap Z^3| = \frac{abc}{6} + \frac{ab+ac+bc+a+b+c}{4} + \frac{1}{12}\left(\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{1}{abc}\right) - S(bc,a) - S(ac,b) - S(ab,c)$$

This formula is useful because it reduces counting the number of integral points to a computation of Dedekind sums, which can be done efficiently.

Method (4), [4]:

Let $P \subset \Re^d$ *be an integral polytope, for positive integer t, let tP={tX : X∈P} denote the dilated polytope P. By [42, p.238] there is a polynomial p(t) called the Ehrhart polynomial of P*

$$|tP \cap Z^d| = p(t)$$

where p(t) = $a_d t^d + a_{d-1} t^{d-1} + ... + a_0$

furthermore, $a_0 = 1$ *and* a_d *is the volume of the polytope P.*

To illustrate these methods, consider the following examples.

Example (1.3.1):

Let us consider three points in two dimensions such that $v_1 = (0,1), v_2 = (1,0)$ *and* $v_3 = (0,0)$*. Then the convex hull of* v_1, v_2 *and* v_3 *is a triangle in two dimensions.*

We compute the number of integral points by these methods. From method (1), one can have

$$|P \cap Z^2| = area(P) + \frac{|\partial P \cap Z^2|}{2} + 1$$

the area of triangle, is area(P)=0.5(1)(1)=0.5 and $|\partial P \cap Z^2| = 3$ *which represents the number of integral points on the boundary of the triangle.*

Then the number of integral points of the triangle is

$$|P \cap Z^2| = 0.5 + 0.5(3) + 1 = 3.$$

In method (2), we have

$$|P \cap Z^2| = \sum_{X \in Z^2} \delta(X, P)$$

$$v_1 = (0,1) \in P \text{ then } \delta(v_1, P) = 1$$

$v_2 = (1,0) \in P \ then \ \delta(v_2, P) = 1$

$v_3 = (0,0) \in P \ then \ \delta(v_3, P) = 1$

therefore $|P \cap Z^2| = 1 + 1 + 1 = 3$*, which is the number of integral points for the triangle.*

Form method (4), one can have

$$|P \cap Z^2| = a_2 + a_1 + 1$$

where a_2 *is the volume of the polytope and* a_1 *is the half number of integral points on the boundary of the polytope.*

In this case,

a_2 *= 0.5(1)(1)=0.5*

and a_2 *= 3/2*

therefore $|P \cap Z^2| = 0.5 + 1.5 + 1 = 3$.

Example (1.3.1):

Let us consider the tetrahedron $\Delta \subset \Re^3$ *with vertices (0,0,0), (3,0,0), (0,5,0), (0,0,7).*

Form method (3), the number of integral points in Δ *can be expressed as,*

$$|P \cap Z^3| = \frac{abc}{6} + \frac{ab + ac + bc + a + b + c}{4} + \frac{1}{12}\left(\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{1}{abc}\right) - S(bc,a) - S(ac,b) - S(ab,c)$$

here a=3, b=5 and c=7. Therefore

$$|P \cap Z^3| = \frac{105}{6} + \frac{15 + 21 + 35 + 3 + 5 + 7}{4} + \frac{1}{12}\left(\frac{21}{5} + \frac{35}{3} + \frac{15}{7} + \frac{1}{105}\right) - S(35,3) - S(21,5) - S(15,7)$$

After simple computation one can get, S(35,3)=-0.05555, S(21,5)=0.2, S(15,7)= 0.35714,

Thus $|P \cap Z^3| = 17.5 + 21.5 + 1.501877 + 0.05555 - 0.2 - 0.35714 = 40.$

الخلاصة

حساب حجم متعدد السطوح وكذلك حساب عدد النقاط الصحيحة في المجال R^n هو موضوع مهم جدا في فروع الرياضيات المختلفة .

يوجد تمثيلان لمتعدد السطوح وهما تمثيل -H وتمثيل - V ولكلا التمثيلين تم إعطاء طرق مختلفة لحساب حجم متعدد الحدود، وعدد النقاط الصحيحة.

تم حساب متعدد حدود ايرهارت باستخدام بعض الطرق. ا حدى هذه الطرق طورت واستنتجنا مبرهنتين لحساب معاملات متعددة الحدود ايرهارت.

كذلك طورت طريقة لحساب عدد النقاط الصحيحة وتم أجراء العمليات الحسابية على مصفوفة التمثيل لمتعدد السطوح ودراسة تأثيرها على عدد هذه النقاط.

جميع الطرق التي استخدمت وضحت بأمثلة مختلفة.

Acknowledgements

My deepest thanks for *Allah,* for his generous by providing me the strength to accomplish this work.

I would like to express my deep appreciation and gratitude to my supervisors *Dr. Adil G. Naoum* and *Dr. Ahlam J. Kaleel* for their help and advice during the preparation of this thesis.

Also, I would like to express my thanks to all members and friends in the Department of Mathematics and Computer Applications.

And all members in the central library of the University of Technology especially Thawara and Hatham.

Shatha Assaad Al-Najjar

Appendix

Before we give the proof of the relationship between the Dedekind sum and the Dedekind cotangent sum, we need the following definitions and lemma.

Definition 1

By means of the discrete Fourier series of Saw tooth function (saw tooth function ((x)) is the first Bernoulli function $\overline{B}_1(x), \overline{B}_1(x) = 0$ if $x \in Z$), and

$$\left(\left(\frac{n}{\mathrm{P}}\right)\right) = \frac{i}{2\mathrm{P}} \sum_{k=1}^{\mathrm{P}-1} \cot\left(\frac{\pi k}{\mathrm{P}}\right) e^{2\pi i k n / \mathrm{P}}$$

Definition 2

For a,b,c,m,n $\in$ N

$S_{m,n}(a,b,c) = \sum_{k \bmod b} \overline{B}_m\left(\frac{kb}{a}\right)\overline{B}_n\left(\frac{kc}{a}\right)$ is called Dedekind Bernoulli sum.

Definition 3

Let $a_1, a_2, \ldots, a_d \in N$ be relatively prime to $a_0 \in N$ define the higher-dimensional Dedekind sum as

$$S(a_0, a_1, \ldots, a_d) = \frac{(-1)^{d/2}}{a_0} \sum_{k=1}^{a_0-1} \cot\left(\frac{\pi k a_1}{a_0}\right) \cot\left(\frac{\pi k a_2}{a_0}\right) \cdots \cot\left(\frac{\pi k a_d}{a_0}\right).$$

Note that this sum is zero if d is odd, since the cotangent is an odd function.

Lemma

For m≥2

$$\overline{B}_m\left(\frac{n}{\mathrm{P}}\right) = \frac{B_m}{(-\mathrm{P})^m} + m\left(\frac{i}{2\mathrm{P}}\right)^m \sum_{k=1}^{\mathrm{P}} \cot^{(m-1)}\left(\frac{\pi k}{\mathrm{P}}\right) e^{2\pi k n i / \mathrm{P}}$$

These discrete Fourier expansions can be used to rewrite the Dedekind Bernoulli sums in terms of the Dedekind cotangent sums.

Corollary

If a, b, c $\in$ N are pairwise relatively rime and m, n ≥2 are integers with the same parity then

$$S_{m,n}(a,b,c) \overset{def}{=} \sum_{k \bmod a} \overline{B}_m\left(\frac{kb}{a}\right)\overline{B}_n\left(\frac{kc}{a}\right)$$

$$= \mathrm{mn} \frac{(-1)^{(m-n)/2}}{2^{m+n} a^{m+n-1}} \sum_{\mathrm{k}=1}^{\mathrm{a}-1} \cot^{(\mathrm{m}-1)}\left(\frac{\pi k c}{a}\right) \cot^{(\mathrm{n}-1)}\left(\frac{\pi k b}{a}\right) + \frac{\mathrm{B_m B_n}}{\mathrm{a}^{\mathrm{m+n-1}}}$$

We note that the parity assumption on m and n is no restriction; since the sums vanish if m+n is odd it is worth mentioning that a close relative of these sums namely

$$\sum_{k=1}^{a-1} \overline{B}_m\left(\frac{k}{a}\right)\overline{B}_n\left(\frac{kb}{a}\right) = \mathrm{m}\frac{(-1)^{(m-n)/2}}{2^{m+1}a^m}\sum_{k=1}^{a-1}\cot\left(\frac{\pi k}{a}\right)\cot^{(m-1)}\left(\frac{\pi kb}{a}\right)$$

Proof

By lemma we get

$$S_{m,n}(a,b,c) \overset{def}{=} \sum_{k \bmod a} \overline{B}_m\left(\frac{kb}{a}\right)\overline{B}_n\left(\frac{kc}{a}\right)$$

$$= \sum_{k \bmod a}\left[\frac{B_m}{(-a)^m} + m\left(\frac{i}{2a}\right)^m \sum_{j=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)e^{2\pi j\left(\frac{kb}{a}\right)}\right]\left[\frac{B_n}{(-a)^n} + n\left(\frac{i}{2a}\right)^n \sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{2\pi l\left(\frac{kl}{a}\right)}\right]$$

$$= \sum_{k \bmod a}\left[\frac{B_m B_n}{(-a)^{m+n}} + \frac{B_m n}{(-a)^m}\left(\frac{i}{2a}\right)^n \sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{2\pi l\left(\frac{kc}{a}\right)} + \frac{B_m n}{(-a)^m}\left(\frac{i}{2a}\right)^n \sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{2\pi l\left(\frac{kc}{a}\right)}\right]$$

$$+ \frac{B_n m}{(-a)^n}\left(\frac{i}{2a}\right)^m \sum_{l=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)e^{2\pi j\left(\frac{kb}{a}\right)} + mn\left(\frac{i}{2a}\right)^{m+n}\sum_{j=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)e^{2\pi j\left(\frac{kb}{a}\right)}\sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{2\pi l\left(\frac{kc}{a}\right)}$$

$$= mn\left(\frac{i}{2a}\right)^{m+n}\sum_{k \bmod a}\sum_{j,l=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)\sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{\frac{2\pi ki}{a}(jb+lc)}$$

$$= m\left(\frac{i}{2a}\right)^m \sum_{k \bmod a}\sum_{j,l=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)e^{2\pi j\left(kb/a\right)i} + n\left(\frac{i}{2a}\right)^n \frac{B_m}{(-a)^m}\sum_{k \bmod a}\sum_{l=1}^{a-1}\cot^{(n-1)}\left(\frac{\pi l}{a}\right)e^{2\pi l\left(kc/a\right)i} +$$

$$\sum_{k \bmod a}\frac{B_m B_n}{(-a)^{m+n}}$$

We use the fact that m+n=even.

Note:

a,b,c are relatively prime

$$\sum_{k \bmod a} e^{\left(2\pi ink/a\right)} = \begin{cases} \mathrm{a} & \text{if } a/n \\ 0 & \text{else} \end{cases}$$

if a|(jb+lc) then $l = jbc^{-1} \bmod a$ therefore

$$S_{m,n}(a,b,c) = mn\frac{(-1)^{(m+n)/2}}{(2a)^{m+n}}\sum_{j=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)\cot^{(n-1)}\left(\frac{-\pi jbc^{-1}}{a}\right) + (-a)\frac{B_m B_n}{(-a)^{m+n}}$$

$$= mn\frac{(-1)^{(m+n)/2}}{(2a)^{m+n}}\sum_{j=1}^{a-1}\cot^{(m-1)}\left(\frac{\pi j}{a}\right)\cot^{(n-1)}\left(\frac{-\pi jbc^{-1}}{a}\right) + \frac{B_m B_n}{(-a)^{m+n-1}}$$

Committee Certification

We, the examining committee, certify that we read this thesis and have examined the student ***Shatha Assaad Al-Najjar*** *in its contents and that, in our opinion it is adequate as a thesis for the degree of Ph-D of Science in Mathematics.*

Chairman

Signature:
Name: Dr. Nadar J. Mansour
Professor
Date: / 10 / 2005

Member

Signature:
Name: Dr. Mohamad S. Kerdar
Professor
Date: / 10 / 2005

Member

Signature:
Name: Dr. Akram M. Al-bood
Assist professor
Date: / 10 / 2005

Member

Signature:
Name:Dr. Norii F. Athab
Assist professor
Date: / 10 / 2005

Member (supervisor)

Signature:
Name: Dr. Adil G. Naoum
Professor
Date: / 10 / 2005

Member

Signature:
Name: Dr. Sahab K. Al-Saadi
Assist professor
Date: / 10 / 2005

Member(supervisor)

Signature:
Name: Dr. Ahlam J. Kaleel
Assist professor
Date: / 10 / 2005

I hear be certify upon the decision of the examining committee.

Signature:

Name: Dr. LAITH-ABDUL AZIZ AL-ANI
Dean of the College
Date: /10 /2005

Supervisors Certification

We certify that this thesis was prepared under our supervision at the University of Al-Nahrian, Colloge of Science as a partial fulfillment of the Requirements of Doctor of Philosophy of Science in Mathematics.

Signature:
Name: Dr. Adil G. Naoum
Professor
Date: / / 2005

Signature:
Name: Dr. Ahlam J. Kaleel
Assist professor
Date: / / 2005

In view of the available recommendations, I forward this thesis for debate by examining committee.

Signature:
Name: Dr. Akram M. Al-Abood
Assist professor

Date: / /2005

Contents

List of symbols

$\Re$	the set of all real numbers.
Z	the set of all integers.
$\Re^d$	the vector space of d-vectors.
$\Re^{m\times d}$	an $m\times d$ real matrix.
Z^d	the standard integer lattice.
$\Re^d_+$	d-space of vectors with positive components.
Vol(P)	volume of P.
ext(P)	extreme points of P.
$\lfloor x\rfloor$	greatest integer ≤x.
$\lceil x\rceil$	least integer ≥ x.
vo(P)	Voronoi cell of p.
nb(S,v)	nearest neighbor set of v in S.
conv(nb(S,v))	Delaunay cell of v.
$\left\|P\cap Z^d\right\|$	number of integral points of a polyhedron.
$\left\|\partial P\cap Z^2\right\|$	number of integral points on the boundary of the polyhedron.
$\delta(x,p)$	delta function of x and p.
$\\|x\\|$	the Euclidean norm of a vector x.
rank(A)	rank of the matrix A.
dim(P)	dimension of the polytope P.
$\Delta(v_0,v_1,...,v_d)$	simplex in $\Re^d$ with vertices $v_0,v_1,...,v_d\in\Re^d$.
$\det(v_1-v_0,...,v_d-v_0)$	determinant of $d\times d$ matrix whose columns are $v_1-v_0,...,v_d-v_0$.
$P=\bigcup_+ p_i$	signed union of p_i.
$\prod_{i=1}^n x_i$	multiplication symbol of $x_1,x_2,...,x_n$.
L(P,t)	the Ehrhart polynomial of a polytope P.
S(a,b)	the Dedekind sum of a and b.
Res(f(z), z=a)	residue of f(z) about z = a.
$S_1(m,n)$	Stirling number of the first kind.
$S_2(m,n)$	Stirling number of the second kind.

$$((x))=\begin{cases} x-\lfloor x\rfloor-\frac{1}{2} & if\ \ x\notin Z \\ 0 & if\ \ x\in Z \end{cases}$$

P°	interior of a polytope P.
B_n	Birkhoff polytope.
$H_n(t)$	Ehrhart polynomial of the Birkhoff polytope.
int |Z| = r	interior of the circle |Z| = r.

Dedication

To those they covered me with their love and kindness

My mother & My father

My brothers, sisters and My aunt

To my Dear husband Mahmood who supported me by all means

To my children's Rukia, Nayarh and Ali

With love.

Introduction

A wide variety of pure and applied mathematics involve the problem of counting the number of integral points inside a region in space. Applications range from the very pure: number theory, toric Hilbert functions, Kostant's partition function in representation theory, Ehrhart polynomial in combinatorics to the very applied: cryptography, integer programming, statistical contingency, mass spectroscope analysis. Perhaps the most basic case is when the region is a convex bounded polyhedron. Convex polyhedra, i.e., the intersections of a finite number of half spaces of the space $\Re^d$, are important objects in various areas of mathematics and other disciplines as seen before. In particular, the compact ones among them (polytopes), which, can equivalently be defined as the convex hulls of finitely many points in $\Re^d$, have been studied since ancient times, for example, platonic solids, diamonds, the great pyramids in Egypt etc., [27]. polytopes appear as building blocks of more complicated structures, e.g. in combinatorial, topology, numerical mathematics and computed aided designs. Even in physics polytopes are relevant e.g., in crystallography or string theory, [31].

Probably the most important reason of the tremendous growth of interest in the theory of convex polyhedra in the second half of 20'th century was the fact that linear programming i.e., optimizing a linear function over the solutions of a system of linear inequalities became a wide spread tool to solve practical problems in industry and military. Dantzig's simplex algorithm, developed in the 40's of the last century, showed that geometric and combinatorial knowledge of polyhedra (as the domains of linear programming problems), is quite helpful for finding and analyzing solution procedures for linear programming problems, [31].

Since the interest in the theory of convex polyhedra to a large extent comes from algorithmic problems, it is not surprising that many algorithmic questions on polyhedra rose in the past, but also inherently, convex polyhedra (in particular: polytopes) give rise to algorithmic questions, because they can be treated as finite objects by definition; this makes it possible to investigate the smaller ones among them by computer programs like the polymake - system written by Gawilow and Jowing, [24].

Once chosen to exploit this possibility one immediately finds oneself confronted with many algorithmic challenges.

Also, the notion of the volume of a polytope is basic and intuitive; its computation has raised a lot of problems. In this thesis we attempt to answer some fundamental and practical question on volume computation of higher dimensional convex polytopes given by their vertices and / or facets. In particular, we study through extensive computational experiment typical behavior of the exact methods, including Delaunay and boundary triangulation, the triangulation scheme described by Cohen and Hickey and the methods presented by Lawrence, [13].

This thesis consists of three chapters.

In chapter one we try to give a short introduction, provide a sketch of what bounded polyhedron looks like and how they behave with many examples. Also we recall some methods for finding the number of integral points inside a convex polytope, [13], [25] and [4].

In chapter two we present some methods for computing the coefficients of Ehrhart polynomial that depend on the concepts of Dedekind sum and residue theorem in complex analysis. Also, a method for counting these coefficients is introduced. The polytope that we take are with V-representations, [5] and [60]. We give a method for computing the coefficients of the Ehrhart polynomial, c_{d-3}, c_{d-4} until c_{d-9} also we give a formula for the differentiation of the given method.

In chapter three, a method for finding the volume of H-representation of a polytope using Laplace transform is presented and some basic concepts and remarks about the Birkhoff polytope and their volumes are discussed with their Ehrhart polynomials, [33], [11], [7], [8] and [9]. We make a change on the matrix, which represents the polytopes and finds a general formula for the number of integral points; also we make a change of matrix operation and study the effect of this change on the number of integral points of the polytopes. To the best of our knowledge, this result seems to be new.

References

[1] J. Abate, G. L. Choudhury and W. Whitt, Numerical inversion of multidimensional Laplace transforms by the Laguerre method, http://citeseer.ist.psu.edu/abate96numerical.html,(1996), 1-24.

[2] T. M. Apostol, Modular functions and Dirichlet series in number theory, Springer–Verlag Inc. 1976.

[3] D. Avis and D. Bremner, How good are convex hull algorithms, http://citeseer.ist.psu.edu/avis95how.html , (1995), 1-24.

[4] A. Barvinok and J. E. Pommersheim, An algorithmic theory of lattice points in polyhedra, new perspectives in geometric combinatorics, MSRI publications, 38, (1999), 91-147.

[5] M. Beck, The Reciprocity law for Dedekind sums via the constant Ehrhart coefficient, AMM, 106, (5), (1999), 459-462.

[6] M. Beck, Counting lattice points by means of the residue theorem, RamanuJan J. 4, (3), (2000), 299-310.

[7] M. Beck and D. Pixton, The Ehrhart polynomial of the Birkhoff polytope,www.math.binghamton.edu/dennis/papers/birkhoff.html-3k,(2002),1-14

[8] M. Beck, J. A. DE. Loera, M. Develin, J. Peifle and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, conference on integer points in polyhedra (13-17) July in Snowbird, (2003), 1-24.

[9] M. Beck and D. Pixton, The Volume of the 10^{th} Birkhoff polytope, www.math.binghamton.edu/dennis/papers/birkhoff.html-3k,(2003)

[10] A. Bemporad, K. Fukuda and F. D. Torrisi, Convexity recognition of the union of polyhedra, http://citeseer.ist.psu.edu/300246.html,(2000),1-16.

[11] I. Bengtesson, A. Ericsson, M.Kuś, W. Tadej and K. Życzkowski, Birkhoff `s polytope and unistochastic matrices N=3 and N=4, arXiv: math.CO/0402325 v2 24 Feb 2004, (2004), 1-30.

[12] M. Brion and M. Vergne, Lattice points in simple polytopes, J. Amer. Math. Soc., 10, (2), (1997), 371-392.

[13] B. Büeler and A. Enge and K. Fukuda, Exact volume computation for polytopes: A practical study, http://citeseer.ist.psu.edu/article/bueler98exact.html, (1998), 1-17.

[14] C. S. Chan, D. P. Robbins and D. S. Yuen, on the volume of a certain polytope, experiment Math.9, (1), (2000), 91-99.

[15] C. S. Chan and D. P. Robbins, on the volume of the polytope of doubly stochastic matrices, arXiv:math.CO/9806076 v1, 13 Jun 1998, (2002), 1-17.

[16] J. A. De Loera, R. Hemmecke, J. Tauzer and R.Yoshida, effective lattice points counting in rational convex polytopes, J. Symbolic computation, (2003), 1-33.

[17] J. A. De Loera, D. Haws, R. Hemmecke, A user´s Guide for LattE v1.1,http://www.math.ucdavis.edu/~Latte/manual/v1.1/pdf.,(2003), 1-22.

[18] R. Diaz and S. Robins, The Ehrhart polynomial of a lattice n-simplex, Electronic research announcements of AMS. 2, (1), (1996), 1-6.

[19] R. Diaz, S. Robins, The Ehrhart polynomial of a lattice polytope, Annals of Math. 145, (1997), 503-518.

[20] P. Franklin, Functions of complex variables, Prentice – Hall, INC. Englewood Cliffs, N. J., 1958.

[21] B. A. Fuchs and B. V. Shabat, Functions of a complex variable and some of their applications, Pergamon press LTD, 1964.

[22] K. Fukuda, frequently asked questions in polyhedral computation, http://www.ifor.math.ethz.ch/staff/fukuda/polyfaq/polfaq.html,(2004), 1-30.

[23] S. I. Gass, Linear programming, McGraw-Hill, Inc., 1985.

[24] E. Gawrilow and M. Joswig, polymake: a Freme work for analyzing convex polytopes, http://citeseer.ist.psu.edu/gawrilow99polymake.html, (1999), 1-31.

[25] H. Greenberg, Polyhedral computation, http://gcrg.ucsd.edu/classes/polyhcomp.pdf,(2003), 1-28

[26] P. Gritzmann and V. Klee, on the complexity of some basic problems computational convexity, cmath. phys. sci-440, kluwer Acad. publ. Dordrecht, (1994), 373-466.

[27] J. Gubeladze, Course in information, Math 890 Discrete Geometry Fall(2003),math.sfsu.edu/gubeladze/fall2003/discrete.pdf-63k., (2003).

[28] B. Grünbaum and G.C. Shephard, Pick´s theorem, the Amer. Math. Monthly, 100, (2), (1993), 150-161.

[29] R. Hans, Topics in analytic number theory, Springer - Verlag Berlin Heidelberg New York, 1973.

[30] M. Henk, J. R. Gebert and G.M. Ziegler, basic properties of convex polytopes, http://citeseer.ist.psu.edu/henk97basic.html,(2004), 1-28.

[31] V. Kaibel and M.E. Pfetsch, some algorithmic problems in polytope theory, www.zib.de/pfetsch/publications/polycomplex.pdf, (2002), 1-25.

[32] D. Knuth, The art of computer programming, Vol. 2. Addison – Wesley, Reading, Mass., 1981.

[33] J. B. Lasserre and E. S. Zeron, A Laplace transform algorithm for the volume of a convex polytope, arXiv: math.NA/0106168 v1 Jun 2001, (2001), 1-13.

[34] P. Mcmullen and G. C. Shephard, Convex polytopes and upper bound conjecture, Cambridge University Press, 1971.

[35] G. L. Nemhauser and L. A. Wolsey, Integer and combinatorial optimization, John Wiley & Sons, Inc., 1988.

[36] R.G. Parker and R. L. Radin, discrete optimization, Academic press, Inc. 1988.

[37] I. Pak, four questions on Birkhoff polytope, Annals of combinatorics, 4, (2000), 83-90.

[38] O. Pikhurko, lattice points in lattice polytopes, http://citeseer.ist.psu.edu/oleg2000lattice.html,2, (2000), 1-14.

[39] J. Pommersheim, Toric varieties, lattice points, and Dedekind sums, Math. Ann. 295, (1993), 1-24.

[40] G. Polya and R. E. Tarjan and D.R. Woods, Notes on introductory combinatorics, Birkhaüser Bostan, Inc., 1983.

[41] S. S. Rao, Optimization theory and applications, John Wiley & Sons, 1984.

[42] R. P. Stanley, Enumerative combinatorics , Wadsworth & Brooks / Cole Advanced Books & software, California, 1986.

[43] R. P. Stanley, linear homogeneous Diophantine equations and magic labeling of graphs, Duke Math. J.40, (1973), 607-632.

Chapter One
Preliminaries

Introduction

Convex bounded polyhedrons are fundamental geometric objects that have been investigated since antiquity. The beauty of their theory is nowadays complemented by their importance for many other mathematical subjects, ranging from the integration theory, algebraic topology, and algebraic geometry (toric varieties) to the linear and combinatorial optimization.

In this chapter we try to give a short introduction and provide a sketch of what bounded polyhedron looks like and how they behave with many examples.

A convex polyhedron is an intersection of a finite number of half spaces of the space $\Re^d$, and a convex polytope is a bounded convex polyhedron. Every convex polyhedron has two natural representations, a half space representation (H-representation) and a vertex representation (V-representation). In recent years various techniques of geometric computations associated with convex polyhedron have been discovered, see [13], [31], [16] and [17].

Also, we recall some methods for finding the volume of a convex polytope and other methods for finding the number of integral points inside a convex polytope.

This chapter consists of three sections:

In section one, some basic definitions with some useful remarks about representation of the polyhedron are presented.

In section two, some methods for computing the volume of convex polytopes are given with some illustrative examples. These are classified into two groups: triangulation methods and signed decomposition methods.

The triangulation methods include boundary triangulation, Delaunay triangulation and Cohen & Hickey's triangulation, [13]. The signed decomposition methods include Lawerence's method, [25].

In section three, some methods for finding the number of integral points of a convex polytope are discussed; these methods are demonstrated with some examples.

1.1 Representation of a polyhedron

The volume of a convex bounded polyhedron is not easy to compute and the basic methods for exact computation of this volume can be classified according to whether a half space representation or a vertex representation of it, [33]. Therefore, in this section some basic definitions on a convex bounded polyhedron and its representations are given.

We start this section by the following definitions:

Definition (1.1.1), [35, p.85]:

Let $AX \le b$ where $A \in \Re^{m \times d}$ is a given real matrix, and $b \in \Re^m$ is a known real vector. A set $P = \{X \in \Re^d : AX \le b\}$ is said to be a polyhedron.

A polyhedron P is bounded if there exists $\omega \in \Re_+^1$ such that $\|X\| \le \omega$ for every $X \in P$, [35, p.86].

Definition (1.1.2), [35, p.85]:

Every bounded polyhedron is said to be a polytope.

Definition (1.1.3), [35, p.84]:

Let $S = \{x_1, x_2, ..., x_k\}$ where $x_i \in \Re^d$, $1 \le i \le k$, then S is said to be affinely independent if the unique solution of $\sum_{i=1}^{k} a_i x_i = 0$ and $\sum_{i=1}^{k} a_i = 0$ is $a_i = 0$, for i = 1, …, k.

Recall that a polyhedron P is of dimension k, denoted by dim(P)=k, if the maximum number of affinely independent points in P is k+1. In this case a polyhedron (polytope) is said to be k-polyhedron (k-polytope). On the other hand, a polyhedron is of a full dimensional if dim (P) = d, [35, P.86].

Remark (1.1.1):

If the polyhedron P which is defined by, $P = \{X \in \Re^d : AX \le b\}$ is not full dimensional, then at least one of the inequalities $a_i X \le b_i$, i=1,2,…,k, is satisfied as equality by all points of P, where a_i is the i-*th* row of the matrix A and b_i are the values of the vector b, [35, p.86].

Proposition (1.1.1), [35, p.84]:

Let $P = \{X \in \Re^d : AX \le b\}$ then the following statements are equivalent:

(a) $\{X \in \Re^d : AX \le b\} \ne \phi$.

(b) rank(A)=rank(A|b),

where $A \in \Re^{m \times d}, b \in \Re^m$, A|b is the augmented matrix of the system AX=b and rank(A|b) is the maximum number of linearly independent rows (columns) of A|b.

Now, if P takes the form $P = \{X \in \Re^d : AX \le b\}$, the pair A|b is said to be a half space representation or simply H-representation of P, where $A \in \Re^{m \times d}, b \in \Re^m$, [13].

Proposition (1.1.2), [35, p.87]:

Let $P = \{X \in \Re^d : AX \le b\}$ be a polytope, then:

dim(P)+rank($A^* | b^*$)=d,

where $A^* | b^*$ denotes the corresponding rows of A|b, which represent the equality sets of the representation A|b of P, that is, $P = \{X \in \Re^d : A^* X = b^*\}$, [35,p.86].

Definition (1.1.4), [4]:

Let $P = \{X \in \Re^d : AX \le b\}$ be a polyhedron. If the entries of A and b have integer values then this polyhedron is said to be rational polyhedron.

Recall that for a given convex set S, a point $X \in S$ is said to be vertex (or sometimes extreme point) if it does not lie on a line segment joining two other points of this set. In this case the line joining any two vertices is said to be an edge [41, p.98].

It can be easily seen that any polyhedron is a convex set in $\Re^d$.

Definition (1.1.5), [38]:

A lattice polytope in $\Re^d$ (sometimes called integral polytope) is a polytope whose vertices are lattice points (integral points), that is, points in Z^d. If the lattice polytope is of dimension d then this polytope is said to be a d-dimensional lattice polytope

Definition (1.1.6), [41, p.96]:

Given $\sum_{i=1}^{d} a_i x_i = b$, where a_i and b are known real constants for $1 \le i \le d$. The set of points $X = \{x_i\}_{i=1}^{d}$, which satisfies the above equation, is said to be a hyperplane.

Moreover, the set of points $X = \{x_i\}_{i=1}^{d}$ is called a half- space if it satisfies the inequality $\sum_{i=1}^{d} a_i x_i \ge b$, [36, p.413].

Definition (1.1.7), [10]:

Let P be a polyhedron in $\Re^d$. For $c \in \Re^d$ and $b \in \Re$, the inequality $\sum_{i=1}^{d} c_i y_i \le b$ is called valid for P if it is satisfied by all points in P, where $c = \{c_i\}_{i=1}^{d}$. The faces of P are the sets of the form $P \cap \{Y = \{y_i\}_{i=1}^{d} : \sum_{i=1}^{d} c_i y_i = b\}$ for some valid inequality $\sum_{i=1}^{d} c_i y_i \le b$.

Recall that a face F is said to be proper if $\phi \ne F \ne P$. On the other hand the faces of dimension 0 and 1 are called vertices and edges respectively. However the faces of highest dimension are termed facets.

Definition (1.1.8), [12]:

A polytope in $\Re^d$ is said to be simple if there are exactly d edges through each vertex, and it is called simplicial if each facet contains exactly d vertices.

It is known that a simplex in $\Re^d$ is a d-dimensional polyhedron, which has exactly d+1 vertices, [23, p.37].

Definition (1.1.9), [35, p.83]:

Given a non empty set $S \subseteq \Re^d$, a point $X \in \Re^d$ is a convex combination of points of S if there exists a finite set of points $\{x_i\}_{i=1}^{t}$ in S and $\lambda \in \Re_+^t$ with $\sum_{i=1}^{t} \lambda_i = 1$ and $X = \sum_{i=1}^{t} \lambda_i x_i$.

It is known that the convex hull of S, denoted by conv(S) is the set of all points that are convex combinations of all points in S.

Now, if $V = \{v_0, v_1, ..., v_n\}$ is a finite set of points in $\Re^d$, the convex hull of V denoted by conv(V) is said to be convex polytope. In this case, V is called vertex representation or simply V-representation of P, [13].

1.2 Some methods for the volume computation of a polytope

As mentioned before, computing the volume of a polytope is very important in many real life applications, so in this section we give some methods for finding it. There is a comparative study of various volume computation algorithms for polytopes in [13]. However there is no single algorithm that works well for many different types of them, [22].

For simple polytopes, triangulation-based algorithms are more efficient and for simplicial polytopes sign-decomposition based algorithms are better, [13].

In this section, some methods for volume computation are given with different examples.

We start this section by the following remark.

Remark (1.2.1):

All known algorithms for exact volume computation decompose a given polytope into simplices, and thus they all rely on the volume formula of a simplex which is given by the following proposition, [13]:

Proposition (1.2.1), [13]:

For a polytope represented by its vertices $v_0, v_1, ..., v_d \in \Re^d$, the volume of it is given by

$$\text{Vol}(\Delta(v_0, v_1, ..., v_d)) = \frac{1}{d!}\left|\det(v_1 - v_0, ..., v_d - v_0)\right|$$

Where $\Delta(v_0, v_1, ..., v_d)$ denotes the simplex in $\Re^d$ with vertices $v_0, v_1, ..., v_d \in \Re^d$ and $(v_1 - v_0, ..., v_d - v_0)$ is $d \times d$ matrix whose columns are $v_1 - v_0, ..., v_d - v_0$.

Next, there are two types of methods for exact volume computation of the simple polytopes, which are discussed below:

I.Triangulation methods:

In these methods one has a simple polytope P in $\Re^d$. P is triangulated into simplices $\Delta_i (i=1,2,...,s)$ $P=\bigcup_{i=1}^{s}\Delta_i$. The volume of P is simply the sum of the volumes of the simplices.

$$Vol(P)=\sum_{i=1}^{s}Vol(\Delta_i) \qquad (1.1)$$

The following: boundary triangulation, Delaunay triangulation and Cohen & Hickey's combinatorial triangulation by dimensional recursion named, as triangulations method, [13].

An important difference between these methods is that the former two methods need only a V–representation while the last method requires both V- and H-representations, [13].

Before giving the signed decomposition methods, we need the following definition.

Definition (1.2.1):

Let $\mathrm{P}\in\Re^d$ be a polytope, a signed union of P means, a collection of polytopes $\mathrm{P}_1,\mathrm{P}_2,...,\mathrm{P}_k\subseteq\Re^d$ such that $\mathrm{P}=\bigcup_{i=1}^{k}\mathrm{P}_i$, and $\mathrm{P}_i\cap\mathrm{P}_j$ is a proper face of P_i and P_j, for $i\neq j$ .In this case we write $P=\bigcup_{+}\mathrm{P}_i$, [30].

II. Signed decomposition methods:

Instead of triangulating a polytope P, one can decompose P into signed simplices whose signed union is exactly P. More specifically, P is represented as a signed union of simplices Δ_i, i =1, 2, …, s. This means,

$$\mathrm{P}=\bigcup_{i=1}^{s}\sigma_i\Delta_i \qquad (1.2)$$

Where σ_i is either +1 or –1. The volume of P is, [13].

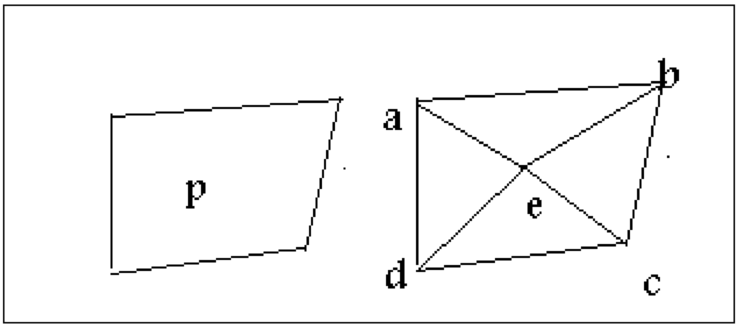
$$Vol(P)=\sum_{i=1}^{s}\sigma_i Vol(\Delta_i)$$

1.2.1 Triangulation methods

In this subsection we discuss briefly some of the known triangulation methods that compute the volume of the polytope.

(i) Boundary triangulation, [13]:

In boundary triangulation, one computes the convex hull of the perturbed points, interpreting the result in terms of the original vertices leads to a triangulation of the boundary, which by linking with a fixed interior point yields, a triangulation of P. For the convex hull computation the reverse search algorithm is chosen, [3], where only the V-representation of a polytope is required. To illustrate this method, consider the polytope which is represented by a set of vertices named {a, b, c , d} as given in figure (1). Using an interior point e where the boundary of a polytope is easily triangulated or already triangulated as in the case of simplicial polytopes. By linking a point e with the vertices a, b, c and d yields four triangles then, volumes of these triangles are found, summing all of these volumes the volume of this polytope is obtained.



(a) (b)

Figure (1):

(a) represents a polytope P.
(b) represents a partition of the given polytope P by using the boundary triangulation method.

(ii) Delaunay triangulation, [13]:

Before we discuss this method, some basic definitions concerning the Delaunay triangulation are needed.

Definition (1.2.2), [22]:

Given a set S of n distinct points in $\Re^d$, Voronoi diagram is the partition of $\Re^d$ into n polyhedron regions (denoted by $vo(\rho), \rho \in S$). Each region $vo(\rho)$ is called Voronoi cell of ρ, which is defined as set of points in $\Re^d$ that are closer to ρ than other points in S, or more precisely

$$vo(\rho) = \{X \in \Re^d : \|X - \rho\| \le \|X - q\|, \forall q \in S - \rho\}.$$

Definition (1.2.3), [22]:

Let S be a set of n points in $\Re^d$. For each point $\nu \in \Re^d$, the nearest neighbor set denoted by $(nb(S,\nu))$ of ν in S is the set of points $\rho \in S-\nu$, which are closest to ν in Euclidean distance.

Definition (1.2.4), [22]:

Let S be a set of n points in $\Re^d$. A point $\nu \in \Re^d$ is said to be a Voronoi vertex of S if $nb(S,\nu)$ is maximal over all nearest neighbor sets.

Definition (1.2.5), [22]:

Let S be a set of n points in $\Re^d$. The convex hull of the nearest neighbor set of Voronoi vertex ν denoted by $conv(nb(S,\nu))$ is said to be a Delaunay cell of ν.

The Delaunay triangulation of S is a partition of the convex hull conv(S) into the Delaunay cells of Voronoi vertices together with their faces, [22].

Now we discuss the method of Delaunay triangulation method that requires only the V–representation of the polytope.

The geometric idea behind a Delaunay triangulation of a d–polytope is to ' lift ' it on a paraboloid in dimension d+1. The following construction is very important to compute the Voronoi diagram, [22].

Let S be a set of n points in $\Re^d$. For each point $\rho \in S \subseteq \Re^d$, consider the hyperplane tangent to the paraboloid $x_{d+1} = x_1^2 + ... + x_d^2$ in $\Re^{d+1}$ at ρ:

This hyperplane is represented by h (ρ) as:

$$\sum_{j=1}^{d} \rho_j^2 - \sum_{j=1}^{d} 2\rho_j x_j + x_{d+1} = 0$$

where ρ_j $(j = 1,2,...,d)$ are the coordinates of ρ, for each point ρ, the equality in the above equation is replaced by the inequalities $(\geq)$, which yields a system of n inequalities that is denoted by $b - AX \geq 0$. The polyhedron P in $\Re^{d+1}$ of all solutions X to the system of inequality is a lifting of the Voronoi diagram to one higher dimensional space. [13], shows that the underlying convex hull algorithm uses the ' beneath – beyond ' method.

Example (1.2.1):

Consider the set of vertices:
$S = \{\rho_1 = (0,0), \rho_2 = (2,1), \rho_3 = (1,2), \rho_4 = (4,0), \rho_5 = (0,4), \rho_6 = (4,4)\}$. Here the volume of the polytope given by these vertices is to be determined. To do so, Delaunay triangulation is used to compute the volume of this polytope.

First, write down the system of linear inequalities in three variables as explained before. That is for each $\rho_j \in S, j = 1,2,...,6$, apply the inequalities:

$$\sum_{j=1}^{2} \rho_j^2 - \sum_{j=1}^{2} 2\rho_j x_j + x_3 \geq 0$$

we get a system of six inequalities

$$\begin{aligned} x_3 &\geq 0 \\ 5 - 4x_1 - 2x_2 + x_3 &\geq 0 \\ 5 - 2x_1 - 4x_2 + x_3 &\geq 0 \\ 16 - 8x_1 + x_3 &\geq 0 \\ 16 - 8x_2 + x_3 &\geq 0 \\ 32 - 8x_1 - 8x_2 + x_3 &\geq 0. \end{aligned}$$

The set of solutions $X \in \Re^3$ of the above inequalities represents a polyhedron P. By applying the cdd+ program [22], the Delaunay cells, $(\rho_1, \rho_3, \rho_5), (\rho_1, \rho_2, \rho_3), (\rho_1, \rho_2, \rho_4), (\rho_2, \rho_3, \rho_6), (\rho_2, \rho_4, \rho_6)$ and (ρ_3, ρ_5, ρ_6) are obtained. The cell (ρ_1, ρ_3, ρ_5) means the triangle which is represented by three vertices ρ_1, ρ_3 and ρ_5, and similarly for the other cells. Therefore six triangles are obtained, summing the volumes of these triangles yields the volume of the polyhedron P is equal to 16.

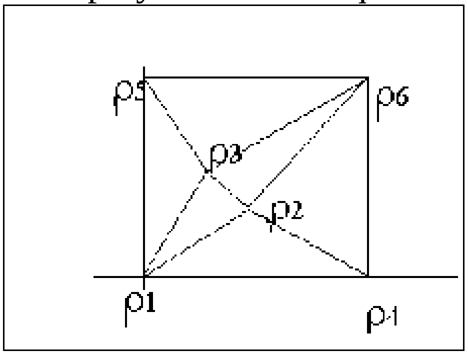


Figure (2): represents the polyhedron P with the set of vertices $\{\rho_1, \rho_2, ..., \rho_6\}$ and Delaunay cells.

(iii) Triangulation by Cohen & Hickey, [13]:

This recursive scheme triangulates a d- polytope P by choosing any vertex $v \in P$ as an apex and connecting it with the (d-1)–dimensional simplices resulting from a triangulation of all facets of P not containing v. To be precise, denote by θ^k, $0 \le k \le d$, the k- dimensional faces of P, and let η be a ' map ' which associates to each face one of its vertices. Then the pyramids with apex $\eta(\theta^d)$ and bases among the facets θ^{d-1} with $\eta(\theta^d) \notin \theta^{d-1}$ form a dissection of the polytope.

Applying the scheme recursively to all θ^{d-1} results in a set of decreasing chains of faces $\theta^0 \subset \theta^1 \subset ... \subset \theta^{d-1} \subset \theta^d$ such that $\eta(\theta^k) \notin \theta^{k-1}$ for $1 \le k \le d$. Then the set of corresponding simplices $\Delta(\eta(\theta^0), \eta(\theta^1), ..., \eta(\theta^d))$ is a triangulation of P.

To implement this recursive method, an extensive use of the double description as V-representation and H-representation is made by representing all faces as sets of vertices.

Note that in the case of Cohen & Hickey compared to a boundary triangulation all simplices in the facets containing the apex v are eliminated and therefore the number of simplices is usually reduced.

Example (1.2.2):

Consider the polytope which is represented by set of vertices $\{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}$ as illustrated in figure (3), let η be the 'map' which assigns to each face of the polytope its vertex with the lowest number, so $\eta(P) = \rho_0$, all facets which do not contain the vertex ρ_0 are examined, that is, II, III and IV. The scheme of the Cohen and Hickey is applied to facet II with $\eta(II) = \rho_1$. II is intersected with all facets not containing the vertex ρ_1, these are III, IV and V. The intersections with IV and V are empty, so this recursion is unsuccessful. The intersection with III yields the vertex ρ_2, and the fixed vertices ρ_0, ρ_1, ρ_2 forms a first simplex. The other simplices obtained from III and IV is also marked in the figure (3). Therefore we have three triangles. Summing the volumes of these triangles yields the volume of the polytope.

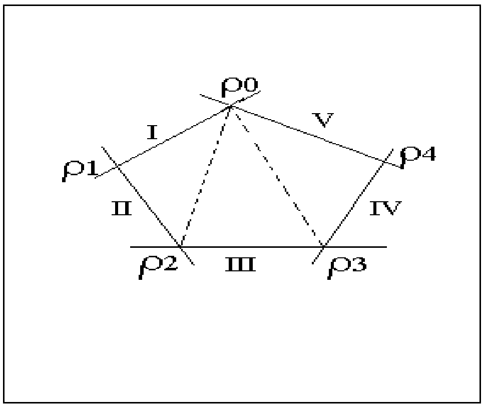


Figure (3): represents the partition of the polytope by Cohen & Hickey's triangulation method.

1.2.2 Signed decomposition method

In this subsection Lawrence's volume formula, which is one of the signed decomposition methods, is discussed.

(i) Lawrence's volume formula, [13]:

Assume the polytope P is simple and choose a vector $C \in \Re^d$ and $q \in \Re$ such that the function $f : \Re^d \to \Re$ which is defined by f(X)=$C^T X + q$ is not constant along any edge that connected the vertices of the polytope P and C^T is the transpose of C. Let V be the set of vertices defining the polytope P. For each vertex $v \in V$, let A_v be the $d \times d$ – matrix composed by the rows of A which are binding at v. Then by using [13], A_v is invertible and $\gamma^v = \left[A_v^T\right]^{-1} C$. The assumption imposed on C assures that none of the entries of γ^v is zero. It is shown that

$$Vol(P) = \sum_{v \in V} \frac{(C^T v + q)^d}{d!\ \left|\det A_v\right| \prod_{i=1}^{d} \gamma_i^v}$$

To illustrate this method, consider the following example.

Example (1.2.4) :

Consider the polytope P which is described by the following constraints

$$-x_1 \le 0$$
$$-x_2 \le 0$$
$$x_1 \le 2$$
$$x_2 \le 2$$
$$x_1 + x_2 \le 3$$

then

$$A=\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix},\ X=\begin{bmatrix}x_1\\ x_2\end{bmatrix} \text{ and } b=\begin{bmatrix}0\\0\\2\\2\\3\end{bmatrix}$$

It is easy to check that the polytope of this example is simple, therefore the Lawerence volume formula can be applied. Define a function f by $f(X)=x_1-x_2$ where $C^T=(1,-1)$ and q = 0. Note that f (X) is non – constant on each edge of the polytope in the figure (4), for example, on edge (1) which connect v_1 and v_2, $x_2=0$ and x_1 varies from 0 to 2. Therefore f (X) = x_1 which varies from 0 to 2 which means that it is nonconstant on edge (1) and similarly on each edge of P. According to figure (4), it is seen that the set of vertices, which represents the polytope, is

$\{v_1=(0,0), v_2=(2,0), v_3=(2,1), v_4=(1,2), v_5=(0,2)\}$.

Now, consider $v_1=(0,0)$, this vertex satisfies the first two constraints, and this implies that

$$A_{v_1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \quad , \quad \text{hence} \quad \left|\det A_{v_1}\right|=1 \quad \text{and} \quad \gamma^{v_1}=\left[A_{v_1}^T\right]^{-1}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$\gamma^{v_1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

and for $v_2=(2,0)$, this vertex satisfies the second and third constraints, that is,

$$A_{v_2}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ \left|\det A_{v_2}\right|=1 \text{ And } \gamma^{v_2}=\left[A_{v_2}^T\right]^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

And similarly for the other vertices we get

$$\gamma^{v_3}=\begin{bmatrix}2\\ -1\end{bmatrix},\ \gamma^{v_4}=\begin{bmatrix}-2\\ 1\end{bmatrix} \text{ And } \gamma^{v_5}=\begin{bmatrix}-1\\ -1\end{bmatrix}$$

Then Lawrence's volume formula is applied to get the volume of P.

$$Vol(P) = \sum_{v \in V} \frac{(C^T v)^2}{2! \left|\det A_v\right| \prod_{i=1}^{2} \gamma_i^v}$$

$$= \frac{0^2}{2!(1)(-1)(1)} + \frac{2^2}{2!(1)(1)(1)} + \frac{1^2}{2!(1)(2)(-1)} + \frac{(-1)^2}{2!(1)(-2)(1)} + \frac{(-2)^2}{2!(1)(-1)(-1)}$$

$$= 0 + 2 + (-1/4) + (-1/4) + 2 = 3\frac{1}{2} \; .$$

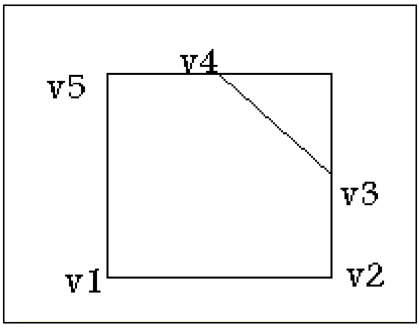


Figure (4): represents a polytope with the vertices v_1, v_2, v_3, v_4 *and* v_5

1.3 Methods for computing integral points

The main objective of this section is to recall some methods for finding integral points of a polyhedron. In this work, we use the symbol $\left|P \cap Z^d\right|$ to denote the number of integral points in the polyhedron P, where Z^d is the integer lattice and P is a rational polyhedron. These methods are:

Method (1), [4]:

For d = 2, $P \subset \Re^2$ and P is an integral polyhedron. The famous formula, [42, p.240] states that

$$\left|P \cap Z^2\right| = area(P) + \frac{\left|\partial P \cap Z^2\right|}{2} + 1$$

That is, the number of integral points in an integral polyhedron is equal to the area of the polyhedron plus half the number of integral points on the boundary of the polyhedron plus one. This formula is useful because it is much more efficient than the direct enumeration of integral points in a polyhedron. The area of P is computed by triangulating the polyhedron. Furthermore, the boundary ∂P is a union of finitely many straight-line intervals, and counting integral points in intervals is easy.

Method (2), [4]:

Let $P \subset \Re^d$ be a polytope, then one can write the number of integral points in P as

$$\left|P \cap Z^d\right| = \sum_{X \in Z^d} \delta(X, P)$$

where
$$\delta(X,P) = \begin{cases} 1 & \text{if } X \in P \\ 0 & \text{if } X \notin P \end{cases}$$

Before we give the next method, we need the following definition.

Definition (1.3.1), [2, p.61]:

The Dedekind sum of two relatively prime positive integers a and b denoted by $S(a,b)$ can be defined as follows,

$$S(a,b) = \sum_{i=1}^{b} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{ai}{b}\right)\right)$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin Z \\ 0 & \text{if } x \in Z \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer $\leq x$.

Remarks (1.3.1):

Dedekind sums appear in various branches of mathematics: the number theory, algebraic geometry and topology. These include the quadratic reciprocity law, random number generators [32], and lattice point problems [19]. More details about Dedekind sums are given in chapter two

Now, we are in a position that we can explain the following method.

Method (3), [4]:

Let $\Delta \subset \Re^3$ be the tetrahedron with vertices (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c) where a, b and c are pairwise coprime positive integers, then the number of integral points in Δ can be expressed as:

$$\left|P\cap Z^3\right| = \frac{abc}{6} + \frac{ab+ac+bc+a+b+c}{4} + \frac{1}{12}\left(\frac{ac}{b}+\frac{bc}{a}+\frac{ab}{c}+\frac{1}{abc}\right) - S(bc,a) - S(ac,b) - S(ab,c)$$

This formula is useful because it reduces counting the number of integral points to a computation of Dedekind sums, which can be done efficiently.

<u>Method (4), [4]:</u>

Let $P \subset \Re^d$ be an integral polytope, for positive integer t, let tP={tX : X∈P} denote the dilated polytope P. By [42, p.238] there is a polynomial p(t) called the Ehrhart polynomial of P

$$\left|tP\cap Z^d\right| = p(t)$$

where p(t) = $a_d\, t^d + a_{d-1}\, t^{d-1} + ... + a_0$

furthermore, $a_0 = 1$ and a_d is the volume of the polytope P.

To illustrate these methods, consider the following examples.

<u>Example (1.3.1):</u>

Let us consider three points in two dimensions such that $v_1 = (0,1), v_2 = (1,0)$ *and* $v_3 = (0,0)$. Then the convex hull of v_1, v_2 *and* v_3 is a triangle in two dimensions.

We compute the number of integral points by these methods.
From method (1), one can have

$$\left|P\cap Z^2\right| = area(P) + \frac{\left|\partial P\cap Z^2\right|}{2} + 1$$

the area of triangle, is area(P)=0.5(1)(1)=0.5 and $\left|\partial P\cap Z^2\right| = 3$ which represents the number of integral points on the boundary of the triangle. Then the number of integral points of the triangle is

$$\left|P\cap Z^2\right| = 0.5 + 0.5(3) + 1 = 3.$$

In method (2), we have

$$\left|P\cap Z^2\right| = \sum_{X\in Z^2} \delta(X,P)$$

$$v_1 = (0,1) \in P \text{ then } \delta(v_1, P) = 1$$
$$v_2 = (1,0) \in P \text{ then } \delta(v_2, P) = 1$$
$$v_3 = (0,0) \in P \text{ then } \delta(v_3, P) = 1$$

therefore $\left|P \cap Z^2\right| = 1+1+1=3$, which is the number of integral points for the triangle.

Form method (4), one can have

$$\left|P \cap Z^2\right| = a_2 + a_1 + 1$$

where a_2 is the volume of the polytope and a_1 is the half number of integral points on the boundary of the polytope.
In this case,

a_2 = 0.5(1)(1)=0.5

and a_2 = 3/2

therefore $\left|P \cap Z^2\right| = 0.5 + 1.5 + 1 = 3$.

Example (1.3.1):

Let us consider the tetrahedron $\Delta \subset \Re^3$ with vertices (0,0,0), (3,0,0), (0,5,0), (0,0,7).

Form method (3), the number of integral points in Δ can be expressed as,

$$\left|P \cap Z^3\right| = \frac{abc}{6} + \frac{ab+ac+bc+a+b+c}{4} + \frac{1}{12}\left(\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{1}{abc}\right) - S(bc,a) - S(ac,b) - S(ab,c)$$

here a=3, b=5 and c=7. Therefore

$$\left|P \cap Z^3\right| = \frac{105}{6} + \frac{15+21+35+3+5+7}{4} + \frac{1}{12}\left(\frac{21}{5} + \frac{35}{3} + \frac{15}{7} + \frac{1}{105}\right) - S(35,3) - S(21,5) - S(15,7)$$

After simple computation one can get, S(35,3)=-0.05555, S(21,5)=0.2, S(15,7)= 0.35714,

Thus $\left|P \cap Z^3\right| = 17.5 + 21.5 + 1.501877 + 0.05555 - 0.2 - 0.35714 = 40.$

Chapter Two
Computing the Volume and Integral Points of V-Representation of a Polytope Using Ehrhart Polynomial

Introduction

As was shown in chapter one, the Ehrhart polynomial of a convex lattice polytope counts the number of integral points in an integral dilate of the polytope. E. Ehrhart proved that the function which counts the number of lattice points that lie inside the dilated polytope tP is a polynomial in t and it is denoted by L(P,t), which is the cardinal of $(tP \cap \mathrm{Z}^d)$ where Z^d is the integer lattice in $\Re^d$, [4]. Many of Ehrhart's valuable results are unknown by mathematician and computer science community at this time, since many of these results have been published in local reports and in French language, [4].

In this chapter we present some methods for computing the coefficients of Ehrhart polynomial that depend on the concepts of Dedekind sum and residue theorem in complex analysis. Also, a method for computing the coefficients of the Ehrhart polynomial is introduced with general formula that counts the derivatives in the introduced method. For our knowledge this method seems to be new. The polytopes that we take are with V – representations.

This chapter consists of six sections. In section one, some basic definitions and remarks concerning the Ehrhart polynomial are given. Section two gives a method for finding the Ehrhart polynomial of the polytope using the formula of the Dedekind sum. Another method for finding the Ehrhart polynomial using residue theorem in complex analysis is presented in section three. In section four the Ehrhart coefficients are computed and in section five, we give a method for computing c_{d-3}, c_{d-4} until c_{d-9} of the Ehrhart polynomial, the general formula for the differentiation is given in section six.

2.1 Basic concepts about Ehrhart polynomials

As seen before, the Ehrhart polynomial is very important in many fields of mathematics. Therefore some methods for finding the coefficients of this polynomial are to be listed. To do so, some basic definitions, theorems and remarks are given in this section.

We start this section by the following definitions

Definition (2.1.1), [19]:

Let $P \subset \Re^d$ be a lattice d-polytope. For $t \in Z^+$, the set tP={tX : X $\in$ P} is said to be the dialeted polytope.

The definition of the Ehrhart polynomial is given below

Definition (2.1.2), [42, p.235]:

Let $P \subset \Re^d$ be a lattice d-polytope. Define a map $L : N \longrightarrow N$ by L(P,t) = card(tP$\cap Z^d$), where 'card' means the cardinality of (tP$\cap Z^d$) and N is the set of natural numbers. It is seen that L(P,t) can be represented as: L(P,t) =1+$\sum_{i=1}^{d} c_i t^i$, this polynomial is said to be the Ehrhart polynomial of a lattice d-polytope P.

Remark (2.1.1):

Let $P \subset \Re^2$ be a lattice 2-polytope, the Ehrhart polynomial of P is given by

$$L\,(P,t) = At^2 + \frac{1}{2}Bt + 1$$

where A is the area of the polytope and B is the number of lattice points on the boundary of P, [28].

Theorem (2.1.1), [8]:

Let $P \subset \Re^d$ be a lattice d-polytope, with the Ehrhart polynomial L(P,t) =$\sum_{i=0}^{d} c_i t^i$. Then c_d is the volume of P, while the constant term is one, which is equal to the Euler characteristic of P.

The other coefficients of L(P,t) are not easily accessible. In fact, a method of computing these coefficients was unknown until quite recently, [4], [12] and [19].

Before we give the next theorems, we need the following definitions:

Definition (2.1.3), [35, p.86]:

Let P be a polytope which is defined by $P=\{X\in\Re^d : AX\le b\}$, X∈ P is said to be an interior point of P if $a_i X < b_i$ for i=1,2,…,m. , where a_i is the i-*th* row of the matrix A, b_i is the i-*th* row of the vector b and m<d.

Definition (2.1.4), [42, p.18]:

Let k and j be two given positive integer and c(k,j) is the number of permutations $\pi=(c_1,...,c_k)$ with exactly j cycles. Then the Stirling numbers of the first kind of k and j, denoted by S_1(k,j), are defined as $S_1(k,j)=(-1)^{k-j}c(k,j)$, and c(k,j) can be found from the recurrence relation c(k,j)=(k-1)c(k-1,j)+c(k-1,j), k,j ≥1,

Theorem(2.1.2),(The reciprocity theorem for Ehrhart polynomial), [42 ,p.238], [19]:

The function L(P,t) satisfies the reciprocity law:

$$L(P,-t) = (-1)^{\dim P} L(P^\circ,t)$$

where P° is the interior of P and t is an integer.

Remark (2.1.2):

From theorem (2.1.2) the study of the polytope P° is essentially equivalent to the study of the polytope P, where P° is the interior of P.

A linear relation satisfied by the coefficients of all Ehrhart polynomials is established by [8] which is a continuation of the pioneering work of [Stanley,1980,1991; Betke&Mcmullen, 1985; Hibi,1995] in [8] who established several relations of linear inequalities for the coefficients and are given in the following theorems.

Theorem (2.1.3), [8]:

Let $P\subset\Re^d$ be a lattice d-polytope with the Ehrhart polynomial L(P,t)=$1+\sum_{i=1}^{d}c_i t^i$ then, $c_r \le (-1)^{d-r}S_1(d,r)c_d + (-1)^{d-r-1}\frac{S_1(d,r+1)}{(d-1)!}$ for r=1,2,…, d-1, where S_1(k, j) denotes the Stirling numbers of the first kind of k and j.

Theorem (2.1.4), [8]:

Let $P \subset \Re^d$ be a lattice d-polytope with the Ehrhart polynomial L(P,t)=$1+\sum_{i=1}^{d} c_i t^i$. Then the following inequalities are valid:

$$c_d \geq \frac{1}{d!}$$

$$c_{d-1} \geq \frac{d+1}{2(d-1)!}$$

$$\sum_{i=0}^{d} (-1)^{d-i} c_i \geq 0 .$$

2.2 Counting integral points using Dedekind sums

In this section we describe the relation between the Dedekind sum and the Ehrhart polynomial of a polytope and discussed a theorem that counts the number of integral points in a polytope.

This section starts by some basic definitions and remarks that are useful to discuss the method for counting the integral points of a polytope.

Recall that the Dedekind sum of two relatively prime positive integers a and b, denoted by S(a,b), is defined as

$$S(a,b) = \sum_{i=1}^{b} ((\frac{i}{b}))((\frac{ai}{b}))$$

where $((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & if \ x \notin Z \\ 0 & if \ x \in Z \end{cases}$

and $\lfloor x \rfloor$ is the greatest integer $\leq x$.

Remark(2.2.1):

The discrete Fourier expansions can be used to rewrite the Dedekind sum in terms of the Dedekind cotangent sum, that is, for two relatively prime positive integers a and b:

$$S(a,b) = \frac{1}{4b} \sum_{k=1}^{b-1} \cot(\frac{\pi k a}{b}) \cot(\frac{\pi k}{b})$$

where S(a,b) is the Dedekind sum of a and b, [2, p. 72].

<u>Theorem (2.2.1), [5]:</u>

Let P denote the simplex in $\Re^d$ with the vertices (0,0,…,0), $(a_1,0,\ldots,0),(0,a_2,0,\ldots,0),\ldots,(0,\ldots,0,a_d)$, where $a_1,\ldots,a_d \in \mathrm{N}$ are relatively prime. Denote the corresponding Ehrhart polynomial by $L(P,t)=\sum_{j=0}^{d} c_j t^j$. Then c_m is the coefficient of $s^{-(m+1)}$ in the Laurent series at s = 0 of

$$\frac{\pi^{m+1}}{m!2^{d-m}\rho}\sum_{r=1}^{\rho}\left(1+\coth(\frac{\pi}{a_1}(s+ir))\right)\left(1+\coth(\frac{\pi}{a_2}(s+ir))\right)\ldots\left(1+\coth(\frac{\pi}{a_d}(s+ir))\right)$$
$$\left(1+\coth(\frac{\pi}{\rho}(s+ir))\right)$$

where $\rho = a_1 \ldots a_d$.

<u>Definition (2.2.2) [34, p. 29]:</u>

Let K be a closed bounded convex set in $\Re^d$. Then a hyperplane H of $\Re^d$ is said to support K if $H \cap K \neq \phi$, and K is contained in one of the closed half - spaces determined by H.

<u>Theorem (2.2.2), (Dedekind's reciprocity law), [5]:</u>

For two relatively prime positive integers a and b

$$S(a,b)+S(b,a)=\frac{-1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{ab}+\frac{b}{a}\right) \tag{2.1}$$

<u>Proof:</u>

Let P denote a simplex in $\Re^2$ with the vertices (0,0), (a,0), (0,b), where a, b$\in$N are relatively prime. From theorem (2.2.1), c_0 is the coefficient of s^{-1} in the Laurent series at s = 0 of

$$\frac{\pi}{4ab}\sum_{r=1}^{ab}\left(1+\coth(\frac{\pi}{a}(s+ir))\right)\left(1+\coth(\frac{\pi}{b}(s+ir))\right)\left(1+\coth(\frac{\pi}{ab}(s+ir))\right) \tag{2.2}$$

The Laurent series of each factor depends on r: that is for any $c \in \mathrm{N}$, we write the series of $(1+\coth\frac{\pi}{c}(s+ir))$ such that c divides r or not.

To illustrate this, we consider these two cases:

<u>First case</u>: if c divides r, then there exists an integer m such that r = mc therefore,

$$\left(1+\coth\frac{\pi}{c}(s+ir)\right)=1+\coth\left(\pi\left(\frac{s}{c}+mi\right)\right)=1+\coth\left(\frac{\pi s}{c}\right)$$

it is known that, Laurent series of coth(z) about z = 0 is:

$$\coth(z)=\frac{1}{z}+\frac{1}{3}z-\frac{1}{45}z^3+...+\frac{B_{2n}}{(2n)!}\frac{(2z)^{2n}}{z}+..., \qquad |z|<\pi$$

where the B_n's are Bernoulli numbers that are defined by the following steps:

Step (1): Expand $\frac{1}{e^z-1}$ as

$$\frac{1}{e^Z-1}=\frac{1}{z}-\frac{1}{2}+\sum_{i=0}^{\infty}A_{2i+1}z^{2i+1}$$

where $A_1=\frac{1}{12}, A_3=\frac{-1}{720}, A_5=\frac{1}{30240},...$

Step (2): Define, the n-*th* Bernoulli number B_n by, [29, p.6]

$$B_1=1,\ A_{2n-1}=(-1)^{n-1}\frac{B_{2n}}{(2n)!} \text{ and } B_{2n+1}=0 \quad \text{for } n=1,2,\ldots$$

By replacing z by $\frac{\pi s}{c}$ in coth(z) we get

$$\coth(\frac{\pi s}{c})=\frac{c}{\pi s}+\frac{\pi}{3c}s+O(s^3), \qquad \left|\frac{s}{c}\right|<1$$

Let $S_c=1+\coth(\frac{\pi s}{c})=\frac{c}{\pi}s^{-1}+1+\frac{\pi}{3c}s+O(s^3)$.

Second case: assume that c does not divide r.

Let $f(s)=(1+\coth\frac{\pi}{c}(s+ir))$

then $f(0)=1+\coth(\frac{\pi ri}{c})$

also

$f'(s)=\frac{\pi}{c}\operatorname{csc}h\left(\frac{\pi}{c}(s+ir)\right)\coth\left(\frac{\pi}{c}(s+ir)\right)$ then

$f'(0)=\frac{\pi}{c}\operatorname{csc}h(\frac{\pi ir}{c})\coth(\frac{\pi ir}{c})$.

Therefore, the Maclaurin series of f(s) takes the form

$R_c=1+\coth(\frac{\pi ri}{c})+O(s)$.

Therefore, from the above one can get

If c | r then $S_c = \frac{c}{\pi}s^{-1} + 1 + \frac{\pi}{3c}s + O(s^3)$ and

If c ∤ r then $R_c = 1 + \coth(\frac{\pi r i}{c}) + O(s)$

Introduce the notation

$$\chi_c = \begin{cases} 1 & \text{if} \quad c \mid r \\ 0 & \text{if} \quad c \mid r \end{cases}$$

so $(1 + \coth\frac{\pi}{c}(s + ir))$ can be written as

$\left(1 + \coth\frac{\pi}{\text{c}}(\text{s} + \text{ir})\right) = \text{S}_\text{c}\chi_\text{c} + \text{R}_\text{c}(1 - \chi_\text{c})$ and (2.2) becomes

$$\frac{\pi}{4ab}\sum_{r=1}^{ab}\big((S_a\chi_a + R_a(1-\chi_a))(S_b\chi_b + R_b(1-\chi_b))(S_{ab}\chi_{ab} + R_{ab}(1-\chi_{ab}))\big)$$

Expand this into eight terms, which are

$$\frac{\pi}{4ab}\sum_{r=1}^{ab}\big(S_a\chi_a S_b\chi_b S_{ab}\chi_{ab} + S_a\chi_a S_b\chi_b R_{ab}(1-\chi_{ab}) + S_a\chi_a S_{ab}\chi_{ab}R_b(1-\chi_b) + S_a\chi_a R_b R_{ab}(1-\chi_{ab})(1-\chi_b) + S_b\chi_b S_{ab}\chi_{ab}R_a(1-\chi_a) + S_b\chi_b R_{ab}R_a\,(1-\chi_a)(1-\chi_{ab}) + S_{ab}\chi_{ab}R_b R_a(1-\chi_a)(1-\chi_b) + R_a R_b R_{ab}(1-\chi_a)(1-\chi_{ab})(1-\chi_b)\big)$$

and consider each term according to the number of S_c factors

1. Terms with one S_c factor are

$$S_a\chi_a R_b(1-\chi_b)R_{ab}(1-\chi_{ab}) = S_a R_b R_{ab}\chi_a(1-\chi_b-\chi_{ab}+\chi_{ab}) = S_a R_b R_{ab}(\chi_a - \chi_{ab}) \qquad (2.3)$$

and similarly

$$S_b\chi_b R_a(1-\chi_a)R_{ab}(1-\chi_{ab}) = S_b R_a R_{ab}\;(\chi_b - \chi_{ab}) \qquad (2.4)$$

The term with S_{ab} is zero (note that $\chi_a\chi_{ab} = \chi_b\chi_{ab} = \chi_{ab}$ and $\chi_a\chi_b = \chi_{ab}$).

To compute the contribution of (2.3) we need the support of $\chi_a - \chi_{ab}$ in {1,2, …, ab} which is $\{\text{ka}: 1 \le \text{k} \le \text{b} - 1\}$;
thus its contribution to (2.2) is,

$$\frac{\pi}{4ab}\frac{a}{\pi}\sum_{r=1}^{b-1}(1+\coth(\frac{\pi ika}{b}))(1+\coth(\frac{\pi ika}{ab}))$$

$$=\frac{1}{4b}\sum_{r=1}^{b-1}(1-i\cot(\frac{\pi ka}{b}))(1-i\cot(\frac{\pi k}{b}))$$

$$=\frac{1}{4b}\sum_{r=1}^{b-1}\left(1-\cot\left(\frac{\pi ka}{b}\right)\cot\left(\frac{\pi k}{b}\right)\right)+i\left(-\cot\left(\frac{\pi ka}{b}\right)-\cot\left(\frac{\pi k}{b}\right)\right)$$

The imaginary part in the preceding sum is zero, because the original generating function is real. The obtained result is

$$\frac{1}{4}-\frac{1}{4b}-S(a,b).$$

Similarly (2.4) gives a contribution of $\frac{1}{4}-\frac{1}{4a}-S(b,a)$.

2. There are no terms with two S_c factors,

$$S_a\chi_a S_b\chi_b R_{ab}(1-\chi_{ab})=S_aS_bR_{ab}\chi_{ab}(1-\chi_{ab})=0$$

and

$$S_a\chi_a R_b(1-\chi_b)S_{ab}\chi_{ab}=S_aR_bS_{ab}\chi_{ab}(1-\chi_b)=0$$

3. Finally, the term $S_a\chi_aS_b\chi_bS_{ab}\chi_{ab} = S_aS_bS_{ab}\chi_{ab}$ has a support {ab}, and gives a contribution of

$$\frac{\pi}{4ab}(\frac{a}{\pi}\frac{b}{\pi}\frac{\pi}{3ab}+\frac{a}{\pi}\frac{ab}{\pi}\frac{\pi}{3a}+\frac{b}{\pi}\frac{ab}{\pi}\frac{\pi}{3a}+\frac{a}{\pi}+\frac{b}{\pi}+\frac{ab}{\pi})$$

$$=\frac{1}{12}(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a})+\frac{1}{4}(\frac{1}{b}+\frac{1}{a}+1).$$

Adding all contributions, with the fact that c_0 is the coefficient of s^{-1} in the Laurent series at s = 0 of (2.2) and $c_0=1$, [42, p.235] we get

$$1=\frac{3}{4}+\frac{1}{12}(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a})-S(a,b)-S(b,a)$$

Then

$$S(a,b)+S(b,a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{ab}+\frac{b}{a}\right)$$

which is the Dedekind reciprocity law.■

2.3 Counting integral points using the residue theorem

This section is concerned with a method given in [6] to count the integral points of a given polytope by means of the residue theorem.

To do so, let $Z^d \subset \Re^d$ be a d–dimensional integer lattice, so the polytope P° is defined as $P^\circ = \left\{(x_1,...,x_d)\in \Re^d : \sum_{k=1}^{d}\frac{x_k}{a_k} < 1 \text{ and } x_k > 0\right\}$, with vertices $(0,0,...,0),(a_1,0,...,0),(0,a_2,0,...,0),...,(0,...,0,a_d)$, where $a_1,a_2,...,a_d$ are positive integers.

Recall that for $t\in N$, L(P,t) is the number of integral points in the dilated polytope tP.

Let us use the notation: $A = a_1a_2...a_d$, $A_k = a_1a_2...\hat{a}_k...a_d$ (where $\hat{a}_k$ means the factor a_k is omitted). Then L(P,t) can be written as,

$$L(P,t)=\text{card}\left\{(m_1,m_2...,m_d)\in Z^d : \sum_{k=1}^{d}\frac{m_k}{a_k}\le t \text{ and all } m_k \ge 0\right\}.$$

Thus

$$\frac{m_1(a_2a_3...a_d)+m_2(a_1a_3...a_d)+...+m_d(a_1a_2...a_{d-1})}{a_1a_2...a_d}\le t$$

In other words,

$$\frac{m_1A_1+m_2A_2+...+m_dA_d}{A}\le t$$

Therefore,

$$\sum_{k=1}^{d} m_kA_k \le At$$

Hence, there exists a non negative real number m such that

$$\sum_{k=1}^{d} m_kA_k + m = At, \qquad m_k, m \ge 0$$

Therefore, by using [6] one can get,

$$\mathrm{L(P,t)} = \mathrm{card}\left\{ (\mathrm{m}_1,...,\mathrm{m}_d,\mathrm{m}) \in Z^{d+1} : \sum_{\substack{k=1 \\ m_k, m \geq 0}}^{d} \mathrm{m}_k \mathrm{A}_k + \mathrm{m} = \mathrm{tA} \right\}$$

L(P,t) can be interpreted as the Taylor coefficient of Z^{tA} for the function

$$\left(\sum_{m_1=0} Z^{m_1 A_1} \cdot \sum_{m_2=0} Z^{m_2 A_2} ... \sum_{m_d=0} Z^{m_d A_d} \cdot \sum_{m=0} Z^{m} \right)$$

$$= (1 + Z^{A_1} + Z^{2A_1} + ...)(1 + Z^{A_2} + Z^{2A_2} + ...)...(1 + Z^{A_d} + Z^{2A_d} + ...)(1 + Z + Z^2 + ...)$$

$$= \frac{1}{1 - Z^{A_1}} \frac{1}{1 - Z^{A_2}} ... \frac{1}{1 - Z^{A_d}} \frac{1}{1 - Z}.$$

Equivalently

$$L(P,t) = \mathrm{Re}\,s\left(\frac{Z^{-tA-1}}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)}, Z = 0 \right)$$

$$= \mathrm{Re}\,s\left(\frac{Z^{-tA-1} - 1/Z + 1/Z}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)}, Z = 0 \right)$$

$$= \mathrm{Re}\,s\left(\frac{Z^{-tA-1} - 1/Z}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)} + \frac{1/Z}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)}, Z = 0 \right)$$

$$= \mathrm{Re}\,s\left(\frac{Z^{-tA} - 1}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)Z}, Z = 0 \right) +$$

$$\mathrm{Re}\,s\left(\frac{1}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)Z}, Z = 0 \right)$$

then

$$L(P,t) = \mathrm{Re}\,s\left(\frac{Z^{-tA} - 1}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)Z}, Z = 0 \right) + 1 \qquad (2.5)$$

Let $f_{-t}(\mathrm{Z}) = \dfrac{\mathrm{Z}^{-tA} - 1}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})...(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}}$

Expression (2.5) counts the number of lattice points in tP. Therefore we need to compute the residue of $f_{-t}(\mathrm{Z})$ at Z=0 and use the residue theorem for the sphere $C \cup \{\infty\}$. In this notation,

$$L(P,t) = \mathrm{Re}\,s(f_{-t}(\mathrm{Z}), \mathrm{Z}=0) + 1 \tag{2.6}$$

Let $\Omega = \{Z \in \mathrm{C} \setminus \{1\} : Z^{\frac{\mathrm{A}}{\mathrm{a_k a_j}}} = 1, 1 \le \mathrm{k} < \mathrm{j} \le \mathrm{d}\}$

The roots of $f_{-t}(\mathrm{Z})$ of unity in Ω, 0 and 1 are the only poles of $f_{-t}(\mathrm{Z})$ Now, by [21, p. 273] one can get

$\mathrm{Re}\,s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\infty) = \mathrm{Re}\,s\left(f_{-t}(\frac{1}{\mathrm{Z}}), \mathrm{Z}=0\right)$. Therefore,

$$f_{-t}(\frac{1}{z}) = \frac{(\mathrm{Z}^{tA}-1)\mathrm{Z}^{A_1+A_2+...+A_d+1}}{(\mathrm{Z}^{A_1}-1)...(\mathrm{Z}^{A_d}-1)(\mathrm{Z}-1)}$$

then, $\mathrm{Re}\,s\left(f_{-t}(\frac{1}{\mathrm{Z}}, \mathrm{Z}=0)\right) = 0$.

The following lemma is needed for proving the next theorem.

<u>Lemma (2.3.1), [20, p.204]:</u>

The sum of the residues of a rational function at all the poles in the finite plane, together with the residue at infinity, is zero.

This theorem appears in [6] without a proof. Here, we prove it for the sake of completeness.

<u>Theorem (2.3.1), [6]:</u>

Let P be a polytope defined as

$$P = \left\{(x_1,...,x_d) \in \Re^d : \sum_{k=1}^{d} \frac{x_k}{a_k} \le 1 \text{ and } x_k > 0\right\}, \tag{2.7}$$

with vertices $(0,0,...,0), (\mathrm{a}_1,0,...,0), (0,a_2,0,...,0),...,(0,...,0,a_d)$,where

$a_1,...,a_d$ are positive integers, and $f_{-t}(\mathrm{Z})$ and Ω are defined as

$$f_{-t}(\mathrm{Z}) = \frac{\mathrm{Z}^{-tA} - 1}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})...(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}},$$

and

$\Omega = \{Z \in \mathrm{C} \setminus \{1\} : Z^{\frac{A}{a_k a_j}} = 1, 1 \le k < j \le d\}$, then

$$L(P,t) = 1 - \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=1) - \sum_{\lambda \in \Omega} \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\lambda).$$

Proof:

Using lemma (2.3.1), the poles of $f_{-t}(\mathrm{Z})$ are at 1, 0 and the roots of unity therefore,

$$\mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=0) + \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=1) + \sum_{\lambda \in \Omega} \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\lambda)$$

$$+ \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\infty) = 0$$

So $\mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=0) = -\mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=1) - \sum_{\lambda \in \Omega} \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\lambda)$

After substations the results in (2.6), we get

$$L(P,t) = 1 - \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=1) - \sum_{\lambda \in \Omega} \mathrm{Re}\, s(f_{-t}(\mathrm{Z}), \mathrm{Z}=\lambda).\ \blacksquare$$

Theorem(2.3.2), [6]:

Let P be a lattice d-polytope given by expression (2.7). Then, the function L(P,t) is a polynomial in t.

Proof:

Let $\lambda \in \Omega$ be a B-*th* root of unity, where B is the product of some of the $a_k's$. Express Z^{-tA} in terms of its power series about $\mathrm{Z}=\lambda$. The coefficients of this power series involve various derivatives of Z^{-tA}, evaluated at $\mathrm{Z}=\lambda$. Introduce a change of variable: $Z = \omega^{1/\mathrm{B}} = \exp(1/\mathrm{B}\log\omega)$. A suitable branch of the logarithm is chosen such that $\exp(1/\mathrm{B}\log(1)) = \lambda$. The terms depending on t in the power series of Z^{-tA} consist therefore the derivatives of the function $\mathrm{Z}^{-tA/B}$, evaluated at z=1. For this, the coefficients of the power series of Z^{-tA} are polynomials in t. The fact that L(P,t) is simply the sum of all these residues, L(P,t) is a polynomial in t. ■

2.4 The Ehrhart coefficients

In this section, some details for deriving formula of Ehrhart coefficients are given. For each coefficient of the Ehrhart polynomial

$$L(P,t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_0 .$$

A formula for finding these coefficients can be derived with a small modification of $f_t(\mathrm{Z})$.

Consider the function,

$$g_k(\mathrm{Z}) = \frac{(\mathrm{Z}^{-tA} - 1)^k}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})\ldots(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}}$$

$$g_k(\mathrm{Z}) = \frac{\sum_{j=0}^{k} \binom{k}{j} \mathrm{Z}^{-tA(k-j)} (-1)^j}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})\ldots(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}}$$

If $-\sum_{j=0}^{k} \binom{k}{j} (-1)^j = 0$ is inserted in the numerator of the above equation, we get

$$g_k(\mathrm{Z}) = \frac{\sum_{j=0}^{k} \binom{k}{j} \mathrm{Z}^{-tA(k-j)} (-1)^j - \sum_{j=0}^{k} \binom{k}{j} (-1)^j}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})\ldots(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}}$$

$$g_k(\mathrm{Z}) = \sum_{j=0}^{k-1} \frac{\binom{k}{j} (-1)^j (\mathrm{Z}^{-tA(k-j)} - 1)}{(1-\mathrm{Z}^{A_1})(1-\mathrm{Z}^{A_2})\ldots(1-\mathrm{Z}^{A_d})(1-\mathrm{Z})\mathrm{Z}}$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_{-t(k-j)}(\mathrm{Z})$$

Recall that, $\mathrm{L(P,t)} = \mathrm{Res}(\mathrm{f}_{-t}(Z), Z = 0) + 1$, using this relation, we obtain,

$$\mathrm{Re}\,s(g_k(\mathrm{Z}), \mathrm{Z} = 0) = \mathrm{Re}\,s(\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_{-t(k-j)}(\mathrm{Z}), \mathrm{Z} = 0)$$

$$= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j \operatorname{Res}(f_{-t(k-j)}(\mathrm{Z}), \mathrm{Z}=0)$$

$$= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j (L(P,(k-j)t)-1)$$

$$= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j (L(P,(k-j)t) - \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j$$

$$g_k(\mathrm{Z}) = \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j (L(P,(k-j)t) + (-1)^k \ .$$

The following lemma is needed to derive the formula of the coefficients of the Ehrhart polynomial. But, before that we give the definition of the Stirling number of the second kind and its properties.

Definition(2.4.1), [42, p. 33]:

The Stirling number of the second kind $S_2(m,k)$ is defined as, the number of partitions of an m –set into k – blocks.

The following properties of $S_2(m,k)$ are known, [42, p. 33-35].

$$S_2(m,k) = 0 \quad \text{if } k > m \tag{2.8}$$

$$S_2(m,1) = 1 \tag{2.9}$$

$$S_2(m,m) = 1 \tag{2.10}$$

$$S_2(m,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} j^m \tag{2.11}$$

$$S_2(m,k) = kS_2(d-1,k) + S_2(d-1,k-1)$$

Lemma(2.4.1), [6]:

Suppose that $L(P,t) = c_d t^d + c_{d-1}t^{d-1} + \ldots + c_0$, then for $1 \le k \le d$

$$\operatorname{Res}(\mathrm{g_k}(Z), Z=0) = \mathrm{k!}\sum_{\mathrm{m=k}}^{\mathrm{d}} \mathrm{S_2(m,k)c_m t^m}$$

where $S_2(m,k)$ denotes the Stirling number of the second kind of m and k and $\mathrm{c_0}$=1.

Proof:

Suppose that

$$\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j L(P,(k-j)t) = \sum_{m=0}^{d} b_{k,m}t^m \tag{2.12}$$

So that for m > 0,

$$b_{k,m} = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j c_m (k-j)^m = c_m \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^m$$

using the identity (2.11) of the Stirling number

so $b_{k,m} = c_m k! S_2(m,k)$ for m > 0.

and by formula (2.8), we conclude that $b_{k,m} = 0$ for $1 \le m < k$. The constant term in formula (2.12) is

$$b_{k,0} = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j c_0 = -c_0 (-1)^k$$

Therefore,

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j L(P,(k-j)t) = \sum_{m=0}^{d} c_m k! S_2(m,k) t^m$$

$$= k! \sum_{m=k}^{d} c_m S_2(m,k) t^m - (-1)^k$$

Then,

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j L(P,(k-j)t) + (-1)^k = k! \sum_{m=k}^{d} c_m S_2(m,k) t^m$$

Therefore,

$$\operatorname{Re} s(g_k(\mathrm{Z}), \mathrm{Z}=0) = k! \sum_{m=k}^{d} S_2(m,k) c_m t^m \, . \ \blacksquare$$

The following theorem appears in [6] without proof. Here we prove it for the sake of completeness.

<u>Theorem (2.4.1), [6]:</u>

Let P be a lattice d-polytope given by expression (2.7), with the Ehrhart polynomial $L(P,t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_0$, then for $1 \le k \le d$

$$\sum_{m=k}^{d} S_2(m,k) c_m t^m = \frac{-1}{k!} (\operatorname{Re} s(g_k(\mathrm{Z}), \mathrm{Z}=1) + \sum_{\lambda \in \Omega_k} \operatorname{Re} s(g_k(\mathrm{Z}), \mathrm{Z}=\lambda))$$

where $\Omega_k = \{Z \in C \setminus \{1\} : Z^{\frac{A}{a_{j_1} \cdots a_{j_{k+1}}}} = 1, 1 \le j_1 < j_2 < \ldots < j_{k+1} \le d\}$

Proof:

Since $\mathrm{Res}(g_k(Z), Z=0) = \mathrm{Res}(\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_{-t(k-j)}(Z), Z=0)$

and from

$$\mathrm{Res}(f_{-t}(Z), Z=0) = -\mathrm{Res}(f_{-t}(Z), Z=1) - \sum_{\lambda \in \Omega} \mathrm{Res}(f_{-t}(Z), Z=\lambda)$$

therefore,

$$\mathrm{Res}(f_{-t(k-j)}(Z), Z=0) = -\mathrm{Res}(f_{-t(k-j)}(Z), Z=1) - \sum_{\lambda \in \Omega} \mathrm{Res}(f_{-t(k-j)}(Z), Z=\lambda)$$

Now

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \mathrm{Res}(f_{-t(k-j)}(Z), Z=0)$$

$$= -\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j [\mathrm{Res}(f_{-t(k-j)}(Z), Z=1) + \sum_{\lambda \in \Omega} \mathrm{Res}(f_{-t(k-j)}(Z), Z=\lambda)]$$

$$\mathrm{Res}(g_k(Z), Z=0) = -\mathrm{Res}(g_k(Z), Z=1) - \sum_{\lambda \in \Omega} \mathrm{Res}(g_k(Z), Z=\lambda)$$

By using lemma(2.4.1) we get

$$\sum_{m=k}^{d} S_2(m,k) c_m t^m = \frac{-1}{k!} (\mathrm{Res}(g_k(Z), Z=1) + \sum_{\lambda \in \Omega_k} \mathrm{Res}(g_k(Z), Z=\lambda)) \cdot \blacksquare$$

The following corollary appears in [6] without proof. Here we prove it for the sake of completeness.

Corollary (2.4.1), [6]:

For m > 0, c_m is the coefficient of t^m in

$$\frac{-1}{m!} (\mathrm{Res}(g_m(Z), Z=1) + \sum_{\lambda \in \Omega_m} \mathrm{Res}(g_m(Z), Z=\lambda))$$

Proof:

By using the fact that $S_2(m,m)=1$ in theorem (2.4.1), we get the result of corollary (2.4.1). ■

Now, we give the following proposition with its proof, which we need for proving the next theorem.

Proposition (2.4.1), [20, p. 265]:

Let f (z) be analytic inside and on a simple closed curve C except at a pole a of order m inside C. The residue of f (z) at a is given by

$$a_{-1}=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \qquad (2.13)$$

Proof:

If f (z) has a pole a of order m, then the Laurent series of f (z) at a is

$$f(z)=\frac{a_{-m}}{(z-a)^m}+\frac{a_{-m+1}}{(z-a)^{m-1}}+\ldots+\frac{a_{-1}}{z-a}+a_0+a_1(z-a)+a_2(z-a)^2+\ldots$$

Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z)=a_{-m}+a_{-m+1}(z-a)+\ldots+a_{-1}(z-a)^{m-1}+a_0(z-a)^m+\ldots \qquad (2.14)$$

From Taylor's theorem one can see that the coefficient of $(z-a)^{m-1}$ in the expansion (2.14) is

$$a_{-1}=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}. \text{ ■}$$

Remark(2.4.1):

We know that from the definitions of the Ehrhart polynomial, the leading coefficient is the volume of the polytope and the constant term is one; these are termed as the trivial coefficient of the Ehrhart polynomial, the other coefficients are nontrivial.

Theorem (2.4.2) [6]:

Let $P \subset \Re^d$ be a lattice d-polytope, with vertices $(0,0,...,0)$, $(a_1,0,...,0),...,(0,0,...,a_d)$ where $a_1, a_2..., a_d$ are pairwise relatively prime integers. The first nontrivial Ehrhart coefficients c_{d-2}, $d \geq 3$ is given by,

$$c_{d-2} = \frac{1}{(d-2)!}(C_d - S(A_1, a_1) - ... - S(A_d, a_d))$$

where $S(a,b)$ denotes the Dedekind sum and

$$C_d = \frac{1}{4}(d + A_{1,2} + ... + A_{d-1,d}) + \frac{1}{12}(\frac{1}{A} + \frac{A_1}{a_1} + ... + \frac{A_d}{a_d}),$$

$A = a_1 a_2 ... a_d$, $A_k = a_1 a_2 ... \hat{a}_k ... a_d$ (where $\hat{a}_k$ means the factor a_k is omitted), and $A_{j,k}$ denotes $a_1 .. \hat{a}_j ... \hat{a}_k ... a_d$.

Proof:

Consider $g_{d-2}(Z) = \dfrac{(Z^{-tA} - 1)^{d-2}}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)Z}$

since $a_1, a_2, ..., a_d$ are pairwise relatively prime, $g_{d-2}(Z)$ has simple poles at the $a_1, a_2, ..., a_d$ - th roots of unite. Let $\lambda^{a_1} = 1 \neq \lambda$. Then,

$$\operatorname{Re}s(g_{d-2}(Z), Z = \lambda) = \frac{1}{(1 - \lambda^{A_1})(1 - \lambda)\lambda} \operatorname{Re}s\left(\frac{(Z^{-tA} - 1)^{d-2}}{(1 - Z^{A_2})(1 - Z^{A_3})...(1 - Z^{A_d})}, Z = \lambda\right)$$

A change of variable $Z = \omega^{1/B} = \exp\left(\frac{1}{B}\log\omega\right)$ is made, where a suitable branch of logarithm such that $\exp\left(\frac{1}{a_1}\log(1)\right) = \lambda$, thus

$$\operatorname{Re}s(g_{d-2}(Z), Z = \lambda) = \frac{1}{(1 - \lambda^{A_1})(1 - \lambda)\lambda} \frac{\lambda}{a_1} \operatorname{Re}s\left(\frac{(\omega^{-tB} - 1)^{d-2}}{(1 - \omega^{B_2})...(1 - \omega^{B_d})}, \omega = 1\right)$$

where $B = a_2, ..., a_d$, $B_k = a_2 ... \hat{a}_k ... a_d$. We change ω by Z.

Claim

$$\operatorname{Re}s\left(\frac{(Z^{-tB} - 1)^{d-2}}{(1 - Z^{B_2})...(1 - Z^{B_d})}, Z = 1\right) = -t^{d-2}$$

to prove this, first note that the Taylor series of $(Z^{-tB} - 1)^{d-2}$ about Z = 1 is

$$(Z^{-tB}-1)^{d-2}=(-tB)^{d-2}(Z-1)^{d-2}+O((Z-1)^{d-2})$$

Now for $m\in N$

$$\operatorname{Res}\left(\frac{1}{1-Z^{m}},Z=1\right)=\lim_{Z\to 1}\frac{Z-1}{1-Z^{m}}=\frac{-1}{m}$$

Putting all of this together, we obtain

$$\operatorname{Res}\left(\frac{(Z^{-tB}-1)^{d-2}}{(1-Z^{B_2})\ldots(1-Z^{B_d})},Z=1\right)=\frac{(-tB)^{d-2}}{(-B_2)\ldots(-B_d)}$$

$$=-\frac{t^{d-2}a_2^{\,d-2}\ldots a_d^{\,d-2}}{a_2^{\,d-2}\ldots a_d^{\,d-2}}=-t^{d-2}$$

as desired. Therefore

$$\operatorname{Res}(g_{d-2}(Z),Z=\lambda)=\frac{-t^{d-2}}{a_1(1-\lambda^{A_1})(1-\lambda)}$$

Adding up all the residues at the a_1-th roots of unity $\neq 1$, we get

$$\sum_{\lambda^{a_1}=1\neq\lambda}\operatorname{Res}(g_{d-2}(Z),Z=\lambda)=\frac{-t^{d-2}}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^{k})}$$

where ξ is a primitive a_1-th roots of unity. This finite sum is practically a Dedekind sum:

$$\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^{k})}=\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(1+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\left(1+\frac{1+\xi^{k}}{1-\xi^{k}}\right)$$

After simple computations one can get,

$$\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^{k})}=\frac{1}{4a_1}(a_1-1)-\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\cot\frac{\pi kA_1}{a_1}\cot\frac{\pi k}{a_1}-$$

$$\frac{i}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi kA_1}{a_1}+\cot\frac{\pi k}{a_1}\right)$$

the imaginary terms disappear and the cotangent sum can be rewrite in terms of the Dedekind sum , so we get

$$\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^{k})}=\frac{1}{4}-\frac{1}{4a_1}-S(A_1,a_1)$$

and therfore

$$\sum_{\lambda^{a_1}=1\neq\lambda}\mathrm{Res}(\mathrm{g}_{d-2}(Z),Z=\lambda)=-\mathrm{t}^{d-2}\left(\frac{1}{4}-\frac{1}{4\mathrm{a}_1}-\mathrm{S}(\mathrm{A}_1,\mathrm{a}_1)\right).$$

Similar expressions for the residues at the other roots of unity are obtained, so that corollary (2.4.1) give us for $d\geq 3$

$$\mathrm{c}_{d-2}=\frac{1}{(\mathrm{d}-2)!}\left(\frac{\mathrm{d}}{4}-\mathrm{C}-\frac{1}{4}\left(\frac{1}{\mathrm{a}_1}+\frac{1}{\mathrm{a}_2}+\ldots+\frac{1}{\mathrm{a}_d}\right)-\mathrm{S}(\mathrm{A}_1,\mathrm{a}_1)-\ldots-\mathrm{S}(\mathrm{A}_d,\mathrm{a}_d)\right) \tag{2.15}$$

where C is the coefficient of t^{d-2} of $\mathrm{Res}(g_{d-2}(\mathrm{Z}),\mathrm{Z}=1)$,

Next, $\mathrm{Res}(g_{d-2}(\mathrm{Z}),\mathrm{Z}=1)=\mathrm{Res}(e^{Z}g_{d-2}(e^{\mathrm{Z}}),\mathrm{Z}=0)$

$$=\mathrm{Res}\left(\frac{(e^{-tA\mathrm{Z}}-1)^{d-2}}{(1-e^{A_1\mathrm{Z}})(1-e^{A_2\mathrm{Z}})\ldots(1-e^{A_d\mathrm{Z}})(1-e^{\mathrm{Z}})},\mathrm{Z}=0\right).$$

Since $(e^{-tA\mathrm{Z}}-1)^{d-2}=(-tA\mathrm{Z})^{d-2}+\mathrm{O}((t\mathrm{Z})^{d-1})$ and

$$\frac{1}{1-e^{\mathrm{Z}}}=-\mathrm{Z}^{-1}+\frac{1}{2}-\frac{1}{12}\mathrm{Z}+\mathrm{O}(\mathrm{Z}^3),$$

the coefficient of t^{d-2} of $\mathrm{Res}(g_{d-2}(\mathrm{Z}),\mathrm{Z}=1)$ turns out to be

$$C=(-A)^{d-2}\left[\frac{1}{12}\left(\frac{(-1)^{d+1}}{A_1\ldots A_d}+\frac{(-1)^{d+1}A_1}{A_2\ldots A_d}+\ldots+\frac{(-1)^{d+1}A_d}{A_1\ldots A_{d-1}}\right)+\right.$$
$$\left.\frac{1}{4}\left(\frac{(-1)^{d-1}}{A_2\ldots A_d}+\ldots+\frac{(-1)^{d-1}}{A_1\ldots A_{d-1}}+\frac{(-1)^{d-1}}{A_3\ldots A_d}+\frac{(-1)^{d-1}}{A_2A_4\ldots A_d}+\ldots\frac{(-1)^{d-1}}{A_1\ldots A_{d-2}}\right)\right]$$

$$=-\frac{1}{4}\left(\frac{1}{a_1}+\ldots+\frac{1}{a_d}+A_{1,2}+\ldots+A_{d-1,d}\right)-\frac{1}{12}\left(\frac{1}{A}+\frac{A_1}{a_1}+\ldots+\frac{A_d}{a_d}\right)$$

By substituting this into equation (2.15) we get the result.■

2.5 Computing c_{d-3} and c_{d-4} in The Ehrhart Polynomial

As seen before, the leading coefficient of the Ehrhart polynomial represents the volume of the polytope, the second coefficient represents half of the surface area of the polytope and the constant term is one, while the other coefficients are unknown.

In this section we find the non trivial coefficients c_{d-3} and c_{d-4} for the d–polytope with d≥4 and d≥5 respectively, where P is represented by a list of vertices $(0,0,...,0),(a_1,0,...,0),\ (0,a_2,0,...,0),...,(0,0,...,0,a_d)$, such that $a_1,...,a_d$ are pairwise relatively prime positive integers.

By corollary (2.4.1), if we define $g_{d-3}(Z)$ as

$$g_{d-3}(Z)=\frac{(Z^{-tA}-1)^{d-3}}{(1-Z^{A_1})(1-Z^{A_2})\cdots(1-Z^{A_d})(1-Z)Z}$$

where $A=a_1a_2\cdots a_d$, $A_k=a_1a_2\cdots\hat{a}_k\cdots a_d$ and $\hat{a}_k$ means that the factor a_k is omitted, then the poles of the function $g_{d-3}(Z)$ are at Z = 0, 1 and the roots of unity.

We find the residues of the function $g_{d-3}(Z)$ at these poles.
Since $a_1,...,a_d$ are pairwise relatively prime therefore $g_{d-3}(Z)$ has simple poles at $a_1,...,a_d$*-th* roots of unity. Let $\lambda^{a_1}=1\neq\lambda$ and since, $A=a_1\cdots a_d$, $A_1=a_2a_3\cdots a_d,\ldots,\ A_d=a_1a_2\cdots a_{d-1}$, therefore

$$g_{d-3}(Z)=\frac{(Z^{-t(a_1\cdots a_d)}-1)^{d-3}}{(1-Z^{a_2\cdots a_d})(1-Z^{a_1a_3\cdots a_d})\cdots(1-Z^{a_1a_2\cdots a_{d-1}})(1-Z)Z}.$$

Now at $Z=\lambda$,

$1-\lambda^{a_2\cdots a_d}\neq 0$ and $1-\lambda\neq 0$.

Therefore

A change of variables $Z=\omega^{1/a_1}=\exp\left(\frac{1}{a_1}\log\omega\right)$ is made, where a suitable branch of logarithm such that $\exp\left(\frac{1}{a_1}\log(1)\right)=\lambda$, thus

$$\operatorname{Re}s(g_{d-3}(Z), Z=\lambda) = \frac{1}{(1-\lambda^{A_1})(1-\lambda)\lambda}\frac{\lambda}{a_1}$$
$$\operatorname{Re}s\left(\frac{(\omega^{-tB}-1)^{d-3}}{(1-\omega^{B_2})\ldots(1-\omega^{B_d})}, \omega=1\right)$$

where $B = a_2, a_3, \ldots, a_d, B_k = a_2 a_3 \ldots \hat{a}_k \ldots a_d$.
since $\operatorname{Re}s(f(Z), Z=1) = \operatorname{Re}s(e^Z f(e^Z), Z=0)$, then

$$\operatorname{Re}s\left(\frac{(Z^{-tB}-1)^{d-3}}{(1-Z^{B_2})\ldots(1-Z^{B_d})}, Z=1\right) = \operatorname{Re}s\left(\frac{e^Z(e^{-tBZ}-1)^{d-3}}{(1-e^{B_2 Z})\ldots(1-e^{B_d Z})}, Z=0\right)$$

Let α = tB, then

$$\operatorname{Re}s\left(\frac{e^Z(e^{-tBZ}-1)^{d-3}}{(1-e^{B_2 Z})\ldots(1-e^{B_d Z})}, Z=0\right) = \operatorname{Re}s\left(\frac{e^Z(e^{-\alpha Z}-1)^{d-3}}{(1-e^{B_2 Z})\ldots(1-e^{B_d Z})}, Z=0\right).$$

By writing the Maclaurin series for exponential function one can get,

$$\operatorname{Re}s\left(\frac{e^Z\left(1-\alpha Z+\frac{(\alpha Z)^2}{2!}-\frac{(\alpha Z)^3}{3!}+\ldots+(-1)\right)^{d-3}}{\left(1-1-B_2 Z-\frac{(B_2 Z)^2}{2!}-\ldots\right)\cdots\left(1-1-B_d Z-\frac{(B_d Z)^2}{2!}-\ldots\right)}, Z=0\right)$$

after simple computations the above residue can be written as,

$$\operatorname{Res}\left(\frac{(-\alpha)^{d-3}e^Z}{(-B_2)\cdots(-B_d)z^{-d+3+d-1}}\left[\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}-\ldots\right)^{d-3}}{\left(1+\frac{(B_2 Z)}{2!}+\frac{(B_2 Z)^2}{3!}\ldots\right)\cdots\left(1+\frac{(B_d Z)}{2!}+\frac{(B_d Z)^2}{3!}+\ldots\right)}\right], Z=0\right)$$

Let

$$I = \left(1-\frac{\alpha}{2!}Z+\frac{\alpha^2}{3!}Z^2-\frac{\alpha^3}{4!}Z^3+\ldots\right)^{d-3}$$

$$J_2 = \left(1+\frac{B_2}{2!}Z+\frac{B_2^2}{3!}Z^2+\frac{B_2^3}{4!}Z^3+\ldots\right)^{-1}$$

$$J_3 = \left(1+\frac{B_3}{2!}Z+\frac{B_3^2}{3!}Z^2+\frac{B_3^3}{4!}Z^3+\ldots\right)^{-1}$$

$\vdots$

$$J_d = \left(1 + \frac{B_d}{2!}Z + \frac{B_d^2}{3!}Z^2 + \frac{B_d^3}{4!}Z^3 + \ldots\right)^{-1}$$

then

$$\mathrm{Res}\left(\frac{e^Z(e^{-\alpha Z}-1)^{d-3}}{(1-e^{B_2 Z})\ldots(1-e^{B_d Z})}, Z=0\right) = \mathrm{Res}\left[\frac{(-\alpha)^{d-3}e^Z}{(-B_2)\cdots(-B_d)Z^2}(IJ_2\cdots J_d), Z=0\right]$$

for the function $\frac{(-\alpha)^{d-3}e^Z}{(-B_2)\cdots(-B_d)Z^2}(IJ_2\cdots J_d)$ we have a pole of order two at zero.

Let $\phi(Z) = e^Z IJ_2J_3\cdots J_d$, and $\gamma = \frac{(\alpha)^{d-3}}{(-B_2)\ldots(-B_d)}$

After simple computations on γ, we get $\gamma = \frac{t^{d-3}}{B}$. By the formula for finding the residues given by (2.13), if we consider

$f(Z) = \frac{\gamma\phi(Z)}{Z^2}$, then $\mathrm{Res}(f(Z), Z=0) = \frac{\phi'(0)\gamma}{1!}$,where

$$\phi'(Z) = \phi(Z) + e^Z I'J_2J_3\cdots J_d + \ldots + e^Z IJ_2J_3\cdots J_d'.$$

Let

$$K_1 = e^Z I'J_2J_3\cdots J_d,\ K_2 = e^Z IJ_2'J_3\cdots J_d, \ldots,\ K_d = e^Z IJ_2J_3\cdots J_d'$$

therefore

$$\phi'(Z) = \phi(Z) + K_1 + K_2 + \ldots + K_d$$

at Z = 0, we compute $\phi'(0)$, after simple computations we get

$$\phi'(0) = 1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d) - \frac{\alpha(d-3)}{2!}$$

therefore,

$$\mathrm{Res}(f(Z), Z=0) = \frac{t^{d-3}}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d) - \frac{\alpha(d-3)}{2!}\right).$$

Let $D = \frac{1}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d)\right)$

Therefore,

$$\operatorname{Re}s(g_{d-3}(\mathrm{Z}),\mathrm{Z}=\lambda\ )=\frac{D}{a_1(1-\lambda^{A_1})(1-\lambda)}t^{d-3}\,.$$

all the a_1-*th* roots of unity ≠ 1 are added up to get

$$\sum_{\lambda^{a_1}=\ 1\neq\lambda}\operatorname{Re}s(g_{d-3}(\mathrm{Z}),\mathrm{Z}=\lambda\ )=\frac{Dt^{d-3}}{a_1}\sum_{\lambda^{a_1}=\ 1\neq\lambda}\frac{1}{(1-\lambda^{A_1})(1-\lambda)}\,.$$

Let ξ be a primitive a_1-th roots of unity, therefore

$$\frac{Dt^{d-3}}{a_1}\sum_{\lambda^{a_1}=\ 1\neq\lambda}\frac{1}{(1-\lambda^{A_1})(1-\lambda)}=\frac{Dt^{d-3}}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^k)}$$

then

$$\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{1}{(1-\xi^{kA_1})(1-\xi^k)}=\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{\xi^{kA_1}-\xi^{kA_1}+1+1}{2(1-\xi^{kA_1})}\cdot\frac{\xi^k-\xi^k+1+1}{2(1-\xi^k)}$$

$$=\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(1+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\cdot\left(1+\frac{1+\xi^k}{1-\xi^k}\right)$$

$$=\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(1+\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\cdot\frac{1+\xi^k}{1-\xi^k}\right)$$

$$=\frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1}1+\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)+\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\cdot\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$

$$=\frac{1}{4a_1}(a_1-1)+\frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)+\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\cdot\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$

Now, since $\xi\ =1^{\frac{1}{a_1}}$, then by using the formula for finding the roots in the complex plane, $r_k=e^{\frac{2k\pi}{a_1}i}$, k = 0,1,…, a_1-1. We obtain

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)=\sum_{k=1}^{a_1-1}\left[\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}}+\frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right].$$

and

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\right)\cdot\left(\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)=\sum_{k=1}^{a_1-1}\left[\left(\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}}\right)\cdot\left(\frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right)\right]$$

But $\cot(z)=\dfrac{\cos(z)}{\sin(z)}=-i\left(\dfrac{1+e^{2iz}}{1-e^{2iz}}\right)$

hence $\displaystyle\sum_{k=1}^{a_1-1}\left[\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}}+\frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right]=\sum_{k=1}^{a_1-1}\frac{-1}{i}\left(\cot\frac{\pi k}{a_1}+\cot\frac{\pi kA_1}{a_1}\right)$

and

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\right)\cdot\left(\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)=\sum_{k=1}^{a_1-1}-\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi kA_1}{a_1}\right)$$

$$=-\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi kA_1}{a_1}\right)$$

therefore

$$\frac{1}{4a_1}(a_1-1)+\frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)+\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\cdot\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$

$$=\frac{1}{4a_1}(a_1-1)+\frac{i}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}+\cot\frac{\pi kA_1}{a_1}\right)-\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi kA_1}{a_1}\right)$$

The imaginary terms disappear, and then the above equation can be written as

$$\frac{1}{4}-\frac{1}{4a_1}-\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi kA_1}{a_1}\right)=\frac{1}{4}-\frac{1}{4a_1}-\frac{4a_1}{4a_1}S(A_1,a_1)$$

where $S(A_1,a_1)$ is the Dedekind sum of A_1 and a_1. Hence

$$\sum_{\lambda^{a_1}=\ 1\neq\lambda}\operatorname{Res}(g_{d-3}(Z),Z=\lambda)=Dt^{d-3}\left(\frac{1}{4}-\frac{1}{4a_1}-S(A_1,a_1)\right)\cdot$$

Similar expressions are obtained for the residues at the other roots of unity.

Now we find the residue at $g_{d-3}(Z)$ at Z = 1, we have

$\mathrm{Res}(g_{d-3}(\mathrm{Z}),\mathrm{Z}=1)=\mathrm{Res}(e^{\mathrm{Z}}g_{d-3}(e^{\mathrm{Z}}),\mathrm{Z}=0)$ then

$$\mathrm{Res}(g_{d-3}(\mathrm{Z}),\mathrm{Z}=1)=\mathrm{Res}\left(\frac{e^{\mathrm{Z}}(e^{-tA\mathrm{Z}}-1)^{d-3}}{(1-e^{A_1\mathrm{Z}})(1-e^{A_2\mathrm{Z}})\ldots(1-e^{A_d\mathrm{Z}})(1-e^{\mathrm{Z}})e^{\mathrm{Z}}},\mathrm{Z}=0\right).$$

By writing the Maclaurin series for exponential function we get,

$$\mathrm{Res}\left(\frac{\left(1-\alpha Z+\frac{(\alpha Z)^2}{2!}-\frac{(\alpha Z)^3}{3!}+\ldots+(-1)\right)^{d-3}}{\left(1-1-A_2Z-\frac{(A_2Z)^2}{2!}-\ldots\right)\cdots\left(1-1-A_dZ-\frac{(A_dZ)^2}{2!}-\ldots\right)\cdot\left(1-1-Z-\frac{Z^2}{2!}-\ldots\right)},\mathrm{Z}=0\right)$$

where $\alpha=tA$, then the above residue becomes

$$\mathrm{Res}\left(\frac{(\alpha)^{d-3}}{(A_1)\cdots(A_d)Z^4}\left[\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}-\ldots\right)^{d-3}}{\left(1+\frac{(A_1Z)}{2!}+\frac{(A_1Z)^2}{3!}\ldots\right)\cdots\left(1+\frac{(A_dZ)}{2!}+\frac{(A_dZ)^2}{3!}+\ldots\right)\left(1+\frac{Z}{2!}+\frac{Z^2}{3!}+\ldots\right)}\right],\mathrm{Z}=0\right)$$

the function for which we want to find the residue has a pole of order four at zero.

Let

$$\phi(Z)=\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}+\ldots\right)^{d-3}}{\left(1+\frac{(A_1Z)}{2!}+\frac{(A_1Z)^2}{3!}+\ldots\right)\cdots\left(1+\frac{(A_dZ)}{2!}+\frac{(A_dZ)^2}{3!}\ldots\right)\cdot\left(1+\frac{Z}{2!}+\frac{Z^2}{3!}\ldots\right)},$$

$\gamma=\dfrac{\alpha^{d-3}}{A_1\ldots A_d}$ and $f(z)=\dfrac{\gamma\phi(\mathrm{z})}{z^4}$

By the formula for finding the residue given by (2.13), we get

$$\mathrm{Res}(\mathrm{f}(Z),Z=0)=\frac{\phi^{(3)}(0)\gamma}{3!}.$$

Let

$$I=\left(1-\frac{\alpha}{2!}Z+\frac{\alpha^2}{3!}Z^2-\frac{\alpha^3}{4!}Z^3+\ldots\right)^{d-3},\ \mathrm{h}=\left(1+\frac{1}{2!}\mathrm{Z}+\frac{1}{3!}\mathrm{Z}^2+\frac{1}{4!}\mathrm{Z}^3+\ldots\right)^{-1}$$

$J_1 = \left(1 + \frac{A_1}{2!}Z + \frac{A_1^2}{3!}Z^2 + \frac{A_1^3}{4!}Z^3 + \ldots\right)^{-1}, J_2 = \left(1 + \frac{A_2}{2!}Z + \frac{A_2^2}{3!}Z^2 + \frac{A_2^3}{4!}Z^3 + \ldots\right)^{-1}, \ldots,$

and $J_d = \left(1 + \frac{A_d}{2!}z + \frac{A_d^2}{3!}z^2 + \frac{A_d^3}{4!}z^3 + \ldots\right)^{-1}$.

Then $\phi(Z) = IJ_1J_2 \cdots J_d h$

and

$$\phi'(Z) = I'J_1J_2 \cdots J_d h + IJ_1'J_2 \cdots J_d h + \ldots + IJ_1J_2 \cdots J_d' h + IJ_1J_2 \cdots J_d h'$$

let

$K_1 = I'J_1J_2 \cdots J_d h, K_2 = IJ_1'J_2 \cdots J_d h, \ldots, K_{d+1} = IJ_1J_2 \cdots J_d' h$ and

$K_{d+2} = IJ_1J_2 \cdots J_d h'$

hence

$\phi'(Z) = K_1 + K_2 + \ldots + K_{d+1} + K_{d+2}$ and $\phi'''(Z) = K_1'' + K_2'' + \ldots + K_{d+1}'' + K_{d+2}''$

Now,

$$I = \left(1 - \frac{tA}{2!}Z + \frac{(tA)^2}{3!}Z^2 - \frac{(tA)^3}{4!}Z^3 + \ldots\right)^{d-3}$$

therefore

$$I'(Z) = (d-3)\left[\left(1 - \frac{tA}{2!}Z + \frac{(tA)^2}{3!}Z^2 - \frac{(tA)^3}{4!}Z^3 + \ldots\right)^{d-4} \cdot \left(-\frac{tA}{2!} + \frac{2(tA)^2}{3!}Z + \ldots\right)\right]$$

Differentiating I' to get I'' and I''', then put Z = 0 in the obtained expression to get

$I(0) = 1$

$I'(0) = (d-3)\left(\frac{-tA}{2!}\right)$

$I''(0) = (d-3)\left[(d-4)(-\frac{tA}{2!})^2 + \frac{2(tA)^2}{3!}\right]$

$$I'''(0)=(d-3)\left[(d-5)(d-6)\left(-\frac{tA}{2!}\right)^3+\right.$$

$$\left.(d-4)(-tA)\left(\frac{2(tA)^2}{3!}+(d-4)\right)\left(\frac{-tA}{2}\right)\left(\frac{2(tA)^2}{3!}+\left(\frac{-3!(tA)^3}{4!}\right)\right)\right]$$

For

$$J_1=\left(1+\frac{A_1}{2!}Z+\frac{A_1^2}{3!}Z^2+\frac{A_1^3}{4!}Z^3+\dots\right)^{-1}$$

$$J_1'(Z)=-\left(1+\frac{A_1}{2!}Z+\frac{A_1^2}{3!}Z^2+\frac{A_1^3}{4!}Z^3+\dots\right)^{-2}\left(\frac{A_1}{2!}+\frac{2A_1^2}{3!}Z+\frac{3A_1^3}{4!}Z^2+\dots\right)$$

Differentiate J' to get J'' and J''', then put Z = 0 in the obtained expressions to get

$J_1(0)=1$, $J_1'(0)=-\frac{A_1}{2!}$, $J_1''(0)=\frac{A_1^2}{3!}$ and $J_1'''(0)=0$.

In a similar way, we get the other differentiation of $J_2, J_3, \dots, J_d$ and h, then

$$\operatorname{Res}\left(\frac{(tA)^{d-3}}{(A_1)\cdots(A_d)z^4}\phi(Z), Z=0\right)=\frac{(A)^{d-3}}{(A_1)\cdots(A_d)}\frac{t^{d-3}}{3!}\phi^{(3)}(0)\cdot$$

Let $C=\dfrac{(A)^{d-3}}{(A_1)\cdots(A_d)}\cdot\dfrac{\phi^{(3)}(0)}{3!}$

So by corollary (2.4.1) we get for d≥4, c_{d-3}, which is the coefficient of t^{d-3} of

$$\frac{-1}{(d-3)!}(\operatorname{Res}(g_{d-3}(\mathrm{Z}),\mathrm{Z}=1)+\sum_{\lambda\in\Omega_{d-3}}\operatorname{Res}(g_{d-3}(\mathrm{Z}),\mathrm{Z}=\lambda))$$

So

$$c_{d-3}=\frac{-1}{(d-3)!}\left[D\left(\frac{d}{4}-\frac{1}{4}\left(\frac{1}{a_1}+\dots+\frac{1}{a_d}\right)-S(A_1,a_1)-\dots-S(A_d,a_d)\right)-C\right]$$

Thus we have proved the following:

Theorem (2.5.1):

Let P denote the polytope in $\Re^d (d \geq 4)$ with vertices $(0,0,...,0),(a_1,0,...,0),...,(0,0,...,a_d)$ where $a_1,...,a_d$ are pairwise relatively prime positive integers. Then c_{d-3} is given by

$$c_{d-3} = \frac{-1}{(d-3)!}\left[D\left(\frac{d}{4} - \frac{1}{4}\left(\frac{1}{a_1} + ... + \frac{1}{a_d}\right) - S(A_1,a_1) - ... - S(A_d,a_d)\right) - C\right]$$

where S(a,b) is the Dedekind sum of a and b,

$$D = \frac{1}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + ... + B_d)\right),$$

$$C = \frac{(A)^{d-3}}{(A_1)\cdots(A_d)} \cdot \frac{\phi^{(3)}(0)}{3!},$$

$$\phi(Z) = \frac{\left(1 - \frac{(tBZ)}{2!} + \frac{(tBZ)^2}{3!} + ...\right)^{d-3}}{\left(1 + \frac{(A_1Z)}{2!} + \frac{(A_1Z)^2}{3!} + ...\right)\cdots\left(1 + \frac{(A_dZ)}{2!} + \frac{(A_dZ)^2}{3!} + ...\right)\cdot\left(1 + \frac{Z}{2!} + \frac{Z^2}{3!} + ...\right)}.$$

$A = a_1a_2...a_d$, $A_k = a_1a_2...\hat{a}_k...a_d$, $\hat{a}_k$ means the factor a_k is omitted, $B = a_2a_3...a_d$ and $B_k = a_2a_3...\hat{a}_k...a_d$.

In a similar way, we get c_{d-4} for the d–polytope P (d≥5), where P is represented by a list of vertices $(0,...,0),(a_1,0,...,0),...,(0,0,...,a_d)$, $a_1,...,a_d$ are pairwise relatively prime positive integers. By corollary (2.4.1), if we define $g_{d-4}(Z)$ as

$$g_{d-4}(Z) = \frac{(Z^{-tA} - 1)^{d-4}}{(1 - Z^{A_1})(1 - Z^{A_2})...(1 - Z^{A_d})(1 - Z)Z}$$

where $A = a_1 \cdots a_d$, $A_k = a_1 \cdots \hat{a}_d \cdots a_d$ and $\hat{a}_k$ means that the factor a_k is omitted. The poles of the function $g_{d-4}(Z)$ are at Z = 0, 1 and the roots of unity.

Now we compute the residues at these poles.

Since $a_1,...,a_d$ are pairwise relatively prime therefore $g_{d-4}(Z)$ has simple poles at $a_1,...,a_d$-*th* roots of unity.

Let $\lambda^{a_1} = 1 \neq \lambda$ and since, $A = a_1 \cdots a_d$, $A_1 = a_2a_3 \cdots a_d$

Therefore

$$g_{d-4}(Z)=\frac{(Z^{-t(a_1\cdots a_d)}-1)^{d-4}}{(1-Z^{a_2\cdots a_d})(1-Z^{a_2a_3\cdots a_d})\cdots(1-Z)Z}.$$

Now at $z=\lambda$

$1-\lambda^{A_1}\neq 0$ and $1-\lambda\neq 0$.

Therefore

$$\operatorname{Res}(g_{d-4}(Z),Z=\lambda)=\frac{1}{(1-\lambda^{A_1})(1-\lambda)\lambda}\operatorname{Res}\left(\frac{(Z^{-tA}-1)^{d-4}}{(1-Z^{A_2})(1-Z^{A_3})\ldots(1-Z^{A_d})},Z=\lambda\right)$$

We make a change of variables $Z=\omega^{1/a_1}=\exp\left(\frac{1}{a_1}\log\omega\right)$, where a suitable branch of logarithm such that $\exp\left(\frac{1}{a_1}\log(1)\right)=\lambda$, thus,

$$\operatorname{Res}(g_{d-4}(Z),Z=\lambda)=\frac{1}{(1-\lambda^{A_1})(1-\lambda)\lambda}\frac{\lambda}{a_1}\operatorname{Res}\left(\frac{(\omega^{-tB}-1)^{d-4}}{(1-\omega^{B_2})\ldots(1-\omega^{B_d})},\omega=1\right)$$

where $B=a_2a_3\ldots a_d, B_k=a_2a_3\ldots\hat{a}_k\ldots a_d$.
By the same steps followed before, we obtain with $\alpha=tB$

$$\operatorname{Res}\left(\frac{e^{Z}(e^{-\alpha Z}-1)^{d-4}}{(1-e^{B_2Z})\ldots(1-e^{B_dZ})},Z=0\right)$$

$$=\operatorname{Res}\left(\frac{(-\alpha)^{d-4}e^{Z}}{(-B_2)\cdots(-B_d)Z^{-d+4+d-1}}\left(\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}-\ldots\right)^{d-4}}{\left(1+\frac{(B_2Z)}{2!}+\frac{(B_2Z)^2}{3!}\ldots\right)\cdots\left(1+\frac{(B_dZ)}{2!}+\frac{(B_dZ)^2}{3!}+\ldots\right)}\right),Z=0\right)$$

let

$$I=\left(1-\frac{\alpha}{2!}Z+\frac{\alpha^2}{3!}Z^2-\frac{\alpha^3}{4!}Z^3+\ldots\right)^{d-4}$$

$$J_2=(1+\frac{B_2}{2!}Z+\frac{B_2^2}{3!}Z^2+\frac{B_2^3}{4!}Z^3+\ldots)^{-1}$$

$$\vdots$$

$$J_d=(1+\frac{B_d}{2!}Z+\frac{B_d^2}{3!}Z^2+\frac{B_d^3}{4!}Z^3+\ldots)^{-1}$$

then

$$\operatorname{Res}\left(\frac{e^{Z}(e^{-\alpha Z}-1)^{d-4}}{(1-e^{B_2 Z})\cdots(1-e^{B_d Z})}, Z=0\right)=\operatorname{Res}\left(\frac{(-\alpha)^{d-4}e^{Z}}{(-B_2)\cdots(-B_d)Z^{3}}(IJ_2\cdots J_d), Z=0\right)$$

So for the function in the above residue we have a pole of order three at zero.

Let $\phi(Z)=e^{Z}IJ_2J_3\cdots J_d$ and $\gamma=\dfrac{(\alpha)^{d-4}}{(-B_2)\ldots(-B_d)}$

after simple commutations on γ we get $\gamma=-\dfrac{t^{d-4}}{B^{2}}$

By the formula for finding the residues given by (2.13), if we consider $f(Z)=\dfrac{\gamma\phi(Z)}{Z^{3}}$, then $\operatorname{Res}(f(Z),Z=0)=\dfrac{\phi''(0)\gamma}{2!}$

Let $D=\dfrac{1}{B^{2}\,2!}\phi''(0)$

Therefore,

$$\operatorname{Res}(g_{d-4}(Z),Z=\lambda)=\frac{-D}{a_1(1-\lambda^{A_1})(1-\lambda)}t^{d-4}.$$

all the a_1-*th* roots of unity $\neq 1$ are added up to get

$$\sum_{\lambda^{a_1}=1\neq\lambda}\operatorname{Res}(g_{d-4}(Z),Z=\lambda)=\frac{-Dt^{d-4}}{a_1}\sum_{\lambda^{a_1}=1\neq\lambda}\frac{1}{(1-\lambda^{A_1})(1-\lambda)}.$$

With the same procedure that we used to get C_{d-3}, we obtain

$$\sum_{\lambda^{a_1}=1\neq\lambda}\operatorname{Res}(g_{d-4}(Z),Z=\lambda)=-Dt^{d-4}\left(\frac{1}{4}-\frac{1}{4a_1}-S(A_1,a_1)\right).$$

Similar expressions are obtained for the residues at the other roots of unity.

Now we find the residue of $g_{d-4}(Z)$ at Z = 1 by using

$$\operatorname{Res}(g_{d-4}(Z),Z=1)=\operatorname{Res}(e^{Z}g_{d-4}(e^{Z}),Z=0)$$

then

$$\operatorname{Res}(g_{d-4}(Z),Z=1)=\operatorname{Res}\left(\frac{e^{Z}(e^{-tAZ}-1)^{d-4}}{(1-e^{A_1Z})(1-e^{A_2Z})\ldots(1-e^{A_dZ})(1-e^{Z})e^{Z}}, Z=0\right).$$

By the Maclaurin series of the exponential function with $\alpha=tA$, the above residue becomes

$$\operatorname{Res}\left(\frac{-(\alpha)^{d-4}}{(A_1)\cdots(A_d)Z^5}\left[\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}-\ldots\right)^{d-4}}{\left(1+\frac{(A_1Z)}{2!}+\frac{(A_1Z)^2}{3!}+\ldots\right)\cdots\left(1+\frac{(A_dZ)}{2!}+\frac{(A_dZ)^2}{3!}+\ldots\right)\cdot\left(1+\frac{Z}{2!}+\frac{Z^2}{3!}+\ldots\right)}\right],Z=0\right)$$

for the function in the above residue, we have a pole of order five at zero.

Let $\gamma=-\frac{\alpha^{d-4}}{A_1\ldots A_d}$ and $f(Z)=\frac{\gamma\phi(Z)}{Z^5}$

where

$\phi(Z)=$

$$\frac{\left(1-\frac{(\alpha Z)}{2!}+\frac{(\alpha Z)^2}{3!}+\ldots\right)^{d-4}}{\left(1+\frac{(A_1Z)}{2!}+\frac{(A_1Z)^2}{3!}+\ldots\right)\cdots\left(1+\frac{(A_dZ)}{2!}+\frac{(A_dZ)^2}{3!}+\ldots\right)\cdot\left(1+\frac{Z}{2!}+\frac{Z^2}{3!}+\ldots\right)}$$

by the formula for finding the residues given by (2.14), we get

$$\operatorname{Res}(f(Z),Z=0)=\frac{\phi^{(4)}(0)\cdot\gamma}{4!}$$

Let

$$I=\left(1-\frac{\alpha}{2!}Z+\frac{\alpha^2}{3!}Z^2-\frac{\alpha^3}{4!}Z^3+\ldots\right)^{d-4},\ h=\left(1+\frac{1}{2!}Z+\frac{1}{3!}Z^2+\frac{1}{4!}Z^3+\ldots\right)^{-1}$$

$$J_1=\left(1+\frac{A_1}{2!}Z+\frac{A_1^2}{3!}Z^2+\frac{A_1^3}{4!}Z^3+\ldots\right)^{-1},\ldots,J_d=\left(1+\frac{A_d}{2!}Z+\frac{A_d^2}{3!}Z^2+\frac{A_d^3}{4!}Z^3+\ldots\right)^{-1}$$

then $\phi(Z)=IJ_1J_2\cdots J_dh$

Now for

$$I=\left(1-\frac{tA}{2!}Z+\frac{(tA)^2}{3!}Z^2-\frac{(tA)^3}{4!}Z^3+\ldots\right)^{d-4}$$

The derivative is

$$I'(Z)=(d-4)\left[\left(1-\frac{tA}{2!}Z+\frac{(tA)^2}{3!}Z^2-\frac{(tA)^3}{4!}Z^3+\ldots\right)^{d-4}\cdot\left(-\frac{tA}{2!}+\frac{2(tA)^2}{3!}Z+\ldots\right)\right]$$

Differentiating I' to get I'', I''' and $I^{(4)}$, then put Z = 0 in the obtained expression to get

$I(0)=1$

$$I'(0)=(d-4)\left(\frac{-tA}{2!}\right)$$

$$I''(0)=(d-4)\left[(d-5)\left(-\frac{tA}{2!}\right)+\frac{2(tA)^2}{3!}\right]$$

$$I'''(0)=(d-4)\left[(d-5)(d-6)\left(-\frac{tA}{2!}\right)^3\right.$$

$$\left.+(d-5)(-tA)\left(\frac{2(tA)^2}{3!}\right)+(d-5)\left(\frac{-tA}{2}\right)\left(\frac{2(tA)^2}{3!}+\left(\frac{-3!(tA)^3}{4!}\right)\right)\right]$$

$$I^{(4)}(0)=(d-4)[(d-5)(d-6)(d-7)(-\frac{tA}{2!})^4+(d-5)(d-6)(\frac{3(tA)^2}{2^2})(\frac{2(tA)^2}{3!})+$$

$$(d-5)(d-6)(\frac{-2tA}{2})(\frac{2(tA)^2}{3!})+(d-5)(2)(\frac{2(tA)^2}{3!})^2+(d-5)(2)(\frac{-tA}{2})(\frac{-3!(tA)^3}{4!})+$$

$$(d-5)(d-6)((-\frac{tA}{2!})^2(\frac{2(tA)^2}{3!})+(d-5)(\frac{2(tA)^2}{3!})^2+(d-5)(\frac{-tA}{2})(\frac{-3!(tA)^3}{4!})]$$

For

$$J_1=\left(1+\frac{A_1}{2!}Z+\frac{A_1^2}{3!}Z^2+\frac{A_1^3}{4!}Z^3+\ldots\right)^{-1}$$

$$J_1'(Z)=-\left(1+\frac{A_1}{2!}Z+\frac{A_1^2}{3!}Z^2+\frac{A_1^3}{4!}Z^3+\ldots\right)^{-2}\cdot\left(\frac{A_1}{2!}+\frac{2A_1^2}{3!}Z+\frac{3A_1^3}{4!}Z^2+\ldots\right)$$

Differentiating J' to get J'', J''' and $J_1^{(4)}$, then put Z = 0 in the obtained expression to get

$$J_1(0)=1,\ J_1'=-\frac{A_1}{2},\ J_1''(0)=\frac{A_1^2}{3!},\ J_1'''(0)=0 \text{ and } J_1^{(4)}(0)=-\frac{A_1^4}{30}$$

In a similar way, we get the other differentiation of $J_2, J_3, \ldots, J_d$ and h, then

$$\operatorname{Res}\left(\frac{(tA)^{d-4}}{(A_1)\cdots(A_d)Z^5}\phi(Z), Z=0\right)=\frac{-(A)^{d-4}}{(A_1)\cdots(A_d)}\frac{t^{d-4}}{4!}\phi^{(4)}(0)$$

Let $C=-\dfrac{(A)^{d-4}}{(A_1)\cdots(A_d)}\cdot\dfrac{\phi^{(4)}(0)}{4!}$

So by corollary (2.4.1) we get for d≥5, c_{d-4}, which is the coefficient of t^{d-4} of

$$\frac{-1}{(d-4)!}(\operatorname{Re}s(g_{d-4}(Z),Z=1)+\sum_{\lambda\in\Omega_{d-4}}\operatorname{Re}s(g_{d-4}(Z),Z=\lambda))$$

So

$$c_{d-4}=\frac{-1}{(d-4)!}\left[D\left(\frac{d}{4}-\frac{1}{4}\left(\frac{1}{a_1}+\frac{1}{a_2}...+\frac{1}{a_d}\right)-S(A_1,a_1)-...-S(A_d,a_d)\right)-C\right].$$

Thus we have proved the following:

Theorem (2.5.2):

Let P denote the polytope in $\Re^d\,(d\geq 5)$ with vertices $(0,0,...,0),(a_1,0,...,0),...,(0,0,...,a_d)$ where $a_1,...,a_d$ are pairwise relatively prime positive integers. Then c_{d-4} is given by

$$c_{d-4}=\frac{-1}{(d-4)!}\left[D\left(\frac{d}{4}-\frac{1}{4}\left(\frac{1}{a_1}+\frac{1}{a_2}...+\frac{1}{a_d}\right)-S(A_1,a_1)-...-S(A_d,a_d)\right)-C\right]$$

where S(a,b) denotes the Dedekind sum of a and b,

$$D=\frac{\frac{d^2}{dZ^2}(e^Z\phi(Z))}{B^2\,2!}\Big|_{Z=0},\quad C=-\frac{(A)^{d-4}}{(A_1)\cdots(A_d)}\cdot\frac{\phi^{(4)}(0)}{4!},$$

$$\phi(Z)=\frac{(1-\frac{(tBZ)}{2!}+\frac{(tBZ)^2}{3!}+...)^{d-4}}{(1+\frac{(A_1Z)}{2!}+...)...(1+\frac{(A_dZ)}{2!}+...)\cdot(1+\frac{Z}{2!}+...)}.$$

$A=a_1...a_d$, $A_k=a_1...\hat{a}_k...a_d$, $\hat{a}_k$ means the factor a_k is omitted and $B=a_2a_3...a_d$ and $B_k=a_2...\hat{a}_k...a_d$.

With the same procedure we get the coefficients $c_{d-5},c_{d-6},...,c_{d-9}$ under certain conditions. From the above methods, we note that $\phi(z)$ with their definitions are needed to differentiate more than once, so we obtain a general form for this differentiation given in sec. (2.6).

Example (2.5.1):

Let P be a 4-polytope with vertices (0,0,0,0), (7,0,0,0), (0,2,0,0), (0,0,3,0) and (0,0,0,5). It is easy to check that this polytope satisfies the conditions of theorem (2.5.1).

In this case the Ehrhart polynomial of the polytope is given by

$$L(P,t) = c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0 .$$

by theorem (2.5.1) c_1 can be computed as

$$\mathrm{c}_1 = \frac{-1}{(4-3)!}\Big[\mathrm{D}\Big(\frac{4}{4} - \frac{1}{4}\Big(\frac{1}{7}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}\Big) - \mathrm{S}(\mathrm{A}_1,7) - \mathrm{S}(\mathrm{A}_2,2) - \mathrm{S}(\mathrm{A}_3,3) - \mathrm{S}(\mathrm{A}_4,5) - \mathrm{C}\Big]$$

where $\mathrm{D} = \frac{1}{\mathrm{B}}\left(1 - 0.5(\mathrm{B}_2 + \mathrm{B}_3 + \mathrm{B}_4)\right)$,

$$\phi(\mathrm{Z}) = \frac{\left(1 - \frac{(\mathrm{tBZ})}{2!} + \frac{(\mathrm{tBZ})^2}{3!} + \ldots\right)^{\mathrm{d}-3}}{\left(1 + \frac{(\mathrm{A}_1\mathrm{Z})}{2!} + \frac{(\mathrm{A}_1\mathrm{Z})^2}{3!} + \ldots\right)\cdots\left(1 + \frac{(\mathrm{A}_\mathrm{d}\mathrm{Z})}{2!} + \frac{(\mathrm{A}_\mathrm{d}\mathrm{Z})^2}{3!} + \ldots\right)\cdot\left(1 + \frac{\mathrm{Z}}{2!} + \frac{\mathrm{Z}^2}{3!} + \ldots\right)}$$

After simple computations one can get

$A = 210, A_1 = 30, A_2 = 105, A_3 = 70, A_4 = 42$

$\mathrm{B} = 30, \mathrm{B}_2 = 15, \mathrm{B}_3 = 10, \mathrm{B}_4 = 6$, $\mathrm{D} = \frac{1}{30}$.

$C = \frac{1}{900} \cdot \frac{\phi^{(3)}(0)}{3!}$ where

$$\phi(Z) = \frac{1 - 15tZ + \ldots}{(1 + 15Z + \ldots)\left(1 + \frac{15}{2}Z + \ldots\right)(1 + 5Z + \ldots)(1 + 3Z + \ldots)\left(1 + \frac{1}{2!}Z + \ldots\right)}$$

let

$I = 1 - 15tZ + \ldots$, $\mathrm{J}_1 = (1 + 15\mathrm{Z} + \ldots)^{-1}$, $\mathrm{J}_2 = \left(1 + \frac{15}{2}\mathrm{Z} + \ldots\right)^{-1}$, $\mathrm{J}_3 = (1 + 5\mathrm{Z} + \ldots)^{-1}$,

$J_4 = (1 + 3Z + \ldots)^{-1}$ and $\mathrm{h} = \left(1 + \frac{1}{2!}\mathrm{Z} + \ldots\right)^{-1}$

then $\phi(Z) = IJ_1J_2J_3J_4h$, we find $\phi'(Z)$, $\phi''(Z)$ and $\phi^{(3)}(Z)$ then put Z=0 in $\phi^{(3)}(Z)$ to get the value of C which is equal to

$\mathrm{C} = \frac{1}{900}\frac{1}{3!}\phi^{(3)}(0)$

Also we need to compute S(30,7), S(105,2), S(70,3) and S(42,5), which are equal to 0.071428, 0, 0.055555 and 0 respectively. By substituting c_1 in the above formula one can get:

$$c_1 = \frac{-1}{(4-3)!}\left[\left(\frac{-29}{60}\right)\left(\frac{4}{4}-\frac{1}{4}\left(\frac{1}{7}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}\right)-S(30,7)-S(105,2)-S(70,3)\right.\right.$$
$$\left.\left.-S(42,5)\right)-\left(296.52482\right)\right]$$

$= -296.39864$.

From, $c_{d-2} = \dfrac{1}{(d-2)!}(C_d - S(A_1,a_1) - ... - S(A_d,a_d))$, where

$$C_d = \frac{1}{4}(d + A_{1,2} + ... + A_{d-1,d}) + \frac{1}{12}(\frac{1}{A} + \frac{A_1}{a_1} + ... + \frac{A_d}{a_d})$$

one can get the value of c_2, which is equal to,

$$c_2 = \frac{1}{(4-2)!}[C_4 - S(A_1,7) - S(A_2,2) - S(A_3,3) - S(A_4,5)]$$

where

$$C_4 = \frac{1}{4}\left(4 + A_{1,2} + A_{2,3} + A_{3,4}\right) + \frac{1}{12}\left(\frac{1}{A} + \frac{A_1}{a_1} + \frac{A_2}{a_2} + \frac{A_3}{a_3} + \frac{A_4}{a_4}\right)$$

$A_{1,2} = a_3 a_4 = 15$, $A_{2,3} = a_1 a_4 = 35$, and $A_{3,4} = a_1 a_2 = 14$, therefore

$$c_2 = \frac{1}{2!}\left(24.37698 - 0.12698\right) = 12.125000.$$

2.6 General formula for the differentiation of $I, J_1, J_2, ..., J_d, h$

In this section, we get a general form for the differentiation of the terms $I, J_1, J_2, ..., J_d$ and h that appears throughout the process of finding the coefficients of the Ehrhart polynomial, we begin by considering

$$I^{[j]} = e^Z I^{(j)} J_2 J_3 \cdots J_d h \qquad j = 1,2,..$$

where $I^{[j]}$ means that only I in the expression $e^Z I J_2 J_3 \cdots J_d h$ is differentiated j times.

Let $E_1 = 1 + \dfrac{J_2'}{J_2} + \ldots + \dfrac{J_d'}{J_d}$, then

$$I'' = I^{[2]} + E_1 I',$$

$$I''' = I^{[3]} + E_1(I^{[2]} + I'') + E_1' I',$$

$$I^{(4)} = I^{[4]} + E_1(2I^{[3]} + I''') + E_1^2 I^{[2]} + E_1'(I^{[2]} + 2I'') + E_1 I',$$

$$\begin{aligned} I^{(5)} &= I^{[5]} + E_1(3I^{[4]} + I^{(4)}) + 3E_1^2 I^{[3]} + 3E_1'(I^{[3]} + I''') + 3E_1 E_1' I^{[2]} \\ &\quad + E_1''(I^{[2]} + 3I'') + E_1''' I' + E_1^3 I^{[2]}, \end{aligned}$$

$$\begin{aligned} I^{(6)} &= I^{[6]} + E_1(4I^{[5]} + I^{(5)}) + 6E_1^2 I^{[4]} + 4E_1^3 I^{[3]} + E_1^4 I^{[2]} + E_1'(6I^{[4]} + 4I^{(4)}) \\ &\quad + 3E_1'^2 I^{[2]} + 12E_1 E_1' I^{[3]} + 6E_1^2 E_1' I^{[2]} + 4E_1 E_1'' I^{[2]} + E_1''(4I^{[3]} + 6I''') \\ &\quad + E_1'''(I^{[2]} + 4I'') + E_1^{(4)} I' + E_1^4 I^{[2]}, \end{aligned}$$

$$\begin{aligned} I^{(7)} &= I^{[7]} + E_1(5I^{[6]} + I^{(6)}) + 10E_1^2 I^{[5]} + 10E_1^3 I^{[4]} + 5E_1^4 I^{[3]} + E_1^5 I^{[2]} \\ &\quad + E_1'(10I^{[5]} + I^{(5)}) + E_1''(10I^{[4]} + 10I^{(4)}) + E_1'''(5I^{[3]} + 10I''') \\ &\quad + E_1^{(4)}(I^{[2]} + 5I'') + E_1^{(5)} I' + 30E_1 E_1' I^{[4]} + 30E_1^2 E_1' I^{[3]} + 10E_1^3 E_1' I^{[2]} \\ &\quad + 20E_1 E_1'' I^{[3]} + 10E_1^2 E_1'' I^{[2]} + 5E_1 E_1''' I^{[2]} + 15E_1'^2 I^{[3]} + 15E_1 E_1'^2 I^{[2]} \\ &\quad + 10E_1' E_1'' I^{[2]}, \end{aligned}$$

$$\begin{aligned} I^{(8)} &= I^{[8]} + E_1(6I^{[7]} + I^{(7)}) + 5E_1^2 I^{[6]} + 20E_1^3 I^{[5]} + 15E_1^5 I^{[3]} + E_1^6 I^{[3]} \\ &\quad + E_1'(15I^{[6]} + 6I^{(6)}) + E_1''(20I^{[5]} + 15I^{(5)}) + E_1'''(15I^{[4]} + 20I^{(4)}) \\ &\quad + E_1^{(4)}(6I^{[3]} + 15I''') + E_1^{(5)}(I^{[2]} + 6I'') + E_1^{(6)} I' + 60E_1 E_1' I^{[5]} + 90E_1^2 E_1' I^{[4]} \\ &\quad + 60E_1^3 E_1' I^{[3]} + 15E_1^4 E_1' I^{[2]} + 60E_1 E_1'' I^{[4]} + 60E_1^2 E_1'' I^{[3]} + 20E_1^3 E_1'' I^{[2]} \\ &\quad + 30E_1 E_1''' I^{[3]} + 15E_1^2 E_1''' I^{[2]} + 6E_1 E_1^{(4)} I^{[2]} + 45E_1'^2 I^{[4]} + E_1''^2 I^{[2]} + 90E_1 E_1'^2 I^{[3]} \\ &\quad + 45E_1^2 E_1'^2 I^{[2]} + 60E_1' E_1'' I^{[3]} + 15E_1' E_1''' I^{[2]} + 60E_1 E_1' E_1'' I^{[2]} + 15(E_1')^3 I^{[2]}, \end{aligned}$$

In order to differentiate $J_2, J_3, \ldots, J_d$ we need to find a general formula for these differentiations so we work on these elements and find a general formula. To illustrate this, consider for example,

$$J_2 = (1 + \frac{w_2}{2!} Z + \frac{w_2^2}{3!} Z^2 + \frac{w_2^3}{4!} Z^3 + ...)^{-1} = \frac{w_2 Z}{e^{w_2 Z} - 1}$$

then $e^{w_2 Z} J_2 - J_2 = w_2 Z$. By assuming the implicit differentiation for both sides of the above equation, we get

$$\frac{d}{dZ}(e^{w_2 Z} J_2) - \frac{d}{dZ}(J_2) = w_2$$

and the second derivative of the above equation is

$$\frac{d^2}{dZ^2}(e^{w_2 Z} J_2) - \frac{d^2}{dZ^2}(J_2) = 0$$

when we differentiate $e^{w_2 Z} J_2$ d-times we get a shape like a *binomial formula* $(a + b)^d = a^d + da^{d-1}b + \frac{d(d-1)}{2!} a^{d-2}b^2 + ... + b^d$.

Therefore,

$$e^{w_2 Z}(J_2 + w_2)^m - J_2^{(m)} = 0$$

where $J_2^{(m)}$ is the m-*th* derivative of J_2, since w_2 is constant therefore w_2^m means w_2 raised to the power m. For example,

let $h = J_2 e^{w_2 Z}$,

then

$$h' = J_2' e^{w_2 Z} + w_2 e^{w_2 Z} J_2 = e^{w_2 Z}(J_2' + w_2 J_2),$$

$$h'' = J_2'' e^{w_2 Z} + w_2 J_2' e^{w_2 Z} + w_2^2 e^{B_2 z} J_2 + w_2 J_2' e^{B_2 z} = e^{w_2 Z}(J_2'' + 2w_2 J_2' + w_2^2 J_2),$$

$$h''' = e^{w_2 Z}(J_2''' + 3w_2 J_2'' + 3w_2^2 J_2' + w_2^3 J_2).$$

And so on. Therefore

$J_2' = e^Z I J_2' \cdots J_d = J_2^{[1]}$,

$J_2'' = J_2^{[2]} + E_2 J_2'$ where $E_2 = 1 + \frac{I'}{I} + ... + \frac{J_d'}{J_d}$.

$J_2''' = J_2^{[3]} + 2E_2 J_2^{[2]} + E_2^2 J_2' + E_2' J'$.

$J_2^{(4)} = J_2^{[4]} + 3E_2 J_2^{[3]} + 3E_2^2 J_2^{[2]} + E_2^3 J_2' + 3E_2' J_2^{[2]} + E_2'' J_2' + 3E_2 E_2' J_2'$.

and similarly for highest derivative. By arranging them together we obtain

$$J_2' = e^Z I J_2' \cdots J_d \,,$$

$$J_2'' = J_2^{[2]} + E_2 J_2',$$

$$J_2''' = J_2^{[3]} + E_2(J_2^{[2]} + J_2'') + E_2' J' \,,$$

$$J_2^{(4)} = J_2^{[4]} + E_2(2J_2^{[3]} + J_2''') + E_2^2 J_2^{[2]} + E_2'(J_2^{[2]} + 2J_2'') + E_2'' J_2' \,,$$

$$\begin{aligned} J_2^{(5)} = {} & J_2^{[5]} + E_2(3J_2^{[4]} + J_2^{(4)}) + E_2^2(3J_2^{[3]}) + E_2^3 J_2^{[2]} + E_2'(3J_2^{[3]} + 3J_2''') \\ & + E_2''(J_2^{[2]} + 3J_2'') + E_2''' J_2' + 3E_2 E_2' J_2^{[2]}, \end{aligned}$$

$$\begin{aligned} J_2^{(6)} = {} & J_2^{[6]} + E_2(4J_2^{[5]} + J_2^{(5)}) + 6E_2^2 J_2^{[4]} + 4E_2^3 J_2^{[3]} + E_2^4 J_2^{[2]} + \\ & E_2'(6J_2^{[4]} + 4J_2^{(4)}) + E_2''(4J_2^{[3]} + 6J_2^{(3)}) + E_2'''(J_2^{[2]} + 4J_2'') + \\ & E_2^{(4)} J_2' + 12E_2 E_2' J_2^{[3]} + 6E_2' E_2^2 J_2^{[2]} + 4E_2 E_2'' J_2^{[2]} + 3(E_2')^2 J_2^{[2]}, \end{aligned}$$

$$\begin{aligned} J_2^{(7)} = {} & J_2^{[7]} + E_2(5J_2^{[6]} + J_2^{(6)}) + 10E_2^2 J_2^{[5]} + 10E_2^3 J_2^{[4]} + 5E_2^4 J_2^{[3]} \\ & + E_2^5 J_2^{[2]} + E_2'(10J_2^{[5]} + 5J_2^{(5)}) + E_2''(10J_2^{[4]} + 10J_2^{(4)}) \\ & + E_2'''(5J_2^{[3]} + 10J_2''') + E_2^{(4)}(J_2^{[2]} + 5J_2'') + E_2^{(5)} J_2' + 30E_2 E_2' J_2^{[4]} \\ & + 30E_2' E_2^2 J_2^{[3]} + 10E_2' E_2^3 J_2^{[2]} + 20E_2 E_2'' J_2^{[3]} + 10E_2^2 E_2'' J_2^{[2]} \\ & + 5E_2 E_2''' J_2^{[2]} + 15(E_2')^2 J_2^{[3]} + 15(E_2')^2 E_2 J_2^{[3]} + 10E_2' E_2'' J_2^{[2]}, \end{aligned}$$

$$\begin{aligned} J_2^{(8)} = {} & J_2^{[8]} + E_2(6J_2^{[7]} + J_2^{(7)}) + 15E_2^2 J_2^{[6]} + 20E_2^3 J_2^{[5]} + 15E_2^4 J_2^{[4]} + \\ & 6E_2^5 J_2^{[3]} + E_2^6 J_2^{[2]} + E_2'(15J_2^{[6]} + 6J_2^{(6)}) + E_2''(20J_2^{[5]} + 15J_2^{(5)}) + \\ & E_2'''(15J_2^{[4]} + 20J_2^{(4)}) + E_2^{(4)}(6J_2^{[3]} + 13J_2^{(3)}) + E_2^{(5)}(J_2^{[2]} + 6J_2'') + \\ & E_2^{(6)} J_2' + 60E_2 E_2' J_2^{[5]} + 90E_2' E_2^2 J_2^{[4]} + 60E_2^3 E_2'' J_2^{[2]} + 15E_2^4 E_2' J_2^{[2]} + \\ & 60E_2 E_2'' J_2^{[4]} + 60E_2^2 E_2'' J_2^{[3]} + 20E_2^3 E_2'' J_2^{[2]} + 30E_2 E_2''' J_2^{[3]} + 15E_2^2 E_2''' J_2^{[2]} + \\ & 6E_2 E_2^{(4)} J_2^{[2]} + 45(E_2')^2 J_2^{[4]} + 15(E_2')^3 J_2^{[2]} + 90E_2 (E_2')^2 J_2^{[3]} + 45E_2^2 (E_2')^2 J_2^{[2]} + \\ & 60E_2' E_2'' J_2^{[3]} + 15E_2' E_2''' J_2^{[2]} + 60E_2 E_2' E_2'' J_2^{[2]} + 10(E_2'')^2 J_2^{[2]}, \end{aligned}$$

$$\begin{aligned} J_2^{(9)} = {} & J_2^{[9]} + E_2(7J_2^{[8]} + J_2^{(8)}) + 21E_2^2 J_2^{[7]} + 35E_2^3 J_2^{[6]} + 35E_2^4 J_2^{[5]} + 21E_2^5 J_2^{[4]} + \\ & 7E_2^6 J_2^{[3]} + E_2^7 J_2^{[2]} + E_2'\left(21J_2^{[7]} + 7J_2^{(7)}\right) + E_2''\left(35J_2^{[6]} + 21J_2^{(6)}\right) + \\ & E_2'''\left(35J_2^{[5]} + 35J_2^{(5)}\right) + E_2^{(4)}\left(21J_2^{[4]} + 35J_2^{(4)}\right) + E_2^{(5)}\left(7J_2^{[3]} + 21J_2^{(3)}\right) + \\ & E_2^{(6)}\left(J_2^{[2]} + 7J_2''\right) + E_2^{(7)} J_2' + 105E_2E_2'J_2^{[6]} + 210E_2'E_2^2 J_2^{[5]} + 210E_2^3E_2'J_2^{[4]} + \\ & 105E_2^4E_2'J_2^{[3]} + 21E_2^5E_2'J_2^{[2]} + 140E_2E_2''J_2^{[5]} + 210E_2^2E_2''J_2^{[4]} + 140E_2^3E_2''J_2^{[3]} + \\ & 35E_2^4E_2''J_2^{[2]} + 105E_2E_2'''J_2^{[4]} + 105E_2^2E_2'''J_2^{[3]} + 35E_2^3E_2'''J_2^{[2]} + 42E_2E_2^{(4)}J_2^{[3]} + \\ & 21E_2^2E_2^{(4)}J_2^{[2]} + 7E_2E_2^{(5)}J_2^{[2]} + 105(E_2')^2 J_2^{[5]} + 315E_2(E_2')^2 J_2^{[4]} + 315E_2^2(E_2')^2 J_2^{[3]} + \\ & 105E_2^3(E_2')^2 J_2^{[2]} + 105(E_2')^3 J_2^{[3]} + 105E_2(E_2')^3 J_2^{[2]} + 210E_2'E_2''J_2^{[4]} + 105E_2'E_2'''J_2^{[3]} + \\ & 21E_2'E_2^{(4)}J_2^{[2]} + 70(E_2'')^2 J_2^{[3]} + 10E_2(E_2'')^2 J_2^{[2]} + 35E_2''E_2'''J_2^{[2]} + 420E_2E_2'E_2''J_2^{[3]} + \\ & 210E_2^2E_2'E_2''J_2^{[2]} + 105E_2E'E_2'''J_2^{[2]}. \end{aligned}$$

Since our work is for finding the coefficients of the Ehrhart polynomial until c_{d-9} , so the derivatives that we are needed are until 9-th derivative.

By similar procedure we get the derivatives of $J_3, J_4, \ldots, J_d$ and h that are used in the definition of ϕ(z) in the preceding sections. When we arrange the obtained results we get a triangle like a Polya triangle [40, p.20] where the contents of the triangle are the coefficients of $E_2{}^2, E_2^3, \ldots$ in the expression $J_2^{(4)}, J_2^{(5)}, \ldots$

	E_2^2														
$J_2^{(4)}$	1	E_2^3	$J_2^{[2]}$												
$J_2^{(5)}$	3	1	E_2^4	$J_2^{[3]}$	$J_2^{[2]}$										
$J_2^{(6)}$	6	4	1	E_2^5	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$								
$J_2^{(7)}$	10	10	5	1	E_2^6	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(8)}$	15	20	15	6	1	E_2^7	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(9)}$	21	35	35	21	7	1	E_2^8	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(10)}$	28	56	70	56	28	8	1	E_2^9	$J_2^{[8]}$	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

Also, the first terms of the coefficients of $E_2', E_2'', \ldots$ in the expression of $J_2^{(4)}, J_2^{(5)}, \ldots$ are

	E_2'												
$J_2^{(4)}$	1	E_2''	$J_2^{[2]}$										
$J_2^{(5)}$	3	1	E_2'''	$J_2^{[3]}$	$J_2^{[2]}$								
$J_2^{(6)}$	6	4	1	$E_2^{(4)}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(7)}$	10	10	5	1	$E_2^{(5)}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(8)}$	15	20	15	6	1	$E_2^{(6)}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(9)}$	21	35	35	21	7	1	$E_2^{(7)}$	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

The second terms of the coefficients of $E_2', E_2'',...$ in the expression of $J_2^{(4)}, J_2^{(5)},...$ are

	E_2'	E_2''												
$J_2^{(4)}$	1	1	E_2'''	J_2''	J_2'									
$J_2^{(5)}$	3	3	1	$E_2^{(4)}$	J_2'''	J_2''	J_2'							
$J_2^{(6)}$	4	6	4	1	$E_2^{(5)}$	$J_2^{(4)}$	J_2'''	J_2''	J_2'					
$J_2^{(7)}$	5	10	10	5	1	$E_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J_2'''	J_2''	J_2'			
$J_2^{(8)}$	6	15	20	15	6	1	$E_2^{(7)}$	$J_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J_2'''	J_2''	J_2'	
$J_2^{(9)}$	7	21	35	35	21	7	1	$E_2^{(8)}$	$J_2^{(7)}$	$J_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J_2'''	J_2'' J_2'

The coefficients of E_2E_2', $E_2^2E_2',...$ in the expression of $J_2'', J_2''',...$ are arranged as follows.

J_2''	0										
J_2'''	0										
$J_2^{(4)}$	0										
	E_2E_2'										
$J_2^{(5)}$	3	$E_2^2E_2'$	$J_2^{[2]}$								
$J_2^{(6)}$	12	6	$E_2^3E_2'$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(7)}$	30	30	10	$E_2^4E_2'$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(8)}$	60	90	60	15	$E_2^5E_2'$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(9)}$	105	210	210	105	21	E_2^6E'	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

The diagonal of the above results is the second column of the preceding Polya triangle, and the first column for the above results is obtained as follows:

By multiplying the diagonal by 4,5,6… we get the line under the diagonal, which are:
(3)(4)=12,
(6)(5)=30,
(10)(6)=60,
(15)(7)=105,

The general formula of the differentiation is given by

$$J_2^{(m)} = J_2^{[m]} + E_2\left((m-2)J_2^{[m-1]} + J_2^{(m-1)}\right) + W$$

where 1<m≤8 and W can be obtained from the given tables as follow

when m=3 then

$$J_2^{(3)} = J_2^{[3]} + E_2\left(J_2^{[2]} + J_2^{(2)}\right)$$

when m=4 then

$$J_2^{(4)} = J_2^{[4]} + E_2\left(2J_2^{[3]} + J_2^{(3)}\right) + W$$

from the tables, W can be found as follows

$$W = E_2^2 J_2^{[2]} + E_2'\left(J_2^{[2]} + 2J_2^{(2)}\right) + E_2'' J_2'.$$

Chapter Three
The Ehrhart Polynomial of H-representation Of a Polytope

Introduction

As seen before the computation of the volume of a polytope is very important in many applications. For this importance many researches concerning the volume of the polytopes and in particular with the Birkhoff polytopes are found.

In this chapter a method for computing the coefficients of the Ehrhart polynomial of the Birkhoff polytope is discussed. Make a change on the matrix, which represents the polytope and find a formula for the number of integral points. These changes of the matrix are matrix operations on the rows (columns) of the matrix. We try to lessen the effect of their changes on the number of integral points. We further discuss a method for finding the volume of a polytope using Laplace transform.

Chapter three consists of four sections. In section one, a method for finding the volume of H-representation of polytope using Laplace transforms. In section two some basic concepts and remarks about the Birkhoff polytopes and their volumes are given. In section three, the Ehrhart polynomial of Birkhoff polytope is presented and in section four, further properties about the number of integral points are obtained under certain conditions, also we give the relation between the number of integral points of the original polytopes and the changing polytopes.

3.1 Computing the volume of a polytope using Laplace transform, [33]

This section is devoted to the computation of the volume of a polytope with H-representation using Laplace transform.

The idea of the method is to consider the volume of $P = \{X \in \Re^d_+ : AX \le b\}$, $A \in \Re^{n \times d}$ and $y \in \Re^n$, as a function g(b) where $g : \Re^n \to \Re$ which provides a simple expression of its Laplace transform $G : C^n \to C$, and the inverse Laplace transform to G, which, in the next section, can efficiently be done by repeated applications of Cauchy residue theorem for the evaluation of one–dimensional complex integrals.

Let $y \in \Re^n$ and $A \in \Re^{n \times d}$ such that the convex polyhedron,

$$P(y) = \{X \in \Re_+^d : AX \le y\} \qquad (3.1)$$

is compact, that is, P(y) is a convex polytope. The symbol $\Re_+$ stands for the semi closed interval $[0,\infty) \subset \Re$.

Now consider the function $g : \Re^n \to \Re$ defined by

$$g(y) = \int_{P(y)} dx = Vol(P(y)) \qquad (3.2)$$

and let $G : C^n \to C$ be its n-dimensional Laplace transform, that is, [1]

$$G(\lambda) = \int_{\Re^n} e^{-<\lambda, y>} g(y) dy \qquad (3.3)$$

where $< \cdot , \cdot >$ is the Euclidean product defined on $\Re^n$.

Theorem (3.1.1), [33]:

Let $P(y) = \{X \in \Re_+^d : AX \le y\}$ be a polytope, and let g and G be defined by

$$g(y) = \int_{P(y)} dx = Vol(P(y)), \; G(\lambda) = \int_{\Re^n} e^{-<\lambda, y>} g(y) dy$$

respectively. Assume that X=0 is the only solution of the system $\{X \ge 0 : AX \le 0\}$. Then:

$$G(\lambda) = \frac{1}{\prod_{i=1}^{n} \lambda_i \prod_{j=1}^{d} (A^T \lambda)_j}, \quad \text{with} \quad \begin{cases} \mathrm{Re}(\lambda) > 0 \\ \mathrm{Re}(A^T \lambda) > 0 \end{cases} \qquad (3.4)$$

Moreover

$$g(y) = \frac{1}{(2\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} e^{<\lambda, y>} G(\lambda) d\lambda \qquad (3.5)$$

where the real constants $c_1, c_2, ..., c_n$ are chosen such that c>0 satisfies $A^T c > 0$.

Proof:

Apply the definition of G given by (3.3) to obtain

$$G(\lambda) = \int_{\Re^n} e^{-<\lambda, y>} \left[\int_{x \ge 0, Ax \le y} dx \right] dy$$

$$= \int_{\Re_+^d} \left[\int_{y \geq Ax} e^{-<\lambda, y>} dy \right] dx$$

$$= \frac{1}{\prod_{i=1}^{n} \lambda_i} \int_{\Re_+^d} e^{-<A^T\lambda, x>} dx, \quad \text{with } \operatorname{Re}(\lambda) > 0$$

then after simple computations, one can get

$$G(\lambda) = \frac{1}{\prod_{i=1}^{n} \lambda_i \prod_{j=1}^{d} (A^T\lambda)_j}, \text{ with } \begin{cases} \operatorname{Re}(\lambda) > 0 \\ \operatorname{Re}(A^T\lambda) > 0 \end{cases}$$

and (3.5) is obtained by a direct application of the inverse Laplace transform, [1]. It remains to show that, the domain $\{\operatorname{Re}(\lambda) > 0, \operatorname{Re}(A^T\lambda) > 0\}$ is non empty. This flows from the fact that a special version of Farka's lemma due to Carver, [35, p.30], which states that $\{u > 0 : A^T u > 0\}$ has a solution $u \in \Re^n$ if and only if (x, y) = 0 is the only solution of the system $\{Ax + y = 0 : x \geq 0, y \geq 0\}$. In other words, x = 0 is the only solution of $\{x \geq 0 : Ax \leq 0\}$, which is indeed the condition given by this theorem. ■

Suppose that we want to compute the volume of the convex polytope $P = \{X \in \Re_+^d : AX \leq b\}$ with b>0, that is, we must evaluate g(y) at the point $y = b$. Without a loss of generality, we may assume that $y_i = 1$ for every i=1,…,n.

The problem is then computing h(1) of the function $h : \Re_+ \to \Re$ given by,

$$h(z) = g(ze_n) = \frac{1}{(2\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} e^{z<\lambda, e_n>} G(\lambda) d\lambda \qquad (3.6)$$

where $e_i = (1, 1, ...)$ be the vector in $\Re^i$ for $i \geq 1$, such that its element is one, and the real vector $0 < c \in \Re^n$, satisfies $A^T c > 0$.

Computing the complex integral (3.5) can be done in two ways, directly as given in sec. (3.1.1) or indirectly as given in sec. (3.1.2).

3.1.1 The direct method, [33]:

For better understanding the direct method, consider the case of a polytope P with n=2.

Let $A\in\Re^{2\times d}$ be such that X=0 is the only solution of $\{X\geq 0 : AX\leq 0\}$. Moreover, suppose that $A^T=[a|b]$ with $a,b\in\Re^d$ and assume that:

i) $a_jb_j\neq 0$ and $a_j\neq b_j$ for all j =1,2,…,d.

ii) $a_j|b_j\neq a_k|b_k$ for all j, k =1,2,…,d.

then:

$$G(\lambda)=\frac{1}{\lambda_1\lambda_2\prod_{j=1}^{d}(a_j\lambda_1+b_j\lambda_2)},\ \text{with}\begin{cases}\text{Re}(\lambda)>0\\ \text{Re}(a\lambda_1+b\lambda_2)>0\end{cases}$$

Next, fix c_1 and $c_2>0$ such that $a_jc_1+b_jc_2>0$ for every j =1,2,…,d, and compute the integral (3.6) as follows. First evaluate the integral,

$$I_1=\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{e^{z\lambda_1}}{\lambda_1\prod_{j=1}^{d}(a_j\lambda_1+b_j\lambda_2)}d\lambda_1 \tag{3.7}$$

by using Cauchy residue technique. That is:

a) Close the path of integration by adding a semicircle Γ of radius R large enough.
b) Evaluate the closed integral using Cauchy's residue theorem.
c) Show that the integral along Γ converges to zero when $R\to\infty$.

Now, since we are integrating with respect to λ_1 and we must evaluate h(z) at z=1, the semicircle Γ must be added on the left side of the integration path $\text{Re}(\lambda_1)=c_1$ because $e^{z\lambda_1}$ converges to zero when $\text{Re}(\lambda_1)\to-\infty$. Therefore, we must consider only poles of $G(\lambda_1,\cdot)$ whose real parts is strictly less than c_1(with λ_2 being fixed). Recall that $\text{Re}(-\lambda_2b_j/a_j)=-c_2b_j/a_j<c_1$ for each j=1,2,…,d, and $G(\lambda_1,\cdot)$ has only poles of the first order (with λ_2 being fixed). Then, the evaluation of (3.7) follows, and

$$I_1=\frac{1}{\lambda_2^d\prod_{j=1}^{d}b_j}+\sum_{j=1}^{d}\frac{-e^{-(b_j/a_j)z\lambda_2}}{b_j\lambda_2^d\prod_{k\neq j}(-a_kb_j/a_j+b_k)}$$

Therefore,

$$h(z)=\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{e^{z\lambda_2}I_1(\lambda_2)}{\lambda_2}d\lambda_2$$

$$=\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{e^{z\lambda_2}}{\lambda_2^{d+1}\prod_{j=1}^{d}b_j}d\lambda_2-\sum_{j=1}^{d}\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{a_j^d e^{(1-b_j/a_j)z\lambda_2}}{\lambda_2^{d+1}a_j b_j\prod_{k\neq j}(b_k a_j-a_k b_j)}d\lambda_2$$

these integrals must be evaluated according to whether $(1-b_j/a_j)y$ is positive or negative. Thus recalling that Z > 0, each integral is equal to:

i) its residue at the pole $\lambda_2=0<c_2$ when $1-b_j/a_j$ is positive, and

ii) zero if $1-b_j/a_j$ is negative because there is no pole on the right side of $\mathrm{Re}(\lambda_2)=c_2$ that is

$$h(z)=\frac{z^d}{d!}\left[\frac{1}{\prod_{j=1}^{d}b_j}-\sum_{b_j/a_j<1}\frac{(a_j-b_j)^d}{a_j b_j\prod_{k\neq j}(b_k a_j-a_k b_j)}\right] \qquad (3.8)$$

Observe that the above formula is not symmetrical in the parameters a, b. This is because we have chosen to integrate first with respect to λ_1; and the set $\{j:b_j/a_j<1\}$ is different from the set $\{j:a_j/b_j>1\}$, which would have been considered if we have to integrate first with respect to λ_2.

In the latter case, we would have obtained.

$$h(z)=\frac{z^d}{d!}\left[\frac{1}{\prod_{j=1}^{d}a_j}-\sum_{a_j/b_j<1}\frac{(b_j-a_j)^d}{a_j b_j\prod_{k\neq j}(a_k b_j-b_k a_j)}\right]$$

which is (3.8) by interchanging a and b.

<u>Example(3.1.1),[33]:</u>

Let $e_3=(1,1,1)$ be a vector in $\Re^3$ and let $P(ze_3)\subset\Re^2$ be the polytope, which is defined by,

$P(ze_3) = \{X \in \Re_+^2 : x_1 + x_2 \le z; -2x_1 + 2x_2 \le z; 2x_1 - x_2 \le z\}$. To find the volume of this polytope, write $P(ze_3)$ as

$$P(ze_3) = \{X \in \Re_+^2 : AX \le z\}, \text{ where } A = \begin{pmatrix} 1 & 1 \\ -2 & 2 \\ 2 & -1 \end{pmatrix},$$

choose $c_1 = 3, c_2 = 2$ and $c_3 = 1$, so that $c_1 > 2c_2 - 2c_3$ and $c_1 > c_3 - 2c_2$ that satisfies the conditions c>0 and $A^T c > 0$.

$$h(z) = \frac{1}{(2\pi i)^3} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_3 - i\infty}^{c_3 + i\infty} e^{(\lambda_1 + \lambda_2 + \lambda_3)z} G(\lambda) d\lambda,$$

with $G(\lambda) = \dfrac{1}{\lambda_1 \lambda_2 \lambda_3 (\lambda_1 - 2\lambda_2 + 2\lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_3)}$,

Integrate first with respect to λ_1; that is, evaluate the residue of $G(\lambda)$ at the poles $\lambda_1 = 0, \lambda_1 = 2\lambda_2 - 2\lambda_3$ and $\lambda_1 = \lambda_3 - 2\lambda_2$ since 0<z, $0 < c_1$, $\mathrm{Re}(2\lambda_2 - 2\lambda_3) < c_1$ and $\mathrm{Re}(\lambda_3 - 2\lambda_2) < c_1$. We obtain,

$$h(z) = \frac{1}{(2\pi i)^2} \int_{c_2 - i\infty}^{c_2 + i\infty} \cdots \int_{c_3 - i\infty}^{c_3 + i\infty} (I_2 + I_3 + I_4) d\lambda_2 d\lambda_3.$$

where,

$$I_2 = \frac{e^{-(\lambda_2 + \lambda_3)z}}{2\lambda_2 \lambda_3 (\lambda_3 - \lambda_2)(\lambda_3 - 2\lambda_2)}$$

$$I_3 = \frac{e^{(3\lambda_2 - \lambda_3)z}}{6\lambda_2 \lambda_3 (\lambda_3 - \lambda_2)(\lambda_3 - 4/3\lambda_2)}$$

$$I_4 = \frac{e^{(2\lambda_2 - \lambda_3)z}}{3\lambda_2 \lambda_3 (\lambda_3 - 2\lambda_2)(\lambda_3 - 4/3\lambda_2)}$$

Next, integrate I_2 with respect to λ_3, we must consider the poles of I_2 on the left side of $\mathrm{Re}(\lambda_3) = 1$, that is the pole $\lambda_3 = 0$, since $\mathrm{Re}(\lambda_2) = 2$. Thus, we get $\dfrac{-e^{z\lambda_2}}{4\lambda_2^3}$, and the next integration with respect to λ_2 yields $\dfrac{-z^2}{8}$. When integrate I_3 with respect to λ_3, we have to consider the poles $\lambda_3 = \lambda_2$ and $\lambda_3 = \dfrac{4\lambda_2}{3}$, on the right side of $\mathrm{Re}(\lambda_3) = 1$; and we get

$$\frac{-1}{\lambda_2^3}\left[-\frac{e^{2z\lambda_2}}{2}+\frac{3e^{5z\lambda_2/3}}{8}\right]$$

Recall that the path of integration has a negative orientation, so the negative values of residues have to be considered. The next integration with respect to λ_2 yields $z^2\left(1-\frac{25}{48}\right)$.

Finally, when integrating $I_4(\lambda_2,\lambda_3)$ with respect to λ_3, we must consider only the pole $\lambda_3=0$, we get $\frac{-e^{z\lambda_2}}{8\lambda_2^3}$; the next integration with respect to λ_2 yields zero. Hence, adding up the above three partial results, yields,

$$h(z)=z^2\left[-\frac{1}{8}+1-\frac{25}{48}\right]=\frac{17z^2}{48}.$$

<u>3.1.2 The indirect method, [33]:</u>

The indirect method permits avoid evaluating integrals of exponential function in (3.6) which will be illustrated in this section. We want to compute (3.6) where $b\neq 0$ and c>0 are real vectors in $\Re^n$ with $A^T c>0$. From (3.1) it is deduced that g (b)=0 whenever $b\leq 0$, so that the last entry $b_n>0$; and the following simple change of variables are done.

Let $s=<\lambda,b>$ and $d_1=<c,b>$ so that,

$$\lambda_n=\frac{\left(s-\sum_{j=1}^{n-1}b_j\lambda_j\right)}{b_n}$$ and

$$h(z)=\frac{1}{(2\pi i)^n}\int_{c_1-i\infty}^{c_1+i\infty}\cdots\int_{c_{n-1}-i\infty}^{c_{n-1}+i\infty}\left[\int_{d_1-i\infty}^{d_1+i\infty}e^{zs}\hat{G}\,ds\right]d\lambda_1\ldots d\lambda_{n-1}$$

where $\hat{G}(\lambda_1,\ldots,\lambda_{n-1},s)=\frac{1}{b_n}G\left(\lambda_1,\ldots,\lambda_{n-1},\frac{s}{b_n}-\sum_{j=1}^{n-1}\frac{b_j}{b_n}\lambda_j\right)$

h(z) can be rewrite as follows:

$$h(z)=\frac{1}{(2\pi i)}\int_{d_1-i\infty}^{d_1+i\infty}e^{zs}H(s)ds$$, with

$$H(s)=\frac{1}{(2\pi i)^{n-1}}\int_{c_1-i\infty}^{c_1+i\infty}\cdots\int_{c_{n-1}-i\infty}^{c_{n-1}+i\infty}\hat{G}\,d\lambda_1\cdots d\lambda_{n-1}$$

where H(s) is the Laplace transform of h(z)=g(zb), and is called the associated transform of $G(\lambda)$.

Let $A \in \Re^{2\times n}$ such that X=0 is the only solution of $\{X \geq 0 : AX \leq 0\}$. Write $A^T = [a|b]$ with $a,b \in \Re^d$, assume that $a_j b_j \neq 0$ for all j=1,…,d and $a_j|b_j \neq a_k|b_k$ for all $j \neq k$

Then,

$$G(\lambda) = \frac{1}{\lambda_1 \lambda_2 \prod_{j=1}^{d} (a_j \lambda_1 + b_j \lambda_2)}, \qquad \text{with} \begin{cases} \operatorname{Re}(\lambda) > 0, \\ \operatorname{Re}(A^t \lambda) > 0 \end{cases}$$

fix $\lambda_2 = s - \lambda_1$ and choose real constants $c_1 > 0$ and $c_2 > 0$ such that $a_j c_1 + b_j c_2 > 0$ for every j=1,2,…,d. Notice that $\operatorname{Re}(s) = c_1 + c_2$. H(s) is obtained by integrating $G(\lambda_1, s - \lambda_1)$ along the line $\operatorname{Re}(\lambda_1) = c_1$, which yields,

$$H(s) = \frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{1}{\lambda_1 (s - \lambda_1) \prod_{j=1}^{d} ((a_j - b_j)\lambda_1 + b_j s)} d\lambda_1$$

Next, determine which poles of $G(\lambda_1, s - \lambda_1)$ are on the left(right) side of the integration path $\{\operatorname{Re}(\lambda_1) = c_1\}$ in order to apply the Cauchy residue theorem. Let $J_+ = \{j : a_j > b_j\}, J_0 = \{j : a_j = b_j\}$ and $J_- = \{j : a_j < b_j\}$. Then the poles on the left side of $\{\operatorname{Re}(\lambda_1) = c_1\}$ are $\lambda_1 = 0$ and $\lambda_1 = -b_j s/(a_j - b_j)$ for all $j \in J_+$ since $-b_j \operatorname{Re}(s)/(a_j - b_j) < c_1$. Besides, the poles on the right side of $\{\operatorname{Re}(\lambda_1) = c_1\}$ are $\lambda_1 = s$ and $\lambda_1 = -b_j s/(a_j - b_j)$ for all $j \in J_-$.

Finally, notice that $G(\lambda_1, s - \lambda_1)$ has only poles of the first order. Hence, computing the residues of poles on the left side of $\{\operatorname{Re}(\lambda_1) = c_1\}$ yields,

$$H(s) = \frac{1}{\prod_{j \in J_0} s b_j} \left[\frac{1}{s \prod_{j \notin J_0} s b_j} + \sum_{j \in J_+} \frac{-(a_j - b_j)^{d - |j_0|}}{s^2 a_j b_j \prod_{k \notin j_0, k \neq j} (-s b_j a_k + s a_j b_k)} \right]$$

After moving terms around, we obtain

$$H(s)=\frac{1}{s^{d+1}}\left[\frac{1}{\prod_{j=1}^{d} b_j}-\sum_{a_j>b_j}\frac{(a_j-b_j)^d}{a_jb_j\prod_{k\neq j}(a_jb_k-b_ja_k)}\right]$$

Notice that the previous equation holds even for the case $J_0 \neq 0$. Finally, after integration with respect to s, we get

$$h(z)=\frac{z^d}{d!}\left[\frac{1}{\prod_{j=1}^{d} b_j}-\sum_{a_j>b_j}\frac{(a_j-b_j)^d}{a_jb_j\prod_{k\neq j}(a_jb_k-b_ja_k)}\right]$$

Now, computing the negative value of residues of poles on the right side of $\{\mathrm{Re}(\lambda_1)=c_1\}$ yields,

$$H(s)=\frac{1}{s^{d+1}}\left[\frac{1}{\prod_{j=1}^{d} a_j}-\sum_{b_j>a_j}\frac{(b_j-a_j)^d}{a_jb_j\prod_{k\neq j}(b_ja_k-a_jb_k)}\right]$$

and after integration with respect to s, one also get the following

$$h(z)=\frac{z^d}{d!}\left[\frac{1}{\prod_{j=1}^{d} a_j}-\sum_{a_j/b_j<1}\frac{(b_j-a_j)^d}{a_jb_j\prod_{k\neq j}(a_kb_j-b_ka_j)}\right] \text{ in particular case } J_0=0$$

<u>Example(3.1.2),[33]:</u>

Consider example(3.1.1). By setting $e_3=(1,1,1)$, let $\mathrm{P}(ze_3)\subset\Re^2$ be the polytope which is defined by

$\mathrm{P}(\mathrm{ze}_3)=\{\mathrm{X}\in\Re_+^2 \mid \mathrm{x}_1+\mathrm{x}_2\leq \mathrm{z}; -2\mathrm{x}_1+2\mathrm{x}_2\leq \mathrm{z}; 2\mathrm{x}_1-\mathrm{x}_2\leq \mathrm{z}\}$

We can choose $c_1=c_2=1, c_3=2$ and $\lambda_3=s-\lambda_2-\lambda_1$, so that, $\mathrm{Re}(s)=d_1=4$, and

$$H(s)=\frac{1}{(2\pi i)^2}\int_{1-i\infty}^{1+i\infty}\int_{1-i\infty}^{1+i\infty}M(\lambda,s)\,d\lambda_1 d\lambda_2$$

with $M(\lambda,s)=\dfrac{1}{\lambda_1\lambda_2(s-\lambda_1-\lambda_2)(2s-\lambda_1-4\lambda_2)(2\lambda_1+3\lambda_2-s)}$

We first integrate with respect to λ_1. Only the real parts of poles $\lambda_1 = 0$ and $\lambda_1 = (s - 3\lambda_2)/2$ are less than 1. Therefore, the residue of the 0-pole yields:

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{1}{\lambda_2(s-\lambda_2)(2s-4\lambda_2)(3\lambda_2 - s)} d\lambda_2 \qquad (3.9)$$

whereas the residue of $\left((s-3\lambda_2)/2\right)$-pole yields

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{4}{\lambda_2(s-3\lambda_2)(s+\lambda_2)(3s-5\lambda_2)} d\lambda_2 \qquad (3.10)$$

Applying again the Cauchy residue theorem to (3.9) at the pole $\lambda_2 = 0$ yields $-1/(2s^3)$.

Similarly, applying the Cauchy residue theorem to (3.10) at the poles $\lambda_2 = 0$ and $\lambda_2 = -s$ yields $\frac{29}{24s^3}$, finally we obtain $H(s) = \frac{17}{24s^3}$ and so $h(z) = \frac{17}{48z^2}$, which is the area of the polytope.

3.2 Some basic concepts about Birkhoff polytopes

The set of doubly stochastic $n \times n$ matrices form a convex set called Birkhoff polytope. In this section, we describe the Birkhoff polytope with some methods for finding its volume.

We start this section by the following definition.

Definition (3.2.1), [7]:

The n-*th* Birkhoff polytope B_n is the set of all doubly stochastic $n \times n$ matrices, that is, those matrices with non negative real coefficients in which every row and column sums to one. In other words, the n-*th* Birkhoff polytope B_n is defined as

$$B_n = \left\{ \begin{pmatrix} x_{11} & x_{12} & . & . & . & x_{1n} \\ x_{21} & x_{22} & & & & x_{2n} \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ x_{n1} & x_{n2} & . & . & . & x_{nn} \end{pmatrix} \in \Re_+^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \quad \text{for all } 1 \le k \le n \\ \sum_k x_{jk} = 1 \quad \text{for all } 1 \le j \le n \end{array} \right\}$$

B_n is a convex polytope.

Remarks (3.2.1):

There are different ways for computing the volume of Birkhoff polytopes. One of the recent attempts to compute $vol(B_n)$ relies on the theory of counting functions for the integer points in the polytopes. Recall that Ehrhart has proved that for a polytope $P \subset \Re^n$ with integral vertices, the number, $L(P,t) = card(tP \cap Z^n)$, is a polynomial in the positive integer variable t, this counting function, has three properties, which are [7],

- The degree of $L(P,t)$ is the dimension of P.
- The leading term of $L(P,t)$ is the relative volume of P.
- Since $L(P,t)$ is a polynomial, therefore it can be evaluated at nonpositive integers. These evaluations yield

$$L(P,0) = 1$$
$$L(P,-t) = (-1)^{\dim(P)} L(P^\circ, t) \qquad (3.11)$$

where P° is the interior of P.

We will denote the Ehrhart polynomial of the Birkhoff polytope B_n as $H_n(t) = L(B_n, t)$, where H_n is a polynomial in t of degree $(n-1)^2$, to do this count, note that the last row and column are fixed by the conditions that the row and column sums should equal one. The remaining $(n-1)^2$ entries can be chosen freely; therefore the dimension of B_n is $(n-1)^2$, which is the degree of the Ehrhart polynomial of the Birkhoff polytope.

The first two of these polynomials are easily computed, [7] which are,

$$H_1(t) = 1, H_2(t) = t + 1$$

We will compute these polynomials in the next section, the first nontrivial polynomial was computed see [7], is

$$H_3(t) = -3\binom{t+3}{2} + \binom{t+2}{2}^2$$

The properties of H_n are found in [7]. From (3.11), it is deduced that

$$H_n(-n-t) = (-1)^{n-1} H_n(t)$$

and

$$H_n(-1) = H_n(-2) = \ldots = H_n(-n+1) = 0$$

This allows the following strategy for computing H_n, and therefore, the volume of B_n: compute the first $\binom{n-1}{2}$ values of H_n, use the above symmetry and trivial values of H_n, and calculate the polynomial H_n by interpolation, [7], [17].

3.3 The Ehrhart polynomial of a Birkoff polytope

In this section we give some theorems that can be used to count the Ehrhart polynomial of the Birkhoff polytope. One can view the Birkhoff polytope B_n as given in the form of

$$P = \{X \in \Re^d_+ : AX = b\} \qquad (3.12)$$

where
$$A = \begin{pmatrix}
1 & . & . & . & 1 & 0 & & & & 0 & & & & & & & & 0 \\
0 & & & & 0 & 1 & . & . & . & 1 & & & & & & & & 0 \\
0 & & & & 0 & 0 & & & & 0 & . & & & & & & & 0 \\
0 & & & & 0 & 0 & & & & 0 & & . & & 0 & & & & \\
 & & & & & & & & & & & & . & 0 & & & & 0 \\
 & & & & & & & & & & & & & 1 & . & . & . & 1 \\
1 & & & & & 1 & & & & & & & & 1 & & & & \\
 & . & & & & & . & & & & & & & 0 & . & & & \\
 & & . & & & & & . & & & & & & & & . & & \\
0 & & & . & 0 & 0 & & & . & 0 & . & . & . & & & & . & 0 \\
0 & & & & 1 & 0 & & & 0 & 1 & & & & & & & 0 & 1
\end{pmatrix}$$

We illustrate how to represent the Birkhoff polytope in the form (3.12). For example if n = 2 then from the definition of the Birkhoff polytope,

$$B_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} + a_{12} = 1$$
$$a_{21} + a_{22} = 1$$
$$a_{11} + a_{21} = 1$$
$$a_{12} + a_{22} = 1$$

this means that $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \; and \; b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Then

$$\mathrm{P} = \{\mathrm{X} \in \Re_{+}^{\mathrm{d}} : \mathrm{AX} = \mathrm{b}\}$$

which is the H-representation of the polytope.

Now, some theorems that concern this type of polytopes are given.

Theorem (3.3.1), [7]:

Consider a convex rational polytope P given by (3.12) for some $(\mathrm{m} \times \mathrm{d})$-matrix A and m-dimensional vector b. Denote the columns of A by $c_1, \ldots, c_d$. Then

$$L(P,t) = \frac{1}{(2\pi i)^m} \int_{|Z_1|=\xi_1} \cdots \int_{|Z_m|=\xi_m} \frac{Z_1^{-tb_1-1} Z_2^{-tb_2-1} \cdots Z_m^{-tb_m-1}}{(1-Z^{c_1})(1-Z^{c_2})\cdots(1-Z^{c_d})} dZ$$

where $0 < \xi_1, \ldots, \xi_m < 1$ are distinct real numbers and t is a positive integer.

Proof:

Consider the function

$$f(Z) = \frac{Z_1^{-tb_1-1} Z_2^{-tb_2-1} \cdots Z_m^{-tb_m-1}}{(1-Z^{c_1})(1-Z^{c_2})\cdots(1-Z^{c_d})}$$

Here, (the standard multivariate notation $V^W = v_1^{w_1} \dots v_n^{w_n}$) is used. f(Z) will be integrated with respect to each variable over a circle with a small radius:

$$\int_{|Z_1|=\xi_1} \cdots \int_{|Z_m|=\xi_m} f(Z_1,\dots,Z_m)\, dZ_m \cdots dZ_1 \qquad (3.13)$$

here $0 < \xi_1,\dots,\xi_m < 1$ are chosen such that, all of the $\frac{1}{(1-Z^{c_k})}$ can be expanded into power series about 0. Since the integral over one variable will give the respective residue at 0, integration can be done with respect to one variable at a time. When f is expanded into its Laurent series about 0, each term has the form $Z_1^{r_1-tb_1-1},\dots,Z_m^{r_m-tb_m-1}$, where $r_1,\dots,r_m$ are nonnegative integers. Thus, in the integral (3.13) will give a contribution precisely.■

Remark(3.3.1):

A special case of theorem (3.3.1) can be obtained by putting b = (1,1,…,1) $\in Z^{2n}$, therefore the Ehrhart polynomial of the n-*th* Birkhoff polytope is given by

$$H_n(t) = \frac{1}{(2\pi i)^{2n}} \int \cdots \int \frac{(Z_1 \cdots Z_{2n})^{-t-1}}{(1-Z_1 Z_{n+1})(1-Z_2 Z_{n+2})\cdots(1-Z_n Z_{2n})}\, dZ$$

Here, each integral is over a circle with radius < 1 centered at the origin; all appearing radii should be different.

The following theorem gives a general formula of the Ehrhart polynomial of the Birkhoff polytope with b = (1,1,…,1).

Theorem (3.3.2), [7]:

For any distinct $0 < \xi_1,\dots,\xi_n < 1$,

$$H_n(t) = \frac{1}{(2\pi i)^n} \int_{|Z_1|=\xi_1} \cdots \int_{|Z_n|=\xi_n} (Z_1 \cdots Z_n)^{-t-1} \left(\sum_{k=1}^{n} \frac{Z_k^{t+n-1}}{\prod_{j \neq k} (Z_k - Z_j)} \right)^n dZ_n \cdots dZ_1 .$$

Proof:

From remark (3.3.1), the last n variables of $H_n(t)$ can separate and obtain

$$H_n(t)=\frac{1}{(2\pi i)^n}\int\cdots\int(Z_1Z_2\cdots Z_n)^{-t-1}\left(\frac{1}{2\pi i}\int\frac{Z^{-t-1}}{(1-Z_1Z)(1-Z_1Z)\cdots(1-Z_nZ)}dZ\right)^n dZ_n dZ_{n-1}\cdots dZ_1$$

The radius of integration circle of the inner most integral may be chosen to be smaller than the radii of the other integration paths. Then the innermost integral is computed which is equal to the residue at zero of

$$\frac{1}{Z^{t+1}(1-Z_1Z)\cdots(1-Z_nZ)}$$

and, by residue theorem, equal to the negative of the sum of the residues at $Z_1^{-1},\ldots,Z_n^{-1}$. (Note that here we use the fact that t>0).

The residues at these poles are computed: the one, say, at $\frac{1}{Z_1}$ can be calculated as

$$\lim_{Z\to\frac{1}{Z_1}}\left(Z-\frac{1}{Z_1}\right)\cdot\frac{1}{Z^{t+1}(1-Z_1Z)\cdots(1-Z_nZ)}$$

$$=\lim_{Z\to\frac{1}{Z_1}}\frac{Z-\frac{1}{Z_1}}{1-Z_1Z}\cdot\frac{1}{Z^{t+1}(1-Z_2Z)\cdots(1-Z_nZ)}$$

$$=-\frac{1}{Z_1}\cdot\frac{Z_1^{t+1}}{(1-\frac{Z_2}{Z_1})\cdots(1-\frac{Z_n}{Z_1})}=-\frac{Z_1^{t+n-1}}{(Z_1-Z_2)\cdots(Z_1-Z_n)}$$

Similarly we calculate the residues at the other poles, which yield the Ehrhart polynomial of the Birkhoff polytope.■

In the next example we will illustrate the computation of H_1.

Example (3.3.1):

From theorem (3.3.2) we get,

$$H_1(t)=\frac{1}{2\pi i}\int_{|z_1|=\xi_1}Z_1^{-t-1}Z_1^t dZ_1=\frac{1}{2\pi i}\int_{|z_1|=\xi_1}Z_1^{-1}dZ_1$$

$$=\frac{1}{2\pi i}\cdot 2\pi i=1.$$

The next example will explain the computation of H_3 and hence the volume of B_3.

Example (3.3.2), [7]:

By using theorem (3.3.2) we get

$$H_3(t) = \frac{1}{(2\pi i)^3}\int (Z_1Z_2Z_3)^{-t-1}\left(\frac{Z_1^{t+2}}{(Z_1-Z_2)(Z_1-Z_3)} + \frac{Z_2^{t+2}}{(Z_2-Z_1)(Z_2-Z_3)} + \frac{Z_3^{t+2}}{(Z_3-Z_1)(Z_3-Z_2)}\right)^3 dZ$$

We have to order the radii of the integration paths, for each variable; we choose $0<\xi_3<\xi_2<\xi_1<1$ we use this fact after multiplying out the cubic: integrating, for example, the term $\frac{Z_1^{-t-1}Z_2^{-t-1}Z_3^{2t+5}}{(Z_3-Z_2)^3(Z_3-Z_1)^3}$ with respect to Z_3 gives 0, as this function is analytic at the Z_3- origin and $|Z_1|,|Z_2|>\xi_3$. After using this observation for all the terms stemming from the cubic, the only integrals surviving are

$$\frac{1}{(2\pi i)^3}\int \frac{Z_1^{2t+5}Z_2^{-t-1}Z_3^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^3}dZ$$

and

$$\frac{-3}{(2\pi i)^3}\int \frac{Z_1^{t+3}Z_2Z_3^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^2(Z_2-Z_3)}dZ$$

The first integral factors and yields, again by residue calculus

$$\frac{1}{(2\pi i)^3}\int \frac{Z_1^{2t+5}Z_2^{-t-1}Z_3^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^3}dZ$$

$$= \frac{1}{(2\pi i)^3}\int Z_1^{2t+5}\left(\frac{Z^{-t-1}}{(Z_1-Z)^3}dZ\right)^2 dZ_1$$

$$= \frac{1}{2\pi i}\int Z_1^{2t+5}\left(\frac{-1}{2}(-t-1)(-t-2)Z_1^{-t-3}\right)^2 dZ_1$$

$$= \binom{t+2}{2}^2.$$

for the second integral, it is most efficient to integrate with respect to Z_2 first.

$$\frac{-3}{(2\pi i)^3}\int \frac{Z_1^{t+3}Z_2Z_3^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^2(Z_2-Z_3)}dZ = -\frac{3}{(2\pi i)^2}\int \frac{Z_1^{t+3}Z_3^{-t}}{(Z_1-Z_3)^5}dZ_3 dZ_1$$

$$= -\frac{3}{2\pi i}\int Z_1^{t+3}.\frac{1}{4!}(-t)(-t-1)(-t-2)(-t-3)Z_1^{-t-4}dZ_1 = -3\binom{t+3}{4}$$

Adding up the last two lines finally gives

$$H_3(t) = \binom{t+2}{2}^2 - 3\binom{t+3}{4} = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

In general, the relative volume of the fundamental domain of the sub lattice of Z^{n^2} in the affine space spanned by B_n is n^{n-1}, [7].

Therefore, to obtain the volume of B_3, the leading term of H_3 has to be multiplied by the relative volume of the fundamental domain of the sub lattice of Z^9 in the affine space spanned by B_3 which is equal to 9; hence $\mathrm{vol(B_3)} = \frac{9}{8}$

Now for n = 4, the number of integrals that has to evaluate to compute H_4 is only slightly higher. By theorem (3.3.2)

$$H_4(t) = \frac{1}{(2\pi i)^4}\int\limits_{|Z_1|=\xi_1}\int\limits_{|Z_2|=\xi_2}\int\limits_{|Z_3|=\xi_3}\int\limits_{|Z_4|=\xi_4}(Z_1Z_2Z_3Z_4)^{-t-1}.\left(\sum_{k=1}^{4}\frac{Z_k^{t+3}}{\prod_{j\neq k}(Z_k-Z_j)}\right)^4 dZ.$$

Again we have a choice of ordering the radii; $0<\xi_4<\xi_3<\xi_2<\xi_1<1$ are used. After multiplying out the quadric, five integrals have to be calculated; their evaluation (again straight forward by means of the residue theorem) is as follows:

$$\frac{1}{(2\pi i)^4}\int \frac{Z_1^{3t+11}Z_2^{-t-1}Z_3^{-t-1}Z_4^{-t-1}}{(Z_1-Z_2)^4(Z_1-Z_3)^4(Z_1-Z_4)^4}dZ$$

$$= \frac{1}{(2\pi i)^4}\int Z_1^{3t+11}.\left(\int \frac{Z^{-t-1}}{(Z_1-Z)^4}dZ\right)^3 dZ_1 = \binom{t+3}{3}^3,$$

$$\frac{-4}{(2\pi i)^4}\int \frac{Z_1^{2t+8}Z_2^2Z_3^{-t-1}Z_4^{-t-1}}{(Z_1-Z_2)^4(Z_1-Z_3)^3(Z_1-Z_4)^3(Z_2-Z_3)(Z_2-Z_4)}dZ$$

$$=\frac{-4}{(2\pi i)^4}\int \frac{Z_1^{2t+8}Z_2^2}{(Z_1-Z_2)^4}\cdot\left(\int \frac{Z^{-t-1}}{(Z_1-Z)^3(Z_2-Z)}dZ\right)^2 dZ_1dZ_2$$

$$=\frac{4}{(2\pi i)^2}\int \frac{Z_1^{2t+8}Z_2^{-t+1}}{(Z_1-Z_2)^7}\cdot\left(2\binom{t+2}{2}\frac{Z_1^{-t-3}}{(Z_1-Z_2)}+2(t+1)\frac{Z_1^{-t-2}}{(Z_1-Z_2)^2}+\right.$$

$$\left.2\frac{Z_1^{-t-1}}{(Z_1-Z_3)^3}-\frac{Z_2^{-t-1}}{(Z_1-Z_2)^3}\right)dZ$$

$$=8\binom{t+2}{2}\binom{t+5}{7}+8(t+1)\binom{t+6}{8}+8\binom{t+7}{9}-4\binom{2t+8}{9},$$

$$\frac{4}{(2\pi i)^4}\int \frac{Z_1^{2t+8}Z_2^{-t-1}Z_3^2Z_4^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^4(Z_1-Z_4)^3(Z_2-Z_3)(Z_3-Z_4)}dZ$$

$$=\frac{4}{(2\pi i)^3}\int \frac{Z_1^{2t+8}Z_2^{-t-1}Z_4^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_4)^7(Z_2-Z_4)}dZ$$

$$=\frac{4}{(2\pi i)^2}\int \frac{Z_1^{2t+8}Z_4^{-t-1}}{(Z_1-Z_4)^7}\cdot\left(\binom{t+2}{2}\frac{Z_1^{-t-3}}{(Z_1-Z_4)}+(t+1)\frac{Z_1^{-t-2}}{(Z_1-Z_4)^2}+\frac{Z_1^{-t-1}}{(Z_1-Z_4)^2}\right)dZ$$

$$=4\left(\binom{t+2}{2}\binom{t+5}{7}+(t+1)\binom{t+6}{8}+\binom{t+7}{9}+\binom{t+7}{9}\right),$$

$$\frac{6}{(2\pi i)^4}\int \frac{Z_1^{t+5}Z_2^{t+5}Z_3^{-t-1}Z_4^{-t-1}}{(Z_1-Z_2)^4(Z_1-Z_3)^2(Z_1-Z_4)^2(Z_2-Z_3)^2(Z_2-Z_4)^2}dZ$$

$$=\frac{6}{(2\pi i)^4}\int \frac{Z_1^{t+5}Z_2^{t+5}}{(Z_1-Z_2)^4}\cdot\left(\int \frac{Z^{-t-1}}{(Z_1-Z)(Z_2-Z)^2}dZ\right)^2 dZ_1dZ_2$$

$$=\frac{6}{(2\pi i)^2}\int\frac{Z_1^{t+5}Z_2^{t+5}}{(Z_1-Z_2)^4}\cdot((t+1)^2\frac{Z_2^{-2t-4}}{(Z_1-Z_2)^4}-4(t+1)\frac{Z_2^{-2t-3}}{(Z_1-Z_2)^5}+4\frac{Z_2^{-2t-2}}{(Z_1-Z_2)^6})dZ$$

$$=6(t+1)^2\binom{t+5}{7}-24(t+1)\binom{t+5}{8}+24\binom{t+5}{9},$$

$$=-\frac{12}{(2\pi i)^4}\int\frac{Z_1^{t+5}Z_2^2Z_3^2Z_4^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_3)^3(Z_1-Z_4)^2(Z_2-Z_3)^2(Z_2-Z_4)(Z_3-Z_4)}dZ$$

$$=-\frac{12}{(2\pi i)^3}\int\frac{Z_1^{t+5}Z_2^2Z_4^{-t-1}}{(Z_1-Z_2)^3(Z_1-Z_4)^5(Z_2-Z_4)^3}dZ$$

$$=-\frac{12}{(2\pi i)^2}\int\frac{Z_1^{t+5}Z_4^{-t+1}}{(Z_1-Z_4)^5}\cdot(\frac{1}{(Z_1-Z_4)^3}+6\frac{Z_4}{(Z_1-Z_2)^4}+6\frac{Z_4^2}{(Z_1-Z_4)^5})dZ$$

$$=-12\binom{t+5}{7}-72\binom{t+5}{8}-72\binom{t+5}{9}$$

After simple computations one can get

$$H_4(t)=\frac{11}{11340}t^9+\frac{11}{630}t^8+\frac{19}{135}t^7+\frac{2}{3}t^6+\frac{1109}{540}t^5+\frac{43}{10}t^4+\frac{35117}{5670}t^3+\frac{379}{63}t^2+\frac{65}{18}t+1$$

and hence

$$vol(B_4)=4^3\cdot\frac{11}{11340}=\frac{176}{2835}.$$

3.4 The effect of matrix operations on the number of integral points

Consider the polytope P that is defined as $\{X\in\Re^d : AX\le b\}$, where $A\in\Re^{d\times d}$ and $b\in\Re^d$. In this section we study the change of the number of integral points of P upon performing the usual matrix operations on the matrix A, these operations include interchanging two rows (columns), the addition of a row (column) to another row (column), and the transpose of the matrix A.

The study is when n = 2, 3 and n = 4, the computation shows that there is no change in the number of integral points in the case of transpose and in the interchanging of rows or column under certain conditions.

Now we discuss the method for the polytope $\{X \in \Re^d : AX \le b\}$, where $A \in \Re^{d\times d}$ and $b \in \Re^d$. In two cases which are:

(I) The case of 2×2 ***matrix*:**

Suppose that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $b = (1 \quad 1)^t$. In order to find L(P, t) we use

$$L(P,t) = \frac{1}{(2\pi i)^m} \int\limits_{|Z_1|=\xi_1} \cdots \int\limits_{|Z_m|=\xi_m} \frac{Z_1^{-tb_1-1} \ldots Z_m^{-tb_m-1}}{(1-Z^{c_1})\ldots(1-Z^{c_d})} dZ$$ where $c_1, \ldots, c_d$ are the column vectors of A, here $0 < \xi_m, \ldots, \xi_1 < 1$ are distinct real numbers and t a positive integer.

We use the standard multivariate notation $V^W = v_1^{w_1} \cdots v_n^{w_n}$.

$$L(P,t) = \frac{1}{(2\pi i)^2} \int\limits_{|Z_1|=\xi_1} \int\limits_{|Z_2|=\xi_2} \frac{Z_1^{-tb_1-1} Z_2^{-tb_2-1}}{(1-Z^{c_1})(1-Z^{c_2})} dZ$$, here $0 < \xi_2 < \xi_1 < 1$.

Let $|Z_1| = \xi_1 < 1, |Z_2| = \xi_2 < 1$ and $c_1 = (a_{11} \quad a_{21}), c_2 = (a_{12} \quad a_{22})$.

By assuming t = 1, $b_1 = b_2 = 1$ we get

$$L(P,1) = \frac{1}{(2\pi i)^2} \int\limits_{|Z_1|=\xi_1} \int\limits_{|Z_2|=\xi_2} \frac{Z_1^{-2} Z_2^{-2}}{(1 - Z_1^{a_{11}} Z_2^{a_{21}})(1 - Z_1^{a_{12}} Z_2^{a_{22}})} dZ_2 dZ_1$$

Now let,

$$f(Z) = \frac{1}{Z_1^2 Z_2^2 (1 - Z_1^{a_{11}} Z_2^{a_{21}})(1 - Z_1^{a_{12}} Z_2^{a_{22}})}$$

the poles of f(z) at $Z = Z_2$ are

1. A pole at zero:

   Since $0 \in \operatorname{int}|Z_2| = \xi_2$, therefore it is a pole of order two.

2. $1 - Z_1^{a_{11}} Z_2^{a_{21}} = 0$ then $Z_2^{a_{21}} = \frac{1}{Z_1^{a_{11}}}, Z_2 = (\frac{1}{Z_1^{a_{11}}})^{\frac{1}{a_{21}}}$, this is a pole if $|Z_2| < \xi_2$ therefore,

$$|Z_2| = \frac{1}{|Z_1|^{\frac{a_{11}}{a_{21}}}} = \frac{1}{\xi_1^{\frac{a_{11}}{a_{21}}}} \;,\; \frac{1}{\xi_1} > \xi_2 \text{ (because } 0 < \xi_2 < \xi_1 < 1)$$

then $Z_2 = (\frac{1}{Z_1^{a_{11}}})^{\frac{1}{a_{21}}} \notin \text{int}(|Z_2| = \xi_2)$.

3. $1 - Z_1^{a_{12}} Z_2^{a_{22}} = 0$ then $Z_2^{a_{22}} = \frac{1}{Z_1^{a_{12}}}$, $Z_2 = (\frac{1}{Z_1^{a_{12}}})^{\frac{1}{a_{22}}}$

therefore, $|Z_2| = \frac{1}{\xi_1^{\frac{a_{12}}{a_{22}}}}$, but $\frac{1}{\xi_1} > \xi_2$ (because $0 < \xi_2 < \xi_1 < 1$) then

$Z_2 = (\frac{1}{Z_1^{a_{12}}})^{\frac{1}{a_{22}}} \notin \text{int}(|Z_2| = \xi_2)$.

Therefore we have pole of order two at zero only, and its residue has to be computed as follows,

for $f(Z) = \frac{1}{Z_1^2 Z_2^2 (1 - Z_1^{a_{11}} Z_2^{a_{21}})(1 - Z_1^{a_{12}} Z_2^{a_{22}})}$, we determine the residues on

the circle $|Z_2| = \xi_2$ with radius ξ_2 and center zero, if we consider

$f(Z) = \frac{\Phi(Z)}{Z_2^2}$, where

$$\Phi(Z) = \frac{1}{Z_1^2 (1 - Z_1^{a_{11}} Z_2^{a_{21}})(1 - Z_1^{a_{12}} Z_2^{a_{22}})}$$

Z_1 can be regarded as a constant, then

$$\Phi'(Z) = \frac{1}{Z_1^2} \Big[-(1 - \alpha Z_2^{a_{21}})^{-2} (-\alpha a_{21} Z_2^{a_{21}-1}).(1 - \beta Z_2^{a_{22}})^{-1}$$
$$+ (1 - \alpha Z_2^{a_{21}})^{-1} (-\beta a_{22} Z_2^{a_{22}-1})(1 - \beta Z_2^{a_{22}})^{-2} \Big]$$

$$\Phi'(Z) = \frac{1}{Z_1^2} \Big[\alpha a_{21} Z_2^{a_{21}-1} (1 - \alpha Z_2^{a_{21}})^{-2} \cdot (1 - \beta Z_2^{a_{22}})^{-1}$$
$$+ \beta a_{22} Z_2^{a_{22}-1} (1 - \beta Z_2^{a_{22}})^{-2} \cdot (1 - \alpha Z_2^{a_{21}})^{-1} \Big]$$

where $\alpha = Z_1^{a_{11}}$ and $\beta = Z_1^{a_{12}}$ therefore

1. If $a_{21} > 1$ and $a_{22} > 1$ then $\Phi'(0) = 0$ therefore L(P,1) = 0 .

   Now if $a_{21} = 1$ and $a_{22} > 1$ then

$$\Phi'(Z_2)=\frac{1}{Z_1^2}\left[\alpha(1-\alpha Z_2)^{-2}\cdot(1-\beta Z_2)^{-1}+\beta a_{22}Z_2^{a_{22}-1}(1-\beta Z_2^{a_{22}})^{-2}\cdot(1-\alpha Z_2)^{-1}\right]$$

$$\Phi'(0)=Z_1^{a_{11}-2}$$

then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1} Z_1^{a_{11}-2}dZ_1$$

2. If $a_{11}>2$ or $a_{11}<1$, then L(P,1) = 0.

3. If $a_{11}=1$ then $L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}\frac{1}{Z_1}dZ_1=1$

If $a_{21}>1$ and $a_{22}=1$ then

$$\Phi'(Z_2)=\frac{1}{Z_1^2}\left[\beta(1-\beta Z_2)^{-2}\cdot(1-\alpha Z_2^{a_{21}})^{-1}\right]$$

$$\Phi'(0)=\frac{1}{Z_1^2}[\beta]=Z_1^{a_{12}-2}$$

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1} Z_1^{a_{12}-2}dZ_1$$

4. If $a_{12}>2$ or $a_{12}<1$ then L(P,1) = 0

5. If $a_{12}=1$ then $L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}\frac{1}{Z_1}dZ_1=1$

If $a_{21}=a_{22}=1$, then $\Phi'(Z_1)=\frac{1}{Z_1^2}[\alpha+\beta]$

$\Phi'(Z_1)=Z_1^{a_{11}-2}+Z_1^{a_{12}-2}$ then $L(P,t)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_2}(Z_1^{a_{11}-2}+Z_1^{a_{12}-2})dZ_1$

6. If $a_{11}>2$ or $a_{11}<1$, and $a_{12}>2$ or $a_{12}=1$, then L(P,1) = 0

7. If $a_{11}=1$ and $a_{12}=1$ then L(P,1) = 2

This means we have three cases for n = 2, which are

Case 1: L(P,1) = 0

If

(1) $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{21} > 1 \ and \ a_{22} > 1$

(2) $A=\begin{pmatrix} a_{11} & a_{12} \\ 1 & a_{22} \end{pmatrix}, a_{22} > 1 \ and \ (a_{11} > 2 or a_{11} < 1)$

(3) $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1 \end{pmatrix}, a_{21} > 1 and \ (a_{12} > 2 \ or \ a_{12} < 1)$

(4) $A=\begin{pmatrix} a_{11} & a_{12} \\ 1 & 1 \end{pmatrix}, a_{12}, a_{11} > 2 \ or \ (a_{11}, a_{12} < 1)$

Case 2: L(P,1) = 1

If

(1) $A=\begin{pmatrix} 1 & a_{12} \\ 1 & a_{22} \end{pmatrix}, a_{22} > 1$

(2) $A=\begin{pmatrix} a_{11} & 1 \\ a_{21} & 1 \end{pmatrix}, a_{21} > 1$

(3) $A=\begin{pmatrix} 1 & a_{12} \\ 1 & 1 \end{pmatrix}, a_{12} > 2 \ or \ a_{12} < 1$

(4) $A=\begin{pmatrix} a_{11} & 1 \\ 1 & 1 \end{pmatrix}, a_{11} > 2 \ or \ a_{11} < 1$

Case 3: L(P,1) = 2

If $A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Now we look at the change in L(P,1) when a matrix operation is performed on the matrix A.

That is,

1. We first look at ***the operation of taking the adjoin***, the relation between the number of integral points for $P=\{X \in \Re^2 : AX = b\}$ and $P=\{X \in \Re^2 : A^T X = b\}$ are equal and,

a. L(P,1) = 0 if there exists a row R_n of the matrix A with index n such that $a_{nj} > 1$, where a_{nj} are the elements of the row R_n, and there exists a column C_n of the matrix A with index n such that $a_{in} > 1$, where a_{in} are the elements of the column $C_n, i, j = 1,...,n$.

b. L(P,1) = 1 if there exists a column C_j of the matrix A with element a_{ij} such that $a_{ij} = 1$ and $a_{1k} > 2$, $k \neq j$ and there exists a row R_j of the matrix A with index j such that $a_{ij} = 1, i, j = 1,...,n$.

c. L(P,1) = 2 if there exists two columns C_{j_1}, C_{j_2} of the matrix A such that $a_{ij_1} = 1$ and $a_{ij_2} = 1$, $a_{1k} \geq 2, k \neq j_1, k \neq j_2$ and there exists two rows R_{i_1}, R_{i_2} of the matrix A such that $a_{i_1 j} = 1$ *and* $a_{i_2 j} = 1$ and for $C_{j_1 j_2}$ and

$$R_{i_1 i_2}, a_{mk} > 2, k \neq j_1, j_2 : m \neq i_1, i_2 .$$

2. *Interchanging of rows or columns*:

The number of integral points for the original polytope and the polytope obtained by interchanging of rows or columns of the matrix A are equal and,

a. L(P,1) = 0 if there exists a row R_j of the matrix A with index j such that $a_{ji} > 2$, where a_{ji} are the elements of the row R_j, and column C_i of the matrix A, $i, j = 1,...,n$.

b. L(P,1) = 1 if there exists a column C_j of the matrix A with element a_{ij} such that $a_{ij} = 1$ and $a_{ik} > 2$,

$$\forall k \neq j, \quad i, j = 1,...,n .$$

c. L(P,1) = 2 if there exists two columns of the matrix A such that its elements are equal to one.

3. *Sum of two rows or columns*:

The relation between the numbers of integral points for the original polytope and the polytope obtained by adding two rows or columns of the matrix A are given as follows:

a. L(P,1) = 0, the conditions are the same as L(P, 1) = 0 for interchanging of rows or columns.

b. L(P, 1) = 1, in this case the number of integral points of the original polytopes and the changing polytopes may not be equal under any conditions. For example: $A=\begin{pmatrix}1 & 2\\ 1 & 5\end{pmatrix}$, where L(P, 1) =1, but when we add two columns the result is $B=\begin{pmatrix}3 & 2\\ 6 & 5\end{pmatrix}$ and L(P, 1) = 0.

c. L(P, 1) = 2, also in this case, the numbers of integral points of the original polytopes and the changing polytopes may not be equal.

***c. Multiplying by a constant c > 0*:**

The relation between the numbers of integral points for the original polytope and the polytope obtained by multiplying a row or column by a constant c > 0 are are given as follows:

a. L(P,1)=0, the conditions are the same as L(P,1)=0 for interchanging of rows or columns.

b. L(P, 1) = 1, in this case, the number of integral points of the original polytopes and the changing polytopes may not be equal any where for example: $A=\begin{pmatrix}1 & 2\\ 1 & 5\end{pmatrix}$, where L(P, 1) = 1, but when we multiply by a constant c > 0 the result is $B=\begin{pmatrix}3 & 2\\ 3 & 5\end{pmatrix}$ and L(P, 1) = 0.

c. L(P, 1) = 2, also in this case, the number of integral points of the original polytopes and the changing polytopes may not be equal under any condition.

***(II) The case of* 3×3 *matrix*:**

Now if $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

This means that the polytope $P=\{X\in\Re^3 : AX=b\}$,$b\in Z^3$. By theorem (3.3.1) with t = 1 and $b_1=b_2=b_3=1$, we get

$$L(P,1)=\frac{1}{(2\pi i)^3}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}\int_{|Z_3|=\xi_3}\frac{Z_1^{-2}Z_2^{-2}Z_3^{-2}}{(1-Z^{c_1})(1-Z^{c_2})(1-Z^{c_3})}dZ$$

where

$c_1=(a_{11}\quad a_{21}\quad a_{31}), c_2=(a_{12}\quad a_{22}\quad a_{32}), c_3=(a_{13}\quad a_{23}\quad a_{33}),$
$0<\xi_3,\xi_2,\xi_1<1$.

Then

$$L(P,1)=\frac{1}{(2\pi i)^3}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}\int_{|Z_3|=\xi_3}$$

$$\frac{Z_1^{-2}Z_2^{-2}Z_3^{-2}dZ_3dZ_2dZ_1}{(1-Z_1^{a_{11}}Z_2^{a_{21}}Z_3^{a_{31}})\cdot(1-Z_1^{a_{12}}Z_2^{a_{22}}Z_3^{a_{32}})\cdot(1-Z_1^{a_{13}}Z_2^{a_{23}}Z_3^{a_{33}})}$$

the integrand is

$$\frac{Z_1^{-2}Z_2^{-2}Z_3^{-2}}{(1-Z_1^{a_{11}}Z_2^{a_{21}}Z_3^{a_{31}})\cdot(1-Z_1^{a_{12}}Z_2^{a_{22}}Z_3^{a_{32}})\cdot(1-Z_1^{a_{13}}Z_2^{a_{23}}Z_3^{a_{33}})}$$

now we find the poles for the above function:

1. A pole of order one at zero,

   Since $0\in \operatorname{int}|Z_3|=\xi_3$.

2. $1-Z_1^{a_{11}}Z_2^{a_{21}}Z_3^{a_{31}}=0$ then $Z_1^{a_{11}}Z_2^{a_{21}}Z_3^{a_{31}}=1$

$$Z_3=\left(\frac{1}{Z_1^{a_{11}}Z_2^{a_{21}}}\right)^{\frac{1}{a_{31}}}$$

$$|Z_3|=\frac{1}{\left|Z_1^{a_{11}}Z_2^{a_{21}}\right|^{\frac{1}{a_{31}}}}=\frac{1}{|\xi_1|^{\left(\frac{a_{11}}{a_{31}}\right)}|\xi_1|^{\left(\frac{a_{21}}{a_{31}}\right)}}$$

$$\frac{1}{|\xi_1||\xi_1|}>1$$

$$Z_3\notin \operatorname{int}(|Z_3|=\xi_3)\,.$$

Similarly for the other points.

By assuming

$$\Phi(Z)=\frac{1}{(1-\alpha Z_3^{a_{31}})(1-\beta Z_3^{a_{32}})(1-\gamma Z_3^{a_{33}})}$$

where

$\alpha=Z_1^{a_{11}}Z_2^{a_{21}}, \beta=Z_1^{a_{12}}Z_2^{a_{22}}$ and $\gamma=Z_1^{a_{13}}Z_2^{a_{23}}$

by generalized Cauchy integral formula at Z = 0, we get

$$\begin{aligned}\Phi'(Z)=&-(1-\alpha Z_3^{a_{31}})^{-2}(-\alpha a_{31}Z_3^{a_{31}-1})(1-\beta Z_3^{a_{32}})(1-\gamma Z_3^{a_{33}})^{-1}\\&+(-1)(-\beta a_{32}Z_3^{a_{32}-1})(1-\beta Z_3^{a_{32}})^{-2}(1-\alpha Z_3^{a_{31}})^{-1}(1-\gamma Z_3^{a_{33}})^{-1}\\&+\gamma a_{33}Z_3^{a_{33}-1}(1-\gamma Z_3^{a_{33}})^{-2}(1-\alpha Z_3^{a_{31}})^{-1}(1-\beta Z_3^{a_{32}})^{-1}\end{aligned}$$

regarding Z_1 and Z_2 as constants.

Now

If $a_{31} > 1, a_{32} > 1 \ and \ a_{33} > 1$ then $\Phi'(0) = 0$, therefore

L(P, 1) = 0 with $a_{31} > 1, a_{32} > 1 \ and \ a_{33} > 1$.

I) If $a_{31} = 1 \ and \ a_{32}, a_{33} > 1$

$$\Phi'(Z) = \alpha \ a_{31}(1-\alpha Z_3)^{-2}(1-\beta Z_3^{a_{32}})^{-1}(1-\gamma Z_3^{a_{33}})^{-1} + \beta a_{32} Z_3^{a_{32}-1}(1-\beta Z_3^{a_{32}})^{-2}$$
$$(1-\alpha Z_3^{a_{31}})^{-1}(1-\gamma Z_3^{a_{33}})^{-1} + \gamma a_{33} Z_3^{a_{33}-1}(1-\gamma Z_3^{a_{33}})^{-2} \ (1-\alpha Z_3)^{-1}(1-\beta Z_3^{a_{32}})^{-1}$$

$$\Phi'(0) = \alpha a_{31} = Z_1^{a_{11}} Z_2^{a_{21}}$$

$$L(P,1) = \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} \frac{Z_1^{a_{11}} Z_2^{a_{21}}}{Z_1^2 Z_2^2} dZ_2 dZ_1$$

$$= \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} Z_1^{a_{11}-2} Z_2^{a_{21}-2} dZ_2 dZ_1$$

i) If $a_{21} > 2 \ or \, a_{21} < 1$, then L(P,1) = 0.

ii) If $a_{21} = 1$ then $\Phi'(0) = 1$. And

$$L(P,1) = \frac{1}{(2\pi i)} \int_{|Z_1|=\xi_1} Z_1^{a_{11}-2} dZ_1 \text{ then}$$

a) If $a_{11} > 2 \ or \, a_{11} < 1$ then L(P, 1) = 0.

b) If $a_{11} = 1$ then L(P, 1) = 1.

Π) If $a_{32} = 1 \ and \ a_{31}, a_{33} > 1$ then $\Phi'(Z) = \beta a_{32} = Z_1^{a_{12}} Z_2^{a_{22}}$ and

$$L(P,1) = \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} \frac{Z_1^{a_{12}} Z_2^{a_{22}}}{Z_1^2 Z_2^2} dZ_2 dZ_1$$

$$= \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} Z_1^{a_{12}-2} \int_{|Z_2|=\xi_2} Z_2^{a_{22}-2} dZ_2 dZ_1$$

i) If $a_{22} > 2 \ or \, a_{22} < 1$ then $\Phi'(0) = 0$, this mean that L(P, 1) = 0.

ii) If $a_{22} = 1$ then $\Phi'(0) = 1$ and $L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} Z_1^{a_{12}-2} dZ_1$

a) If $a_{12} > 2 \ or \, a_{11} < 1$ then L(P, 1) = 0.

b) If $a_{12} = 1$ then $\Phi'(0) = 1$ and hence L(P, 1) = 1.

III) If $a_{33} = 1$ *and* $a_{31}, a_{32} > 1$ then

$\Phi'(Z) = \gamma a_{33} = Z_1^{a_{13}} Z_2^{a_{23}}$ and hence

$$L(P,1) = \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} Z_1^{a_{13}-2} Z_2^{a_{23}-2} dZ_2 dZ_1$$

$$= \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} Z_1^{a_{13}-2} \int_{|Z_2|=\xi_2} Z_2^{a_{23}-2} dZ_2 dZ_1$$

i) If $a_{23} > 2$ *or* $a_{23} < 1$ then $\Phi'(0) = 0$, this means that L(P, 1) = 0.

ii) If $a_{23} = 1$ then $\Phi'(0) = 1$ and $L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} Z_1^{a_{13}-2} dZ_1$

Then,

a) If $a_{13} > 2 or a_{13} < 1$ then L(P, 1) = 0.

I1) If $a_{31} = 1, a_{32} = 1$ *and* $a_{33} > 1$

$$\begin{aligned}\Phi'(Z) &= \alpha(1-\alpha Z_3)^{-2}(1-\beta Z_3)^{-1}(1-\gamma Z_3^{a_{33}})^{-1} \\ &+ \beta(1-\beta Z_3^{a_{32}})^{-2}(1-\alpha Z_3^{a_{31}})^{-1}(1-\gamma Z_3)^{-1} \\ &+ \gamma a_{33} Z_3^{a_{33}-1}(1-\gamma Z_3^{a_{33}})^{-2}(1-\alpha Z_3)^{-1}(1-\beta Z_3)^{-1}\end{aligned}$$

then $\Phi'(0) = \alpha + \beta = Z_1^{a_{11}} Z_2^{a_{21}} + Z_1^{a_{12}} Z_2^{a_{22}}$

hence $$L(P,1) = \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} \frac{Z_1^{a_{11}} Z_2^{a_{21}} + Z_1^{a_{12}} Z_2^{a_{22}}}{Z_1^2 Z_2^2} dZ_2 dZ_1$$

$$= \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} (Z_1^{a_{11}-2} Z_2^{a_{21}-2} + Z_1^{a_{12}-2} Z_2^{a_{22}-2}) dZ_2 dZ_1$$

i) If ($a_{21} > 2$ *or* $a_{21} < 1$) and ($a_{22} > 2$ *or* $a_{22} < 1$) then L(P, 1) = 0.

ii) If ($a_{21} = 1$ *and* $a_{22} > 2$ *or* $a_{21} < 1$) then $\Phi'(Z) = Z_1^{a_{11}-2} Z_2^{-1}$,

hence $L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} Z_1^{a_{11}-2} dZ_1$

a) If $a_{11} > 2 or a_{11} < 1$ then $\Phi'(0) = 0$, this means that L(P, 1) = 0.

b) If $a_{11} = 1$ then L(P, 1) = 1.

iii) If $a_{22} = 1$ *and* ($a_{21} > 2$ *or* $a_{21} < 1$) then

$$\Phi'(Z)=Z_1^{a_{12}-2}Z_2^{-1}\text{, hence } L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1} Z_1^{a_{12}-2}dZ_1$$

a) If $a_{12}>2$ *or* $a_{12}<1$ then $\Phi'(0)=0$, this means that L(P, 1) = 0.

b) If $a_{12}=1$ then L(P, 1) = 1.

iv) If $a_{12}=a_{22}=1$, then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}(Z_1^{a_{11}-2}+Z_1^{a_{12}-2})dZ_1$$

a) If $(a_{11}>2 \text{ or } a_{11}<1)$ and $(a_{12}>2 \text{ or } a_{12}<1)$, then L(P, 1) = 0.

b) If $a_{11}=1$ and $(a_{12}>2 \text{ or } a_{12}<1)$ then L(P, 1) = 1.

c) If $a_{12}=1$ and $(a_{11}>2 \text{ or } a_{11}<1)$ then L(P, 1) = 1.

d) If $a_{11}=1$ and $a_{12}=1$ then L(P, 1) = 2.

II1) If $a_{31}=a_{33}=1$ *and* $a_{32}>1$, then

$\Phi'(0)=\alpha+\gamma=Z_1^{a_{11}}Z_2^{a_{21}}+Z_1^{a_{13}}Z_2^{a_{23}}$, then

$$L(P,1)=\frac{1}{(2\pi i)^2}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}\frac{Z_1^{a_{11}}Z_2^{a_{21}}+Z_1^{a_{13}}Z_2^{a_{23}}}{Z_1^2Z_2^2}dZ_2dZ_1$$

$$=\frac{1}{(2\pi i)^2}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}(Z_1^{a_{11}-2}Z_2^{a_{21}-2}+Z_1^{a_{13}-2}Z_2^{a_{23}-2})dZ_2dZ_1$$

i) If $(a_{21}>2 \text{ or } a_{21}<1)$ and $(a_{23}>2 \text{ or } a_{23}<1)$, then L(P, 1) = 0.

ii) If $a_{21}=1$ and $(a_{23}>2 \text{ or } a_{23}<1)$, then

$$\Phi'(Z)=Z_1^{a_{11}-2}Z_2^{-1}\text{, hence } L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1} Z_1^{a_{11}-2}dZ_1$$

a) If $(a_{11}>2 \text{ or } a_{11}<1)$ then $\Phi'(0)=0$ and L(P, 1) = 0.

b) If $a_{11}=1$ then L(P, 1) = 1.

iii) If $a_{23}=1$ and $(a_{21}>2 \text{ or } a_{21}<1)$, then

$$\Phi'(Z)=Z_1^{a_{13}-2}Z_2^{-1}\text{, hence } L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1} Z_1^{a_{13}-2}dZ_1$$

a) If $(a_{13}>2 \text{ or } a_{13}<1)$ then L(P, 1) = 0.

b) If $a_{13}=1$ then L(P, 1) = 1.

iv) If $a_{21} = 1$ *and* $a_{23} = 1$ then

$$L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} (Z_1^{a_{11}-2} + Z_1^{a_{13}-2}) dZ_1$$

a) If $(a_{11} > 2 \ or \ a_{11} < 1)$ and $(a_{13} > 2 \ or \ a_{13} < 1)$, then L(P, 1) = 0.

b) If $a_{11} = 1$ and $(a_{13} > 2 \ or \ a_{13} < 1)$, then L(P, 1) = 1.

c) If $a_{11} = 1$ *and* $a_{13} = 1$ then L(P, 1) = 2.

III1) If $a_{32} = a_{33} = 1$ *and* $a_{31} > 1$ then

$$\Phi'(Z) = \beta + \gamma = Z_1^{a_{12}} Z_2^{a_{22}} + Z_1^{a_{13}} Z_2^{a_{23}}$$, hence

$$L(P,1) = \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} \frac{Z_1^{a_{12}} Z_2^{a_{22}} + Z_1^{a_{13}} Z_2^{a_{23}}}{Z_1^2 Z_2^2} dZ_2 dZ_1$$

$$= \frac{1}{(2\pi i)^2} \int_{|Z_1|=\xi_1} \int_{|Z_2|=\xi_2} (Z_1^{a_{12}-2} Z_2^{a_{22}-2} + Z_1^{a_{13}-2} Z_2^{a_{23}-2}) dZ_2 dZ_1$$

i) If $(a_{22} > 2 \ or \ a_{22} < 1)$ and $(a_{23} > 2 \ or \ a_{23} < 1)$, then L(P, 1) = 0.

ii) If $a_{22} = 1$ and $(a_{23} > 2 \ or \ a_{23} < 1)$, then

$$\Phi'(Z) = Z_1^{a_{12}-2} Z_2^{-1}, \text{ hence } L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} Z_1^{a_{12}-2} dZ_1$$

a) If $a_{12} > 2 \ or \ a_{12} < 1$ then $\Phi'(0) = 0$, this means that L(P, 1) = 0.

b) If $a_{12} = 1$ then L(P, 1) = 1.

iii) If $a_{23} = 1$ and $(a_{22} > 2 \ or \ a_{22} < 1)$, then

$$\Phi'(Z) = Z_1^{a_{13}-2} Z_2^{-1}, \text{ hence } L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} Z_1^{a_{13}-2} dZ_1$$

a) If $a_{13} > 2 \ or \ a_{13} < 1$ then L(P, 1) = 0.

b) If $a_{13} = 1$ then L(P, 1) = 1.

iv) If $a_{22} = a_{23} = 1$ then

$$L(P,1) = \frac{1}{2\pi i} \int_{|Z_1|=\xi_1} (Z_1^{a_{12}-2} + Z_1^{a_{13}-2}) dZ_1$$

a) If $(a_{12} > 2 \ or \ a_{12} < 1)$ and $(a_{13} \geq 2 \ or \ a_{13} < 1)$, then L(P, 1) = 0.

b) If $a_{12} = 1$ and $(a_{13} > 2 \ or \ a_{13} < 1)$, then L(P, 1) = 1.

IV1) If $a_{31}=a_{32}=a_{33}=1$ then

$$\Phi'(Z)=\alpha+\beta+\gamma=Z_1^{a_{11}}Z_2^{a_{21}}+Z_1^{a_{12}}Z_2^{a_{22}}+Z_1^{a_{13}}Z_2^{a_{23}}$$

$$L(P,1)=\frac{1}{(2\pi i)^2}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}\frac{Z_1^{a_{11}}Z_2^{a_{21}}+Z_1^{a_{12}}Z_2^{a_{22}}+Z_1^{a_{13}}Z_2^{a_{23}}}{Z_1^2Z_2^2}dZ_2dZ_1$$

$$=\frac{1}{(2\pi i)^2}\int_{|Z_1|=\xi_1}\int_{|Z_2|=\xi_2}(Z_1^{a_{11}-2}Z_2^{a_{21}-2}+Z_1^{a_{12}-2}Z_2^{a_{22}-2}+Z_1^{a_{13}-2}Z_2^{a_{23}-2})dZ_2dZ_1$$

i) If $(a_{21}>2\ or\ a_{21}<1)$, $(a_{22}>2\ or\ a_{22}<1)$ and $(a_{23}>2\ or\ a_{23}<1)$ then L(P, 1) = 0.

ii) If $a_{21}=1$, $(a_{22}>2\ or\ a_{22}<1)$ and $(a_{23}>2\ or\ a_{23}<1)$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}Z_1^{a_{12}-2}dZ_1$$

a) If $(a_{12}>2\ or\ a_{12}<1)$, then L(P, 1) = 0.

b)If $a_{12}=1$ then L(P, 1) = 1.

iv) If $a_{23}=1$, $(a_{21}>2\ or\ a_{21}<1)$ and $(a_{22}>2\ or\ a_{22}<1)$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}Z_1^{a_{13}-2}dZ_1$$

a) If $(a_{13}>2\ or\ a_{13}<1)$ then L(P, 1) = 0.

b) If $a_{13}=1$ then L(P, 1) = 1.

v) If $a_{21}=a_{22}=1$ and $(a_{23}>2\ or\ a_{23}<1)$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}(Z_1^{a_{11}-2}+Z_1^{a_{12}-2})dZ_1$$

a) If $(a_{11}>2\ or\ a_{11}<1)$ and $(a_{12}>2\ or\ a_{12}<1)$ then L(P, 1) = 0.

b) If $a_{11}=1$ and $(a_{12}>2\ or\ a_{12}<1)$ then L(P, 1) = 1.

c) If $a_{12}=1$ and $(a_{11}>2\ or\ a_{11}<1)$ then L(P, 1) = 1.

d) If $a_{11}=a_{12}=1$ then L(P, 1) = 2.

vi) If $a_{22}=a_{23}=1$ and $(a_{21}>2\ or\ a_{21}<1)$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}(Z_1^{a_{12}-2}+Z_1^{a_{13}-2})dZ_1$$

a) If $(a_{12}>2\ or\ a_{12}<1)$ and $(a_{13}>2\ or\ a_{13}<1)$ then L(P, 1) = 0.

b) If $a_{12}=1$ and $(a_{13}>2\ or\, a_{13}<1)$ then L(P, 1) = 1.

c) If $a_{13}=1$ and $(a_{12}>2\ or\, a_{12}<1)$ then L(P, 1) = 1.

d) If $a_{12}=a_{13}=1$ then L(P, 1) = 2.

vii) If $a_{21}=a_{23}=1$ and $(a_{22}>2\ or\, a_{22}<1)$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}(Z_1^{a_{11}-2}+Z_1^{a_{13}-2})dZ_1$$

a) If $(a_{11}>2\ or\ a_{11}<1)$ and $(a_{13}>2\ or\, a_{13}<1)$ then L(P, 1) = 0.

b) If $a_{11}=1$ and $(a_{13}>2\ or\, a_{13}<1)$ then L(P, 1) = 1.

c) If $a_{13}=1$ and $(a_{11}>2\ or\, a_{11}<1)$ then L(P, 1) = 1.

d) If $a_{11}=a_{13}=1$ then L(P, 1) = 2.

viii) If $a_{21}=a_{22}=a_{23}=1$ then

$$L(P,1)=\frac{1}{2\pi i}\int_{|Z_1|=\xi_1}(Z_1^{a_{11}-2}+Z_1^{a_{12}-2}+Z_1^{a_{13}-2})dZ_1$$

a) If $(\mathrm{a}_{11}>2\ \text{or}\ \mathrm{a}_{11}<1),(\mathrm{a}_{12}>2\ \text{or}\ \mathrm{a}_{12}<1)$ and $(a_{13}\geq 2\ or\ a_{13}<1)$ then

L(P, 1) = 0.

b) If $a_{11}=1$ and $(a_{12}>2\ or\, a_{12}<1)$ then L(P, 1) = 1.

c) If $a_{12}=1,(\mathrm{a}_{13}>2\ \text{or}\ \mathrm{a}_{13}<1)$ and $(\mathrm{a}_{11}>2\ \text{or}\ \mathrm{a}_{11}<1)$ then L(P,1)=1

d) If $a_{13}=1,(a_{11}>2\ or\, a_{11}<1)$ and $(a_{12}>2\ or\, a_{12}<1)$ then L(P, 1)=1

e) If $a_{11}=a_{12}=1$ and $(a_{13}>2\ or\, a_{13}<1)$ then L(P, 1) = 2.

f) If $a_{11}=a_{13}=1$ and $(a_{12}>2\ or\, a_{12}<1)$ then L(P, 1) = 2.

g) If $a_{12}=a_{13}=1$ and $(a_{11}>2 or a_{11}<1)$ then L(P, 1) = 2.

h) If $a_{11}=a_{12}=a_{13}=1$ then L(P, 1) = 3.

The obtained results in the case of n=3 are related to the number of integral points for the original polytope and the changing polytope are the same as the case when n=2.

The general case is:

If $A = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{pmatrix}$ where $a_{ji},\ i, j = 1,...,n$

are positive integers and t=1, $b = (1...1)$. For the polytope

$P = \{X \in \Re^d : AX \le b\}$.

Our obtained results are, The number of integral points is:

1. ***L(P,1)=0***

   This will hold if R_n or any row of the matrix A has an element $a_{in} > 2,\ i = 1,...,n$, where R_n is the n-*th* row of the matrix A, and the number of integral points does not change when:

   a) Interchanging any two columns (rows).

   b) Summing two columns (rows).

   c) the matrix A is transposed, the condition on A is $a_{in} \ge 1$ and $a_{nj} \ge 1,\ i, j = 1,...,n$.

2. ***L(P,1)=1***

   When R_n or any row of the matrix A has an element $a_{in} > 2,\ i = 1,...,n$, where R_n is the n-*th* row of the matrix A, and the number of integral points does not change when:

   a) Interchanging any two columns (rows).

But the number of integral points is change if:

b) Multiplying by a constant c>0.

c) Summing two columns (rows).

d) The matrix A is transposed, the conditions on A that are needed to get the same number of integral points are:

i) The elements of the first row are one.

ii) There exist two rows and columns and where their elements are one.