ABSTRACT

Let H(U) be the set of all holomorphic functions on the unit ball U of the complex plane. The Hardy space H^2 is the set of all functions $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ that belongs to H(U) such that $\sum_{n=0}^{\infty}|\hat{f}(n)|^2 < \infty$. Let φ be a holomorphic self map of U. The composition operator C_φ on H^2 is defined as follows:

$C_\varphi f = f \circ \varphi$, for all $f \in H^2$

Littlewood's principle shows that C_φ is a bounded operator on H^2. Recall that, an operator T on a Hilbert space H is said to be cyclic operator if there exists a vector x in H, such that span $\{T^n x : n = 0, 1, \ldots\}$ is dense in H, the operator T is supercyclic if there is a vector x in H, such that the set $\{\lambda_n T^n x : \lambda_n \in \mathbb{C}, n = 0, 1, \ldots\}$ is dense in H. It may happen that $orb(T, x) = \{T^n x : n = 0, 1, \ldots\}$ is dense in H, in this case T is called a hypercyclic operator.

One of our main concerns in this thesis was to give conditions that are necessary and (or) sufficient for the composition operator to be a cyclic (hypercyclic, supercyclic) operator.We give some known results with details of the proofs, specially when φ is a linear fractional transformation, i.e.

$$\varphi(z) = \frac{az+b}{cz+d}, \ z \in U$$

Where a, b, c and d are complex numbers.

This thesis contains some new results (to the best of our knowledge) for the cyclicity of the operator C_φ^*, where C_φ^* is the adjoint of the composition operator C_φ. .

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Laith Khaleel Shaakir

August, 2005.

SUPERVISOR CERTIFICATION

I certify that the preparation of this thesis entitled "***Cyclic Phenomena for Composition Operators***" was accomplished by ***(Laith Khaleel Shaakir)*** under my supervision at Al-Nahrain University in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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CHAPTER ONE
PRELIMINARIES

INTRODUCTION

In this chapter, we introduce the Hardy space H^2 of all holomorphic functions on the unit ball U of the complex plane with square summable power series coefficients, discuss its norm and some properties. Also, we study the holomorphic functions that take U into itself, state famous theorems and recall some definitions. To each holomorphic function φ that takes U into itself we associate the composition operator C_φ defined by:

$$C_\varphi f = f \circ \varphi \quad (f \in H^2)$$

The Littlewoods a subordination theorem (1.3.1) tells us that the operator C_φ takes the Hardy space H^2 into itself. Littlewood's principle also supplies an estimate which shows that C_φ is a bounded operator on H^2, see [17, 3] for more details.

This chapter consists of five sections. In section one; we recall the definition of Hardy space and some basic theorems. In section two, we give the concept of the radial limit, non-tangential limit, angular derivative; we state Schwarz lemma and prove several results from it. In section three, we state the Littelwood subordination theorem and the definition of composition operators.

In section four, we study the linear fractional transformation. The holomorphic mapping φ is a linear fractional transformation if:

$$\varphi(z) = \frac{az+b}{cz+d}, \ (z \in \hat{C})$$

Where a, b, c and d are complex numbers, $\hat{C}$ is the Riemann sphere, i.e., ($\hat{C}$ = $\mathbb{C} \cup \{\infty\}$).we classify the set of all non-constant linear fractional transformations into parabolic, elliptic, hyperbolic and loxodromic. We state and try to prove some useful results.

In section five, we discuss the compact operators and give some necessary and sufficient conditions for a composition operator to be a compact operator.

1.1 HARDY SPACE H^2

In this section, we define the Hardy space H^2 and prove some basic results. We refer the reader to Duren's book [8] and J. H.shapiro [17] for more details about Hardy space. Let U be the unit ball in the complex plane $\mathbb{C}$, i.e., $U = \{z \in \mathbb{C} : |z| < 1\}$ and let H(U) be the set of all complex valued functions which are holomorphic (i.e., analytic) on U. Since pointwise sums and products of holomorphic functions are again holomorphic, then H(U) is a vector space over the field of the complex numbers. Before we give the definition of the Hardy space H^2, we recall Taylor theorem without proof.

Theorem (1.1.1) (Taylor) [4]:

Let f be analytic at all points within a circle C with center at z_0 and radius r_0. Then at each point z inside C, the power series $\sum_{n=0}^{\infty} \hat{f}(n)(z - z_0)^n$, converges uniformly to f(z),

i.e., $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)(z - z_0)^n$ for all z inside C, where $\hat{f}(n) = \frac{f^{(n)}(z_0)}{n!}$ is said to be the n-th Taylor coefficient of the function f.

Remark (1.1.2):

If the function f belongs to H(U), then by Taylor theorem:

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

Now, we give the definition of Hardy space H^2.

Definition (1.1.3) [17]:

The Hardy space H^2 is the set of all functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H(U)$, such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$, i.e., $H^2 = \{f \in H(U) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$.

We can define an inner product on H^2 as follows:

If $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$, are any functions in H^2, then the inner product of f and g is:

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

One can prove that this series converges [3].

Remark (1.1.4):

If f is any function in H^2, then we define the norm of the function f as follows:

$$\|f\|^2 = \langle f, f\rangle = \sum_{n=0}^{\infty} |\hat{f}(n)|^2$$

i.e., H^2 is a normed space.

Theorem (1.1.5) [15]:

The Hardy space H^2 is Hilbert space.

We give the following lemma without proof since it is well-known.

Lemma (1.1.6) (Cauchy-Schwarz Inequality) [15]:

Let K be an inner product space, then

$|\langle x, y\rangle| \le \|x\| \, \|y\|$, for all x and y in K.

The following theorem appeared in [17], we give the proof for the sake of completeness.

Theorem (1.1.7) (Growth Estimate):

For each $f \in H^2$

$$|f(z)| \le \frac{\|f\|}{\sqrt{1-|z|^2}}, \text{ for each } z \in U$$

Proof:

Upon applying the Cauchy-Schwarz inequality to the power series of f, we obtain for each $z \in U$

$$|f(z)| \le \sum_{n=0}^{\infty} |\hat{f}(n)|\,|z^n|$$

$$\le \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{1/2}$$

$$= \frac{\|f\|}{\sqrt{1-|z|^2}}. \quad \blacksquare$$

The following corollary appeared in [17], we give the proof for the sake of completeness.

Corollary (1.1.8):

Every norm convergent sequence in H^2 converges (to the same limit) uniformly on compact subsets of U.

Proof:

Suppose $\{f_n\}$ is a sequence in H^2 norm-convergent to a function $f \in H^2$, that is, $\|f_n - f\| \longrightarrow 0$

For $0 < R < 1$, the growth estimate above yields for each n:

$$\sup_{|z|\le R} |f_n(z) - f(z)| \le \frac{\| f_n - f \|}{\sqrt{1-R^2}}$$

So $f_n \longrightarrow f$ uniformly on the closed disk $\{|z| \le R\}$, since R is arbitrary, $f_n \longrightarrow f$ uniformly on every compact subset of U. ■

Remark (1.1.9):

It is easily seen that $e_n(z) = z^n$, n = 0, 1, …; is a complete orthonormal basis for H^2, therefore span $\{e_n(z)\}$ is dense in H^2 and hence H^2 is a separable Hilbert space.

Definition (1.1.10) [17]:

Let f be a holomorphic function on U and let $z = re^{i\theta}$, $0 \le r < 1$, then define:

$$M_2(f, r) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^2 \, d\theta \right]^{1/2}$$

The proof of the following proposition appeared in [17].

Proposition (1.1.11):

Suppose that f is holomorphic on U, then $f \in H^2$ if and only if $M_2(f, r)$ is bounded for $0 \le r < 1$.

Note that if $f \in H^2$, then $\|f\| = \lim_{r \to 1^-} M_2(f, r)$, [17].

Let H^∞ be the set of all bounded holomorphic functions on U. Define a norm on H^∞ by:

$$\|f\|_\infty = \sup_{z \in U} |f(z)|$$

One can prove that $H^\infty \subseteq H^2$ and $\|f\| \le \|f\|_\infty$, for all $f \in H^\infty$, [17].

Definitions (1.1.12):

1. The function φ is said to be self map of U if it takes the unit ball U of the complex plane $\mathbb{C}$ into itself.
2. A one-to-one holomorphic map is called univalent.
3. Let φ be a holomorphic self map of U. If φ is univalent and onto U, then φ is said to be a conformal automorphism of U or just automorphism of U.

Now, we give two examples of holomorphic functions:

Example (1.1.13):

For each $p \in U$, define the special automorphism function:

$\alpha_p(z) = \dfrac{p - z}{1 - \bar{p}z}$, for all $z \in \hat{C}$

This function interchanges p with the origin, i.e., $\alpha_p(p) = 0$ and $\alpha_p(0) = p$

The following proposition appeared in [20], we give the details of its proof.

Proposition (1.1.14):

For each $p \in U$, the function α_p is conformal automorphism of U and takes ∂U onto ∂U.

Proof:

It is clear that α_p is holomorphic at all z except at $z = \dfrac{1}{\bar{p}}$, which is outside of U, hence α_p is holomorphic on U. Since:

$$1 - |\alpha_p(z)|^2 = 1 - \alpha_p(z)\overline{\alpha_p(z)}$$

$$= \frac{(1-|p|^2)(1-|z|^2)}{|1-\bar{p}z|^2}$$

This equation is greater than 0 for every $z \in U$ and equals 0 for every $z \in \partial U$, hence $|\alpha_p(z)| < 1$ on U and $|\alpha_p(z)| = 1$ on ∂U

Let $z \in \overline{U} = U \cup \partial U$, then $\alpha_p(\alpha_p(z)) = z$, therefore α_p takes U onto U and ∂U onto ∂U.

Finally, we show that α_p is one-to-one.

Let $\alpha_p(z_1) = \alpha_p(z_2)$, then $\alpha_p(\alpha_p(z_1)) = \alpha_p(\alpha_p(z_2))$, hence $z_1 = z_2$

Thus α_p is conformal automorphism. ■

The following remark shows that the function α_p is self inverse.

Remark (1.1.15):

Let $p \in U$, one can show easily that $\alpha_p(\alpha_p(z)) = z$, for all $z \in U$. Therefore $\alpha_p^{-1} = \alpha_p$.

We need the following lemma, the proof is simple.

Lemma (1.1.16):

For each $p \in U$, $\alpha'_p(0) = -1 - |p|^2$ and $\alpha'_p(p) = \dfrac{1}{-1+|p|^2}$. Therefore, $\alpha'_p(0)\,\alpha'_p(p) = 1$.

We give another example of holomorphic function on U.

Example (1.1.17):

Associated to each point $\alpha \in U$, there is a function of a particular interest to us; the reproducing kernel for α, defined by:

$$k_\alpha(z) = \frac{1}{1-\overline{\alpha}z} = \sum_{n=0}^{\infty} (\overline{\alpha}z)^n$$

which clearly belongs to H^2 and has norm $\dfrac{1}{\sqrt{1-|\alpha|^2}}$.

For each $f \in H^2$, one can prove easily:

$$f(\alpha) = \langle f, k_\alpha \rangle$$

1.2 SCHWARZ LEMMA

In this section, we recall several definitions and state important theorems that we need in the next chapters. In the following theorem, we give the concept of the radial limit.

Theorem (1.2.1) [3]:

Let $f \in H^2$ and $0 \le r < 1$, then the limit $f^*(w) = \lim_{r \to 1^-} f(rw)$, exists at almost every point w on the unit circle.

Definition (1.2.2) [3]:

Let $f \in H^2$ and $w \in \partial U$. The limit $f^*(w)$ is said to be the radial limit of f at w.

Remark (1.2.3):

From now on, we drop the notation $f^*(w)$ and we simply write f(w) for the radial limit of f at w.

The following proposition appeared in [8], we give it without proof.

Proposition (1.2.4):

If f and g are two functions in H^2, then one can find the inner product of f and g as follows:

$$<f, g> = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta$$

Thus:

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta$$

Recall that a point $p \in \mathbb{C}$ is fixed point for the function φ, if $\varphi(p) = p$.

Definitions (1.2.5):

Let φ be a holomorphic function that takes the unit ball U of the complex plane $\mathbb{C}$ into itself and fixes the point p, then:

1. p is an interior fixed point for φ if $p \in U$.
2. p is an exterior fixed point for φ if p lies outside the closed disk $\overline{U}$.
3. p is a boundary fixed point if $p \in \partial U$ and $\varphi(p) = p$ in the sense of radial limits (i.e., $\lim_{r\to 1^-} \varphi(rp) = p$).

Definition (1.2.6):

The holomorphic function φ is said to be rotation about the origin if there exists $w \in \partial U$, such that $\varphi(z) = wz$, $(z \in U)$.

The following is one of the most important theorems in complex analysis.

Schwarz Lemma (1.2.7) [5]:

If φ is a holomorphic self map of U with $\varphi(0) = 0$, then:

1. $|\varphi(z)| \le |z|$, for every z in U, with equality for some $0 \ne z \in U$ if and only if φ is a rotation about the origin.
2. $|\varphi'(0)| \le 1$ with equality if and only if φ is a rotation about the origin.

From Schwarz lemma, we can get several results, we give some of them. The following proposition appeared in [20], we give the proof for the sake of completeness.

Proposition (1.2.8):

If φ is a conformal automorphism of U that fixes the origin, then there exists $w \in \partial U$ such that $\varphi(z) = wz$ for every $z \in U$.

Proof:

Since φ is a holomorphic self-map of U that fixes the origin, part (2) of Schwarz lemma guarantees that $|\varphi'(0)| \leq 1$.

But since φ is an automosphism, it has a compositional inverse ψ that also obeys the hypothesis of Schwarz lemma

Hence $|\psi'(0)| \leq 1$. By the chain rule, $\varphi'(0)\psi'(0) = 1$, hence $|\varphi'(0)| = 1$, and so by the "equality part" of part (2) of Schwarz lemma, φ is a rotation about the origin. ■

We prove the following theorem .

<u>Theorem (1.2.9)]20]:</u>

If φ is a holomorphic self map of U, then for every $p \in U$

$$|\varphi'(p)| \leq \frac{1-|\varphi(p)|^2}{1-|p|^2}$$

with equality if and only if φ is automorphism of U.

Proof:

Let $q = \varphi(p)$ and consider the mapping $\psi = \alpha_q o \varphi o \alpha_p$, where α_p and α_q are the special automorphism mappings defined in page (8).

It is clear that ψ is a holomorphic self map of U and $\psi(0) = 0$, hence by Schwarz lemma $|\psi'(0)| \leq 1$ with equality if and only if ψ is a rotation about the origin.

By the chain rule

$$\psi'(z) = \alpha'_q(\varphi o \alpha_p(z))\varphi'(\alpha_p(z))\alpha'_p(z)$$

Hence:

$$\psi'(0) = \alpha'_q(q)\varphi'(p)\alpha'_p(0)$$

$$= \frac{1}{|q|^2-1}\varphi'(p)(|p|^2-1)$$

Since $|\psi'(0)| \le 1$, then $|\varphi'(p)| \le \frac{1-|q|^2}{1-|p|^2} = \frac{1-|\varphi(p)|^2}{1-|p|^2}$

If the equality holds in the last inequality, then $|\psi'(0)| = 1$, so that by Schwarz lemma, there exists $w \in \partial U$, such that $\psi(z) = wz$ for all $z \in U$, hence ψ is automorphism of U

Since $\psi = \alpha_q o \varphi o \alpha_p$ and α_q, α_p are self inverse, then $\varphi = \alpha_q o \psi o \alpha_p$

Since ψ, α_q, α_p are automorphism, then φ is automorphism.

Conversely, if φ is automorphism, then $\psi = \alpha_q o \varphi o \alpha_p$ is also automorphism

Since $\psi(0) = 0$, then from proposition (1.2.8), there exists $w \in \partial U$, such that $\psi(z) = wz$, for all $z \in U$

Hence $|\psi'(0)| = |w| = 1$, that is $|\varphi'(p)| = \frac{1-|q|^2}{1-|p|^2}$. ■

Corollary (1.2.10) [20]:

If φ is a holomorphic self map of U that fixes a point $p \in U$, then $|\varphi'(p)| \le 1$, with equality if and only if φ is an automorphism.

Proof:

From theorem (1.2.9), $|\varphi'(p)| \le \frac{1-|\varphi(p)|^2}{1-|p|^2}$ with equality if and only if φ is automorphism. Since $\varphi(p) = p$, then $|\varphi'(p)| \le 1$ with equality if and only if φ is automorphism. ■

We give the proof of the following proposition for the sake of completeness.

Proposition (1.2.11) [3, 20]:

No self map U (except the identity function) may have more than one interior fixed point.

Proof:

If $\varphi(0) = 0$ and $\varphi(q) = q$, where $0 \neq q \in U$, then by part (1) of Schwarz lemma, $\varphi(z) = \lambda z$ $(z \in U)$, where $|\lambda| = 1$

Therefore $q = \varphi(q) = \lambda q$, so that $\lambda = 1$. Thus φ is the identity mapping

If p and q are non-zero fixed points for φ, then consider the mapping $\psi = \alpha_p o \varphi o \alpha_p$, where α_p is the special automorphism mapping.

It is clear that $\psi(0) = 0$. Since $q \in U$ and α_p is automorphism then there exists $0 \neq r \in U$, such that $\alpha_p(r) = q$, therefore $\psi(r) = r$

Hence ψ fixes the origin and another point in U, therefore by our proof ψ must be the identity mapping

Since $\varphi = \alpha_p o \psi o \alpha_p$, then φ is the identity mapping. ■

<u>Definitions (1.2.12):</u>

1. A sector in U at a point $w \in \partial U$ is the region between two straight lines in U that meat at w and are symmetric about the radius to w, see figure (1.1), [17, p.49].

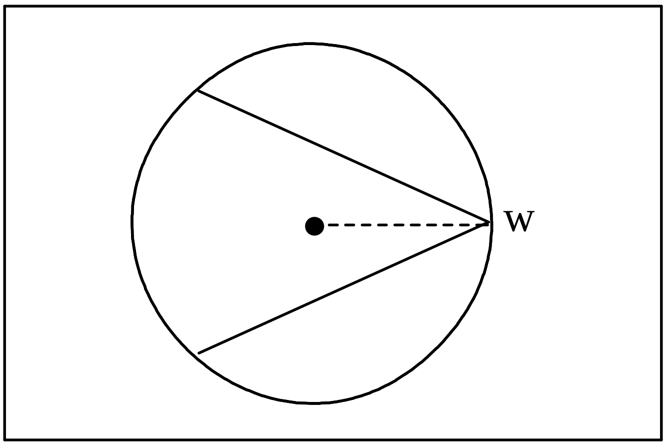

Figure (1.1) Sector at w.

2. If f is a function defined on U and $w \in \partial U$, then $\angle \lim_{z \to w} f(z) = L$, means that $f(z) \longrightarrow L$ as $z \longrightarrow w$ through any sector at w. When this happens, we say that L is the non-tangential (or angular) limit of f at w, [17, p.49].

3. We say a holomorphic self map φ of U has an angular derivative at $w \in \partial U$ if $\angle \lim_{z \to w} \varphi'(z)$ exists (finitely), and that when this happens, we denote the limit by $\varphi'(w)$, [3, p.17].

We end this section by the following theorem. For a proof see [17].

<u>The Julia-Carathéodory Theorem (1.2.13):</u>

Suppose φ is a holomorphic self-map of U, and $w \in \partial U$. Then the following statements are equivalent:

1. $\angle \lim_{z \to w} \frac{\eta - \varphi(z)}{w - z}$ exists for some $\eta \in \partial U$.
2. $\angle \lim_{z \to w} \varphi'(z)$ exists, and $\angle \lim_{z \to w} \varphi(z) = \eta \in \partial U$.

Moreover:

- The boundary point η in parts (1) and (2) are the same.
- The limit of the difference quotient in part (1) coincides with that of the derivative in part (2).

1.3 LITTLEWOOD'S THEOREM

In this section, we explore some links between function theory and operator theory that are created by Littlewood's subordination principle. To each holomorphic function φ that takes the unit ball U of the complex plane C into itself, we associated the composition operator C_φ defined by:

$C_\varphi f = f \circ \varphi$, for all $f \in H^2$

We state the following famous theorem without proof, for a proof see [17].

<u>Littlewood's Subordination Principle (1.3.1):</u>

Suppose φ is a holomorphic self-map of U, with $\varphi(0) = 0$. Then for each $f \in H^2$, $C_\varphi f \in H^2$ and $||C_\varphi f|| \le ||f||$.

The following theorem gives the general case for the map φ (φ does not necessarily fix the origin).

Theorem (1.3.2) [17]:

Let φ be a holomorphic self-map of U, then $f o \varphi \in H^2$,

$$\|f o \varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}\,\|f\|, \text{ for all } f \in H^2$$

We recall that if T is bounded operator on a Hilbert space H, then the norm of such an operator is defined by:

$$\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$$

if $\|T\| \leq 1$, then T is said to be a contraction on H.

From theorem (1.3.2), we get the following corollary:

Corollary (1.3.3):

Let φ be a holomorphic self-map of U, then C_φ is bounded operator on H^2 and $\|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$.

Proof:

We see from theorem (1.3.2) that $f o \varphi \in H^2$, for all $f \in H^2$

Therefore the composition operator C_φ takes the Hardy space H^2 into itself. Also, we have from theorem (1.3.2) that:

$$\|C_\varphi f\| = \|f \circ \varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}\,\|f\|, \text{ for all } f \in H^2$$

Thus C_φ is bounded and $\|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$. ■

Notation:

Let φ be a holomorphic self map of U, then $\varphi_n = \varphi o \varphi o \ldots o \varphi$ (n-times).

Remarks (1.3.4):

1. If φ is holomorphic self map of U that fixes the origin, then from corollary (1.3.3), C_φ is a contraction on H^2
2. One can easily show that $C_\varphi \circ C_\psi = C_{\psi \circ \varphi}$ and hence

$$C_\varphi^n = C_{\varphi\circ\varphi\circ\ldots\circ\varphi} = C_{\varphi_n}$$

3. If φ is a conformal automorphism, then the composition operator C_φ is invertible operator and $C_\varphi^{-1} = C_{\varphi^{-1}}$.

We recall that if H is a Hilbert space and T_1, T_2 are two operators on H, then T_1, T_2 are similar if there is an invertible operator S, such that $T_2 = S^{-1}T_1S$.

Definition (1.3.5) [17, p.93]:

Composition operators C_φ and C_ψ are said to be compositionally similar if there is a conformal automorphism mapping α of the unit ball U, such that:

$$\varphi = \alpha^{-1}\circ\psi\circ\alpha.$$

The following proposition appeared in [17] without proof. We give the proof.

Proposition (1.3.6):

Every compositionally similar composition operators are similar.

Proof:

Let C_φ and C_ψ be two compositionally similar composition operators. Hence by definition, there exists a conformal automorphism mapping α such that $\varphi = \alpha^{-1}\circ\psi\circ\alpha$. Therefore:

$$C_\varphi = C_{\alpha^{-1}\circ\psi\circ\alpha} = C_\alpha \circ C_\psi \circ C_{\alpha^{-1}} = C_\alpha \circ C_\psi \circ C_\alpha^{-1}$$

Thus C_φ and C_ψ are similar operators. ■

We recall that if $\{e_n\}$ is an orthonormal basis for a Hilbert space H, then every operator T on H can be represented by a matrix $A = (a_{ij})$, where $Te_i = \sum_j a_{ji} e_j$, i = 1, 2, … and the converse is true. We shall prove in the following example that the converse of proposition (1.3.6) is not true.

Example (1.3.7):

Let $\varphi(z) = iz$ and $\psi(z) = -iz$, then C_φ and C_ψ are similar, but C_φ and C_ψ are not compositionally similar.

Proof:

We shall find the matrices of the operators C_φ and C_ψ with respect to the orthonormal basis $e_n = z^n$, n = 0, 1, …; $C_\varphi 1 = 1$, $C_\varphi z = \varphi(z) = iz$, $C_\varphi z^2 = (\varphi(z))^2 = -z^2$ and so on.

Thus the operator C_φ has a block matrix A

$$A = \begin{bmatrix} M & & O \\ & M & \\ O & & \ddots \end{bmatrix}$$

where:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

By the same way, the operator C_ψ has a block matrix B:

$$B = \begin{bmatrix} N & & O \\ & N & \\ O & & \ddots \end{bmatrix}$$

Where:

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

Suppose that T is an operator that has a block matrix:

$$C = \begin{bmatrix} P & & O \\ & P & \\ O & & \ddots \end{bmatrix}$$

Where:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is clear that T is invertible operator, where T^{-1} has the block matrix:

$$C^{-1} = \begin{bmatrix} P^{-1} & & O \\ & P^{-1} & \\ O & & \ddots \end{bmatrix}$$

Since $A = C^{-1}BC$, then $C_\varphi = T^{-1}C_\psi T$ and hence C_φ , C_ψ are similar operators. Now, suppose that C_φ , C_ψ are compositionally similar, therefore there exists a conformal automorphism mapping α such that $\varphi = \alpha^{-1} o \psi o \alpha$, that is $\alpha o \varphi = \psi o \alpha$. Thus:

$\alpha(iz) = -i\alpha(z)$, for every $z \in U$ .... (1.1)

Therefore, $\alpha(0) = 0$.

By proposition (1.2.8), there exists $r \in \partial U$, such that $\alpha(z) = rz$, for all $z \in U$.

From equation (1.1) above, $riz = -irz$, for all $z \in U$.

This is a contradiction. Therefore C_φ and C_ψ are not compositionally similar. ■

1.4 LINEAR FRACTIONAL TRANSFORMATIONS

In this section, we present some information about the linear fractional transformations that we use in the next chapters. We refer the reader to [17], [5] for more details about the linear fractional transformations.

Definition (1.4.1):

A linear fractional transformation is a mapping of the form:

$$\varphi(z) = \frac{az+b}{cz+d}$$

Where a, b, c and d are complex numbers.

We prove the following proposition:

Proposition (1.4.2):

Let φ be a linear fractional transformation, then φ is a constant mapping if and only if $ad - bc = 0$.

Proof:

For any two points z_1, z_2, we have:

$$\varphi(z_1) = \varphi(z_2)$$

$$\Leftrightarrow \frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

$$\Leftrightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$\Leftrightarrow (ad - bc)(z_1 - z_2) = 0$$

Therefore, if φ is a constant mapping, then $\varphi(z_1) = \varphi(z_2)$, for any z_1, z_2 and hence $(ad - bc)(z_1 - z_2) = 0$, if we take $z_1 \neq z_2$, then $ad - bc = 0$

Conversely, if $ad - bc = 0$, then $(ad - bc)(z_1 - z_2) = 0$ and hence $\varphi(z_1) = \varphi(z_2)$, for each z_1, z_2; that is φ is a constant mapping. ■

Remarks (1.4.3):

1. One can see from the proof of proposition (1.4.2), that every non-constant linear fractional transformation is one-to-one.

2. We consider a linear fractional transformation $\varphi(z) = \dfrac{az+b}{cz+d}$, with ad – bc ≠ 0 defined on the Riemann sphere $\hat{C} = C \cup \{\infty\}$, where $\varphi(\infty) = \dfrac{a}{c}$ and $\varphi\left(\dfrac{-d}{c}\right) = \infty$ (notice that we cannot have a = 0 = c or d = 0 = c, since ad – bc ≠ 0).

Notation:

1. We denote the set of all linear fractional transformations is subject to the condition ad – bc ≠ 0 by LFT($\hat{C}$).

2. For any $\varphi(z) = \dfrac{az+b}{cz+d} \in$ LFT($\hat{C}$), we some times denote it by $\varphi_A(z)$ where A is the non-singular 2×2 complex matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Remarks (1.4.4):

1. It is clear that if φ_A belongs to LFT($\hat{C}$), then $\varphi_A = \varphi_{\lambda A}$, for any non-zero complex number λ.

2. One can easily prove that $\varphi_A o \varphi_B = \varphi_{AB}$, where o is the composition of maps.

3. If φ_A belongs to LFT($\hat{C}$), then φ_A is invertible where $\varphi_A^{-1} = \varphi_{A^{-1}}$, hence each linear fractional transformation can be regarded as a one-to-one holomorphic mapping of the Riemann sphere $\hat{C}$ onto itself.

From remarks (1.4.4), one can prove the following:

<u>Proposition (1.4.5):</u>

($LFT(\hat{C})$,o) is a group.

<u>Definition (1.4.6):</u>

We say that S, T $\in LFT(\hat{C})$ are conjugate if there exists V $\in LFT(\hat{C})$, such that $S = VTV^{-1}$.

It is easy to prove the following proposition:

<u>Proposition (1.4.7):</u>

Let φ_A and φ_B be any two mappings in $LFT(\hat{C})$. Then A and B are similar matrices if and only if φ_A and φ_B are conjugate.

Next, we study the fixed points for the linear fractional transformation. The proof of the following is simple.

<u>Remarks (1.4.8):</u>

If $\varphi(z) = \dfrac{az+b}{cz+d}$ belongs to $LFT(\hat{C})$, (φ is not the identity mapping), then:

1. φ fixes the point ∞ if and only if $c = 0$.
2. If $c = 0$, then ∞ is the only fixed point if and only if $a = d$ and $b \neq 0$.
3. If $c \neq 0$, then the fixed point equation is quadratic and takes the form $cz^2 + (d - a)z - b = 0$ with solutions:

$$\alpha, \beta = \frac{(a-d) \mp \sqrt{(a-d)^2 + 4bc}}{2c} \quad \text{...} \ (1.2)$$

<u>Definition (1.4.9):</u>

Let $\varphi(z) = \dfrac{az+b}{cz+d}$ be a linear fractional transformation. If $ad - bc = 1$, then we say that φ is in a standard form.

Since every linear fractional transformation φ_A is equal to $\varphi_{\lambda A}$, for every non-zero complex number λ, then one can assume that every linear fractional transformation is in a standard form:

Definition (1.4.10):

If $T(z) = \dfrac{az+b}{cz+d}$ belongs to LFT($\hat{C}$), then define the trace of T to be $X(T) = a + d$.

Remark (1.4.11):

We can write the two fixed points in equation (1.2) in term of the trace:

$$\alpha, \beta = \frac{(a-d) \mp (X(T)^2 - 4)^{1/2}}{2c}$$

Using this equation, one gets the following proposition:

Proposition (1.4.12):

Suppose that T belongs to LFT($\hat{C}$) and ∞ is not a fixed point for T, then T has a unique fixed point in $\hat{C}$ if and only if $|X(T)| = 2$.

The following theorem appeared in [17].

Theorem (Fixed Point and Derivative) (1.4.13):

Suppose that $T \in$ LFT($\hat{C}$) (T is not the identity mapping), then the following are equivalent:

1. $|X(T)| = 2$.
2. T has just one fixed point in $\hat{C}$.
3. $T' = 1$ at a fixed point of T

Definition (1.4.14):

A map $\varphi \in$ LFT($\hat{C}$) is called parabolic if it has a single fixed point in $\hat{C}$.

The proof of the following proposition is in [17].

Proposition (1.4.15):

If $\varphi \in LFT(\hat{C})$ is parabolic, then φ is conjugate to $V(z) = z + c$, $c \neq 0$.

Remark (1.4.16):

If $T \in LFT(\hat{C})$ is not parabolic, then T has two fixed points $\alpha, \beta \in \hat{C}$. Let $S \in LFT(\hat{C})$ that takes α to 0 and β to ∞, then the map $V = SoToS^{-1}$ belongs to $LFT(\hat{C})$ and fixes both 0 and ∞ so it must have the form $V(z) = \lambda z$, λ is said to be the multiplier for T.

Proposition (1.4.17) [17]:

If $T \in LFT(\hat{C})$ has two fixed points α , β then

$T'(\alpha) = \lambda$ and $T'(\beta) = 1/\lambda$ where λ is the multiplier for T .

In the following definition we classify the linear fractional transformations according to their multipliers .

Definition (1.4.18) [17]:

If $\varphi \in LFT(\hat{C})$ is not parabolic and not an identity mapping and $\lambda \neq 1$ is the multiplier for φ, then:

1. φ is elliptic if $|\lambda| = 1$.
2. φ is hyperbolic if $\lambda > 0$.
3. φ is loxodromic if φ is neither elliptic nor hyperbolic.

Our interest here is in LFT(U), the subgroup of $LFT(\hat{C})$ consisting of self maps of the unit ball U (i.e., take U into itself).

Notation:

The notation $f_n \xrightarrow{k} f$, means that the sequence $\{f_n\}$ of functions converges to f uniformly on every compact subset of U.

Now, we state the following theorem without proof:

Theorem (1.4.19) [3]:

Suppose φ is analytic self map of U that is not elliptic linear fractional transformation, (φ not necessary linear fractional transformation).

(a) If φ has a fixed point p ∈ U, then $\varphi_n \xrightarrow{k} p$ and $|\varphi'(p)| < 1$.

(b) If φ has no fixed point in U, then there is a point p ∈ ∂U such that $\varphi_n \xrightarrow{k} p$. Furthermore,

- p is a boundary fixed point of φ, and
- the angular derivative of φ exists at p with $0 < \varphi'(p) \leq 1$.

(c) Conversely, if φ has a boundary fixed point p at which $\varphi'(p) \leq 1$, then φ has no fixed points in U and $\varphi_n \xrightarrow{k} p$.

(Recall that φ_n means φoφo…oφ n-times).

Remark (1.4.20):

The fixed point p for the mapping φ is called the Denjoy Wolff point of φ, or attractive fixed point for φ if for each z in the unit ball U, $\varphi_n(z) \longrightarrow p$ as $n \longrightarrow \infty$, [3].

The following proposition appeared in [17] without proof, we give the proof.

Proposition (1.4.21):

If φ ∈ LFT(U) is parabolic, then φ has its fixed point on ∂U.

Proof:

Since φ is parabolic, then φ has only one fixed point, say p.

From theorem (1.4.19), p∈ U or p ∈ ∂U, if p ∈ U, then from part (a) of (1.4.19), $|\varphi'(p)| < 1$, and this contradicts part (3) of (1.4.13)

Therefore, p ∈ ∂U. ■

We prove the following proposition:

<u>Proposition (1.4.22):</u>

If $\varphi \in$ LFT(U) is parabolic, then for all $z \in \hat{C}$, $\varphi_n(z) \longrightarrow a$ as $n \longrightarrow \infty$, where a is the fixed point for φ.

Proof:

If $\alpha \in \text{LFT}(\hat{C})$ takes the point a to ∞, then $V = \alpha o \varphi o \alpha^{-1}$ belongs to $\text{LFT}(\hat{C})$ and fixes only the point ∞.

Therefore, $V(z) = z + c$, for some non-zero complex number c.

Therefore, $\varphi_n = \alpha^{-1} V_n \alpha$, hence $\varphi_n(z) = \alpha^{-1}(\alpha(z) + nc)$, for all $z \in \hat{C}$

Since $\{\alpha(z) + nc\} \longrightarrow \infty$ as $n \longrightarrow \infty$, then for all $z \in \hat{C}$, $\varphi_n(z) = \alpha^{-1}(\alpha(z) + nc) \longrightarrow \alpha^{-1}(\infty) = a$ as $n \longrightarrow \infty$. ■

The following theorem appeared in [17].

<u>Theorem (1.4.23):</u>

Let φ be a linear fractional self-map of U

1. If φ is hyperbolic, then it has attractive fixed point in $\overline{U}$ with the other fixed point outside U.
2. If φ is loxodromic or elliptic, then it has a fixed point in U and a fixed point outside $\overline{U}$.

The proof of the following proposition appeared in [17].

<u>Proposition (1.4.24):</u>

Let $\varphi \in$ LFT(U) has two fixed points $\alpha, \beta \in \hat{C}$. If α is attractive fixed point for φ and φ is not elliptic, then for all $z \in \hat{C} \backslash \{\beta\}$, $\varphi_n(z) \longrightarrow \alpha$ as $n \longrightarrow \infty$.

We give the following proposition:

Proposition (1.4.25):

Let φ be a linear fractional self map of U, then φ is elliptic if and only if φ is automorphism that has an interior fixed point.

Proof:

($\Rightarrow$) If φ is elliptic, then from theorem (1.4.23), φ has a fixed point $\alpha \in$ U and β outside $\overline{U}$. From proposition (1.4.17), $\varphi'(\alpha) = \lambda$ and $\varphi'(\beta) = 1/\lambda$, where λ is the multiplier for φ

Since φ is elliptic, then $|\lambda| = 1$ and hence $|\varphi'(\alpha)| = |\varphi'(\beta)| = 1$.

Now, if φ is not automorphism, then by corollary (1.2.10), $|\varphi'(\alpha)| < 1$, this is a contradiction, therefore φ is an automorphism.

($\Leftarrow$) Suppose φ is an automorphism of U with an interior fixed point p, let $\psi = \alpha_p o \varphi o \alpha_p$, where α_p is the special automorphism mapping that interchanges the point p with the origin.

Therefore, ψ is automorphism and $\psi(0) = 0$. By proposition (1.2.8), there exists $w \in \partial U$, such that $\psi(z) = wz$, for each $z \in U$. Therefore:

$$w = \psi'(0) = \alpha'_p(\varphi o \alpha_p(0))\, \varphi'(\alpha_p(0)).\alpha'_p(0)$$

$$= \alpha'_p(p)\, \varphi'(p)\, \alpha'_p(0)$$

$$= \varphi'(p)$$

That is the multiplier for φ is w. Since $|w| = 1$, then φ is elliptic. ■

We end this section by the following proposition which appeared in [3].

Proposition (1.4.26):

If $\varphi \in$ LFT(U) has two boundary fixed points, then φ is an automorphism.

1.5 COMPACT COMPOSITION OPERATORS

In this section, we recall the concept of a compact operator and give some necessary and sufficient conditions for a composition operator to be a compact operator. We recall that an operator T on a Hilbert space H is said to be compact if it maps every bounded set into a relatively compact one (one whose closure in H is compact).

The following theorem shows that H^2 supports a lot of compact composition operators.

Theorem (1.5.1) [17]:

Let φ be a holomorphic self map of U. If $\|\varphi\|_\infty < 1$, then C_φ is a compact operator on H^2.

We recall that an operator T on a Hilbert space H is said to be a Hilbert-Schmidt operator if for some orthonormal basis $\{e_n\}$ of H,

$$\sum_{n=0}^{\infty} \| Te_n \|^2 < \infty$$

The following theorem appeared in [17].

Theorem (1.5.2):

Every Hilbert-Schmidt operator is compact.

Corollary (1.5.3):

If $\sum_{n=0}^{\infty} \| \varphi^n \|^2 < \infty$, then C_φ is compact operator.

Proof:

Since $e_n(z) = z^n$, n = 0, 1, …; is orthonormal basis and

$$\sum_{n=0}^{\infty} \| C_\varphi(z^n) \|^2 = \sum_{n=0}^{\infty} \| \varphi^n \|^2 < \infty$$

then C_φ is a Hilbert-Schmidt operator and hence from theorem (1.5.2), C_φ is compact operator. ■

The following theorem appeared in [17].

Theorem (Angular Derivative Criterion for Compactness) (1.5.4):

Suppose φ is a holomorphic self map of U.

(a) If C_φ is compact on H^2, then φ has an angular derivative at no point of ∂U.

(b) If φ is univalent and has no angular derivative at any point of ∂U, then C_φ is compact on H^2.

We are ready now to prove the following proposition.

Proposition (1.5.5) [17]:

If the composition operator C_φ is compact on H^2, then φ has a fixed point in U.

Proof:

If φ has no fixed point in U, then by theorem (1.4.19), φ has an angular derivative at a point $w \in \partial U$ with $0 < \varphi'(w) \leq 1$.

According to the angular derivative criterion, C_φ is not compact. ■

The authors in [13] have studied the compactness of the operator $C_\varphi C^*_\psi$ and showed the following.

Theorem (1.5.6):

Let φ, ψ be univalent self maps of U, then $C^*_\psi C_\varphi$ is compact if and only if $\lim\limits_{|z|\to 1^-} \dfrac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0$.

We end this section by studying the eigenvalues for a composition operator C_φ. The eigenfunction equation for a composition operator C_φ is called Schroder equation:

$$f \circ \varphi = \lambda f$$

The following theorem appeared in [17].

<u>The Eigenfunction Theorem (1.5.7):</u>

Suppose that φ is a holomorphic self map of U for which C_φ is compact on H^2 (i.e., by proposition (1.5.5) there exists an interior fixed point $p \in U$), then the eigenvalues of C_φ are precisely the numbers $\{\varphi'(p)\}_{n=0}^{\infty}$ each has multiplicity one. Moreover, if σ is an eigenfunction for $\varphi'(p)$, then the set $\{\sigma^n\}$ spans the eigenspace for $\varphi'(p)^n$, $n = 0, 1, \ldots$

<u>Remark (1.5.8):</u>

If φ is univalent self map of U, then σ in the previous theorem is also univalent, [17].

CHAPTER TWO

CYCLIC COMPOSITION OPERATORS

INTRODUCTION

In this chapter, we recall the definitions of cyclic, supercyclic and hypercyclic operators on a Hilbert space H and we study the cyclicity of the composition operators. This chapter consists of two sections, in section one we prove an important theorems about the cyclic composition operators, for example we show that if φ is holomorphic self map of U that fixes a point p in U, then C_φ is not a hypercyclic (supercyclic) operator. Although, a composition operator induced by a mapping φ with fixed point in U can never be supercyclic, it can be cyclic (see example (2.1.5)). Also, we study the cyclicity of normal, isometric composition operators.

In section two, we state some conditions for the operator C_φ to be cyclic, for example the univalency of the holomorphic mapping φ and the density of the range of C_φ are necessary conditions for C_φ to be cyclic. In general, if x is a cyclic vector for the operator T, then Tx may not be cyclic vector for T (see example (2.2.14)). We prove that if f is cyclic (hypercyclic, supercyclic) vector for C_φ, then $C_\varphi f$ is a cyclic (hypercyclic, supercyclic) vector for C_φ (theorem (2.2.15)). Also, we prove some new results, to the best of our knowledge, for the cyclicity of the adjoint composition operators.

2.1 CYCLICITY

In this section, we recall a basic concept of cyclicity and we give important theorems about the cyclic composition operators. We begin this section by the following well-known definitions.

Definitions (2.1.1):

Let T be a bounded linear operator on a Hilbert space H, then:

1. T is cyclic if there exists a vector $x \in H$, such that the set span $\{T^n x : n = 0, 1, \ldots\}$ is dense in H.

   The vector x is called a cyclic vector for the operator T.

2. T is a supercyclic operator if there exists a vector $x \in H$, such that the set $\{\alpha_n T^n x : \alpha_n \in \mathbb{C}, n = 0, 1, \ldots\}$ is dense in H.

   The vector x is called a supercyclic vector for the operator T.

3. T is a hypercyclic operator if there exists a vector $x \in H$, such that the orbit, $orb(T, x) = \{T^n x : n = 0, 1, \ldots\}$ is dense in H.

   The vector x is called hypercyclic vector for the operator T.

It is clear from this definition that every hypercyclic operator is a supercyclic operator and every supercyclic operator is a cyclic operator. However, it is known that the opposite implications are false [11].

The proof of the following useful theorem is well-known, thus it is omitted.

Theorem (2.1.2) [12, 10]:

Suppose that S, T, X are bounded operators on a Hilbert space H, such that SX = XT, if T is cyclic (supercyclic, hypercyclic) and X has a dense range, then S is also cyclic (supercyclic, hypercyclic).

Corollary (2.1.3):

If T_1 and T_2 are similar operators, then T_1 is cyclic (supercyclic, hypercyclic) if and only if T_2 is cyclic (supercyclic, hypercyclic).

From [16, proposition (4.5)] and [12, proposition (3.6)], we can prove the following proposition:

Proposition (2.1.4):

Let T be an operator on a Hilbert space H that has diagonal matrix $A = diag(\lambda_1, \lambda_2, \ldots)$ with respect to some orthonormal basis $\{e_n\}$, then T is cyclic if and only if the diagonal entries $\{\lambda_i\}$ are distinct.

Now, we can give an example of a cyclic composition operator.

Example (2.1.5):

Let α be a non-zero complex number with $|\alpha| < 1$ and let $\varphi(z) = \alpha z$, for every z in U, then φ is a holomorphic self map of U. We claim that C_φ is a cyclic operator on H^2. In fact the matrix of C_φ with respect to the orthonormal basis $e_n(z) = z^n$, n = 0, 1, … is diagonal matrix $A = \text{diag}(1, \alpha, \alpha^2, \ldots)$. Since $1 > |\alpha| > |\alpha^2| > \ldots$, then by proposition (2.1.4), C_φ is cyclic.

The following theorem shows that the composition operator C_φ in example (2.1.5) is not hypercyclic (in fact it is not supercyclic, see theorem (2.1.18)).

Theorem (2.1.6) [3]:

Suppose that φ is holomorphic self map of U that fixes a point z_0 in U, then C_φ is not hypercyclic operator. Moreover, if φ is not an elliptic, then for each $f \in H^2$, the only limit point of $\text{orb}(C_\varphi, f)$ is the constant function $f(z_0)$.

Proof:

Suppose that φ fixes a point $z_0 \in U$. If φ is not elliptic, then by theorem (1.4.19), $\varphi_n \longrightarrow z_0$ pointwise on U. Hence, if a function g is a limit point of the orbit of f, say $g = \lim_J f \circ \varphi_{nJ}$.

Then by the continuity of point evaluation function on H^2, we see that for each $z \in U$

$$g(z) = \lim_J f(\varphi_{nJ}(z)) = f(z_0)$$

i.e., $\text{orb}(C_\varphi, f)$ is not dense in H^2

If φ is elliptic, then $\varphi_n(z_0) \longrightarrow z_0$ as $n \longrightarrow \infty$, hence if a function g is a limit point of $\text{orb}(C_\varphi, f)$, then $g(z) = \lim_J f(\varphi_{nJ}(z))$

Therefore $g(z_0) = \lim_J f(\varphi_{nJ}(z_0)) = f(z_0)$

Thus, every function g belongs to the closure of the set orb(C_φ, f) has value $f(z_0)$ at z_0, hence orb(C_φ, f) cannot be dense. ■

Remark (2.1.7):

We can prove theorem (2.1.6) by using Littlewood's subordination principle which asserts that if the fixed point p is the origin, then $\|C_\varphi\| \leq 1$ and hence C_φ is not hypercyclic (every contraction operator is not hypercyclic operator [7]). If $p \neq 0$, we have $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where $\psi(0) = 0$, α_p is the special automorphism mapping. So C_φ is similar to a contraction and therefore still not hypercyclic (corollary (2.1.3)).

We shall prove in theorem (3.2.6) that if φ is automorphism, non-elliptic (i.e., by proposition (1.4.25), φ has no interior fixed point) then C_φ is a hypercyclic operator.

Corollary (2.1.8):

If C_φ is a compact composition operator, then $C_\varphi$is not hypercyclic.

Proof:

Since C_φ is compact, then by proposition (1.5.5), φ has an interior fixed point and hence from theorem (2.1.6), C_φ is not a hypercyclic operator. ■

In fact, much more is true:

Proposition (2.1.9) [17]:

No compact operator on a Hilbert space is hypercyclic.

Remark (2.1.10):

If φ, ψ are univalent holomorphic self maps of U and

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0$$

Then by theorem (1.5.6), $C^*_\psi C_\varphi$ is a compact operator and hence $C^*_\psi C_\varphi$ is not a hypercyclic operator.

The following theorem shows that if C_φ is hypercyclic on H^2, then it is hypercyclic on H(U).

Theorem (2.1.11) [17]:

Suppose E is a linear metric space and F a dense subspace that is itself a linear metric space with a stronger topology. Suppose T is a linear transformation on E that also maps the smaller space F into itself, and is continuous in the topology of each space. If T is hypercyclic on F, then it is also hypercyclic on E and has an E- hypercyclic vector that belong to F.

Corollary (2.1.12):

Any hypercyclic vector for C_φ acting on H^2 is also hypercyclic for $C_\varphi$acting on H(U).

Proof:

Since the set of polynomials is dense in both H^2 and H(U), then H^2 is dense in H(U)

Therefore, any hypercyclic composition operator C_φ acting on H^2 is hypercyclic acting on H(U). ■

The following lemma is proved in [16].

Lemma (2.1.13):

If T is a cyclic operator on H has a matrix A = (a_{ij}) with cyclic vector X = (x_1, x_2, …), then the operator $\overline{T}$ is cyclic with cyclic vector $\overline{X}$ = ($\overline{x}_1, \overline{x}_2$, …) , where $\overline{x}_i$ is the complex conjugate of x_i, for all i and $\overline{T}$ is the operator that has the matrix $\overline{A} = (\overline{a}_{ij})$, ($\overline{a}_{ij}$ is the complex conjugate of a_{ij}).

Notation:

Let $\varphi(z) = \sum_n a_n z^n$, $a_n \in$ C, ∀ n; be a holomorphic self map of U. We denote by $\overline{\varphi}(z)$ to the holomorphic map $\sum_n \overline{a}_n z^n$, where $\overline{a}_n$ is the complex conjugate of a_n.

We give the following proposition:

Proposition (2.1.14):

Let $\varphi(z)$ be a holomorphic self map of U. If C_φ is a cyclic operator with cyclic vector f, then $C_{\overline{\varphi}}$ is a cyclic operator with cyclic vector $\overline{f}$.

Proof:

Let A be the matrix of C_φ with respect to the orthonormal basis $\{e_n(z) = z^n\}$, therefore the matrix of $C_{\overline{\varphi}}$ is $\overline{A} = (\overline{a}_{ij})$, where $\overline{a}_{ij}$ is the complex conjugate of a_{ij}.

If C_φ is cyclic, then by lemma (2.1.13), $C_{\overline{\varphi}}$ is cyclic. ■

Corollary (2.1.15):

Suppose that $\varphi(z) = \dfrac{p_n(z)}{q_m(z)}$ is a holomorphic self map of U, where $p_n(z)$ and $q_m(z)$ are polynomials of degree n and m, respectively. If C_φ is a cyclic operator, then C_ψ is a cyclic operator, where $\psi(z) = \dfrac{\overline{p}_n(z)}{\overline{q}_m(z)}$.

Proof:

One can easily prove that $\overline{\varphi}(z) = \psi(z)$ and hence if C_φ is cyclic, then from proposition (2.1.14), C_ψ is cyclic. ■

Before we give theorem (2.1.18), we need some preliminaries.

Remarks (2.1.16):

1. If T is a cyclic operator, then $\dim[R(T)]^\perp < 2$, where R(T) is the range of T, [9].
2. T is a cyclic operator if and only if $T + \alpha I$ is a cyclic operator for all $\alpha \in \mathbb{C}$, [16].

Lemma (2.1.17) [2]:

Let T be an operator that has the matrix $A = (a_{ij})$ with respect to the orthonormal basis $\{e_n\}$, then the matrix of T* (the adjoint of T) with respect to the same orthonormal basis is $\overline{A}^t = (\overline{a}_{ji})$, where $\overline{a}_{ji}$ is the complex conjugate of a_{ji}.

The author in [1] proves the following theorem, we give another proof.

Theorem (2.1.18):

Let φ be a holomorphic self map of U with interior fixed point p, then C_φ cannot be a suprcyclic operator.

Proof:

Let $\psi = \alpha_p o \varphi o \alpha_p$, where α_p is the special automorphism mapping, therefore $\psi(0) = 0$ and $C_\psi = C_{\alpha p} o C_\varphi o C_{\alpha p}$, that is C_ψ is similar to C_φ.

We claim that C_ψ is not a supercyclic operator. In fact if $w = \psi'(0)$, then the matrix of C_ψ with respect to the orthonormal basis $e_n(z) = z^n$, n = 0, 1, … takes the form:

$$A = (a_{ij}) = \begin{bmatrix} 1 & & & O \\ 0 & w & & \\ & & w^2 & \\ & * & & \ddots \end{bmatrix}$$

Therefore, the matrix of C_ψ^* with the same orthonormal basis takes the form:

$$\overline{A}^t = (\overline{a}_{ji}) = \begin{bmatrix} 1 & 0 & \cdots & \\ & \overline{w} & & * \\ & & \overline{w}^2 & \\ & O & & \ddots \end{bmatrix}$$

If w = 1, then the first and second rows of the matrix A − I are zeros, hence $e_0(z) = 1$ and $e_1(z) = z$ belong to $[R(C_\psi - I)]^\perp$, where $R(C_\psi - I)$ is the range of the operator $C_\psi - I$, therefore $C_\psi - I$ is not cyclic (see remark (2.1.16)), so that C_ψ is not suprcyclic. If $w \neq 1$, then it is clear that 1 and $\overline{w}$ are eigenvalues of the operator C_ψ^* ([16], proposition (4.11)). Thus, C_ψ is not a supercyclic operator (the adjoint of the supercyclic operator has at most one eigenvalue [11]).

Since C_φ and C_ψ are similar, then C_φ is not a supercyclic operator. ■

Corollary (2.1.19):

If C_φ is a compact operator, then C_φ is not a supercyclic operator.

Proof:

Since C_φ is compact, then by proposition (1.5.5), φ has interior fixed point

Hence C_φ is not a supercyclic operator (theorem (2.1.18)). ■

Remark (2.1.20):

Although a composition operator induced by a mapping φ with fixed point in U can never be supercyclic, it can be cyclic (see example (2.1.5)).

We give the following theorem:

Theorem (2.1.21):

Let φ be a conformal automorphism of U and has an interior fixed point p, then C_φ is cyclic if and only if $(\varphi'(p))^n \neq 1$, for all n = 1, 2, …

Proof:

Let $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where α_p is the special automorphism mapping

Therefore, $\psi(0) = 0$ and ψ is automorphism

Hence by proposition (1.2.8), there exists $w \in \partial U$, such that $\psi(z) = wz$, for all $z \in U$, therefore:

$$\psi'(z) = \alpha'_p(\varphi \circ \alpha_p(z))\, \varphi'(\alpha_p(z))\alpha'_p(z)$$

So that $w = \psi'(0) = \varphi'(p)$. It is clear that the matrix of C_ψ with respect to the orthonormal basis $\{z^n\}$ is a diagonal matrix $A = \text{diag}(1, w, w^2, ..)$, therefore, from proposition (2.1.4), C_ψ is cyclic if and only if the diagonal entries $\{w^n : n = 0, 1, \ldots\}$ are distinct, that is $w^n \neq 1$, $n = 1, 2, \ldots$

Since $C_\varphi = C_{\alpha p} \circ C_\psi \circ C_{\alpha p}$, then C_φ and C_ψ are similar so that C_φ is cyclic if and only if C_ψ is cyclic.

Thus C_φ is cyclic if and only if $(\varphi'(p))^n = w^n \neq 1$, $n = 1, 2, \ldots$ ■

Recall that, the operator T on H is called a normal operator if $TT^* = T^*T$ and called isometric if $T^*T = I$.

Theorem (2.1.22) [7]:

Let φ be holomorphic self map of U, then C_φ is normal if and only if $\varphi(z) = \alpha z$, for some α, $|\alpha| \leq 1$.

From this theorem, we have the following results.

Corollary (2.1.23):

Let φ be a holomorphic self map of U. If C_φ is normal, then C_φ and C_φ^* are not hypercyclic (not supercyclic).

Proof:

Since C_φ is a normal operator, the $\varphi(z) = \alpha z$, for some α, $|\alpha| \leq 1$

Therefore, 0 is fixed point for φ. Thus C_φ is not hypercyclic (theorem (2.1.6)), not supercyclic (theorem (2.1.18)).

If A is the matrix of C_φ with respect to the orthonormal basis $\{z^n\}$, then A is a diagonal matrix with diagonal entries $1, \alpha, \alpha^2, \ldots$, therefore the matrix of C_φ^* is a diagonal matrix with diagonal entries $1, \bar{\alpha}, \bar{\alpha}^2, \ldots$

Thus $C_\varphi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z$, $\forall z \in U$

Since 0 is fixed point for ψ, then $C_\varphi^* = C_\psi$ is not hypercyclic (theorem (2.1.6)), not supercyclic (theorem (2.1.18)). ■

Corollary (2.1.24):

Suppose that φ is non-zero holomorphic self map of U (φ is not the identity mapping). If φ is non-elliptic and C_φ is normal operator, then C_φ and C_φ^* are cyclic operators.

Proof:

From theorem (2.1.22), $\varphi(z) = \alpha z$, where $|\alpha| \le 1$

Since φ is non-elliptic and not zero mapping, then $0 < |\alpha| < 1$, hence C_φ is cyclic (see example (2.1.5))

From the proof of corollary (2.1.23), we have $C_\varphi^* = C_{\bar{\alpha}z}$, where $0 < |\bar{\alpha}| = |\alpha| < 1$

Thus C_φ^* is cyclic (example (2.1.5)). ■

Before we characterize the cyclicity of the isometric composition operators, we need the following lemma:

Lemma (2.1.25) [17]:

Let φ be a holomorphic self map of U, then for each $p \in U$, $C_\varphi^* K_p = K_{\varphi(p)}$, where $K_p(z) = \dfrac{1}{1-\bar{p}z}$.

Proof:

We know from chapter one that $\langle g, K_p\rangle = g(p)$, for all $g \in H^2$.

Thus for each $f \in H^2$, we have:

$$\langle f, C_\varphi^* K_p\rangle = \langle C_\varphi f, K_p\rangle = \langle f o\varphi, K_p\rangle = f(\varphi(p)) = \langle f, K_{\varphi(p)}\rangle$$

So that $C_\varphi^* K_p = K_{\varphi(p)}$. ■

Theorem (2.1.26):

Let φ be a holomorphic self map of U (φ is not the identity mapping). If C_φ is isometric, then:

1. C_φ and C_φ^* are not hypercyclic (supercyclic).

2. If φ is not elliptic, then C_φ and C_φ^* are cyclic.

Proof:

We claim that C_φ is normal operator and hence this results follows from corollary (2.1.23) and (2.1.24).

Let $0 \neq p \in U$, then:

$$C_\varphi^* C_\varphi K_p(z) = C_\varphi^* K_p(\varphi(z)) = K_{\varphi(p)}(\varphi(z)) = \frac{1}{1-\overline{\varphi(p)}\varphi(z)}$$

Since C_φ is isometric, then $C_\varphi^* C_\varphi K_p(z) = K_p(z)$

Therefore $\frac{1}{1-\overline{\varphi(p)}\varphi(z)} = K_p(z) = \frac{1}{1-\overline{p}z}$

Thus $\varphi(z) = \frac{\overline{p}}{\overline{\varphi(p)}} z$, for all $z \in U$

Put $\alpha = \frac{\overline{p}}{\overline{\varphi(p)}}$, hence $\varphi(z) = \alpha z$, for all $z \in U$

Since φ is self map of U, then $|\alpha| \leq 1$ and hence from theorem (2.1.22), C_φ is normal operator. ■

Recall that the operator T is hyponormal if $T^*T \geq TT^*$

Theorem (2.1.27) [7]:

Let φ be a holomorphic self map of U. If C_φ is hyponormal, then $\varphi(0) = 0$.

From this theorem, we have the following corollary:

Corollary (2.1.28):

Let φ be a holomorphic self map of U. If C_φ is hyponormal, then C_φ is not hypercyclic (supercyclic) operators.

Proof:

Since 0 is fixed point for φ, then C_φ is not supercyilc operator (theorem (2.1.18)). ■

2.2 SOME CONDITIONS FOR CYCLICITY

In this section, we describe in more details the cyclic composition operators. Also, we study the cyclicity of the operator C^*_φ. , where C^*_φ. is the adjoint of the operator C_φ . The following proposition shows that the univalence of the holomorphic mapping φ is a necessary condition for hypercyclicity.

<u>Proposition (2.2.1) [3]:</u>

Let φ be a holomorphic self map of U. If C_φ is hypercyclic, then φ is univalent on U.

Proof:

If φ identifies two distinct points of U, then so does $f \circ \varphi_n$ for each $f \in H^2$ and each positive integer n and therefore so does every limit point of the orbit of f under C_φ.

It follows that no orbit can be dense in H^2

So C_φ is not hypercyclic. ■

In fact, much more is true as the following theorem shows:

<u>Theorem (2.2.2) [3]:</u>

Let φ be a holomorphic self map of U. If C_φ is cyclic, then φ is univalent on U.

The following important theorem appeared in [3], the proof is long, thus is omitted.

<u>Theorem (2.2.3):</u>

Let φ be a holomorphic self map of U. If C_φ is cyclic, then its range is dense in H^2.

Remark (2.2.4):

The necessary condition for cyclicity discussed in theorem (2.2.3) is not sufficient, we will show in the next chapter that for example if $\psi(z) = \frac{z}{2-z}$, then C_ψ is not cyclic (theorem (3.1.12))

Note, however that C_ψ does have dense range.,as is shown in the following proposition:

Proposition (2.2.5):

Let $\psi(z) = \frac{z}{2-z}$, for all $z \in U$, then C_ψ has dense range.

Proof:

Suppose that f is orthogonal to the range of C_ψ, then because 1 is in the range, $0 = \langle f, 1\rangle = f(0)$, so that $f = zg$, for some g in H^2.

Because ψ^n belongs to the range of C_ψ, we have:

$$0 = \langle f, \psi^n\rangle = \langle zg, \left(\frac{z}{2-z}\right)^n \rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta} g(e^{i\theta}) \frac{\left(\overline{e^{i\theta}}\right)^n}{\left(2-\overline{e^{i\theta}}\right)^n}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta} g(e^{i\theta}) \frac{\left(e^{-i\theta}\right)^n}{\left(2-e^{-i\theta}\right)^n}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta} g(e^{i\theta}) \frac{1}{\left(2e^{i\theta}-1\right)^n}\, d\theta$$

Suppose $\tau = e^{i\theta}$, therefore $d\tau = ie^{i\theta}\, d\theta$, so that:

$$0 = \frac{1}{2\pi i}\int_{\partial U} g(\tau)\frac{1}{(2\tau-1)^n}\,d\tau$$

$$= \frac{1}{2^n(2\pi i)}\int_{dU} \frac{g(\tau)}{\left(\tau-\frac{1}{2}\right)^n}\,d\tau$$

By the general Cauchy integral formula, we get:

$$0 = \frac{1}{2^n}\frac{g^{(n-1)}(\frac{1}{2})}{(n-1)!}$$

and hence $g^{(n-1)}(\frac{1}{2}) = 0$, n = 1, 2, …

Since g and all of its derivatives vanish at the point 1/2, we see that $g \equiv 0$ and hence $f \equiv 0$.

It follows that C_ψ has dense range. ■

Remark (2.2.6):

We shall see in chapter four that if $\|\varphi\|_\infty < 1$, then the converse of theorem (2.2.3) is true (see remarks (4.1.12)).

Although, our main results are set exclusively in the Hardy space H^2 of the unit disk, it is some times convenient to interpret some of the intermediate steps in a more general setting. If G is a simply connected plane domain, and σ is a univalent holomorphic mapping of U onto G, then the Hardy space $H^2(G)$ is the set of functions f holomorphic on G for which $fo\sigma \in H^2$. The inner product of two elements f and g in $H^2(G)$ is defined to be $(f, g) = \langle fo\sigma, go\sigma\rangle$, where $\langle ., .\rangle$ denotes the inner product of the usual Hardy space H^2. We can show that the collection of functions $H^2(G)$ is independent of the particular univalent map σ used above [3].

The following theorem appeared in [3] without proof. We give the proof.

Theorem (2.2.7):

Suppose that $G \subseteq U$ is simply connected and that φ maps U univalently onto G. Then the following are equivalent:

(a) The polynomials are dense in $H^2(G)$.

(b) The polynomials in φ are dense in H^2.

(c) The composition operator $C_\varphi : H^2 \longrightarrow H^2$ has dense range.

Before, we prove this theorem we need the following lemma:

Lemma (2.2.8):

Let φ be a holomorphic self map of U and P is the set of all polynomials in φ, then $\overline{R(C_\varphi)} = \overline{P}$.

Proof:

We prove that $P \subseteq R(C_\varphi) \subseteq \overline{P}$

Let $P(\varphi) \in P$, then $P(\varphi) = a_0 + a_1\varphi + \ldots + a_n\varphi^n$, where $a_i \in C$, i =0, 1, …, n. It is clear that:

$C_\varphi (P(z)) = C_\varphi (a_0 + a_1 z + \ldots + a_n z^n) = P(\varphi)$

that is, $P(\varphi) \in R(C_\varphi)$

We prove now that $R(C_\varphi) \subseteq \overline{P}$, let $f \in R(C_\varphi)$, then there exists $g \in H^2$, such that $f = C_\varphi g = g(\varphi)$

Since $g \in H^2$, then:

$g(z) = b_0 + b_1 z + \ldots$, where $b_i \in C$, i =0, 1, … , so that

$f = b_0 + b_1\varphi + \ldots \in \overline{P}$

Thus $P \subseteq R(C_\varphi) \subseteq \overline{P}$ and hence $\overline{R(C_\varphi)} = \overline{P}$. ■

Proof of the Theorem:

(a) ⇒ (b). We know that $H^2(G)$ is the set of all functions f holomorphic on G, such that $f o\varphi \in H^2$

Let p(z) be any polynomial

Since $C_\varphi : H^2 \longrightarrow H^2$ and $p(z) \in H^2$, then $p(\varphi) = C_\varphi(p(z)) \in H^2$, that is the set of all polynomials in φ belongs to H^2. We will prove that this set is dense in H^2

Suppose that there exists $h \in H^2$, such that $<h, p(\varphi)>=0$, for all polynomials p.

Since φ is univalent mapping of U onto G, then φ^{-1} is univalent mapping of G onto U and hence $ho\varphi^{-1}$ is holomorphic on G with $ho\varphi^{-1}o\varphi = h \in H^2$, therefore $ho\varphi^{-1} \in H^2(G)$

By the definition of the inner product on $H^2(G)$, we get:

$$(ho\varphi^{-1}, p) = <ho\varphi^{-1}o\varphi, po\varphi> = <h, p(\varphi)> = 0$$

for all polynomials p. Since the set of all polynomials is dense in $H^2(G)$, then $ho\varphi^{-1} = 0$, since φ^{-1} is onto, then $h = 0$. It follows that the set of all polynomials in φ is dense in H^2.

(b) ⇒ (c). From lemma (2.2.8), $\overline{P} = \overline{R(C_\varphi)}$, where P is the set of all polynomials in φ. Since $\overline{P} = H^2$, then $\overline{R(C_\varphi)} = H^2$, so that the range of C_φ is dense in H^2.

(c) ⇒ (a). Since $\overline{R(C_\varphi)} = H^2$, then from lemma (2.2.8), we get $\overline{P} = H^2$.

We know that $H^2(G)$ is the set of all holomorphic on G, such that $fo\varphi \in H^2$

Suppose that $h \in H^2(G)$ is orthogonbal to all polynomials p(z), therefore from the definition of the inner product of $H^2(G)$, we get $<ho\varphi, po\varphi> = 0$, for all polynomials p, that is $ho\varphi \in P^\perp$

Since $\overline{P} = H^2$, then $P^\perp = 0$ and hence $ho\varphi = 0$

Since φ is holomorphic mapping of U onto G, we get h =0

So that the polynomials are dense in $H^2(G)$. ■

Combining theorem (2.2.3) and theorem (2.2.7), we obtain:

Corollary (2.2.9):

If C_φ is cyclic, then the set of polynomials in φ is dense in H^2. Equivalently, the set of polynomials in z is dense in $H^2(\varphi (U))$.

Let us say that a function $f \in H^2$ is univalent almost everywhere on ∂U provided that there is a set $E \subset \partial U$ having zero Lebesgue measure, such that f is univalent on $\partial U \backslash E$, [3].

The proof of the following theorem appeared in [3].

Theorem (2.2.10):

If C_φ is cyclic, then φ is univalent almost everywhere on ∂U.

We give the following proposition:

Proposition (2.2.11):

Let φ be analytic self map of U. If $\varphi'(0) = 0$, then C_φ is not cyclic.

Proof:

Let $A = (a_{ij})$ be the matrix of the operator C_φ with respect to the orthonormal basis $\{z^n\}_{n\geq 0}$, then the second row of this matrix is zero, hence $e_1(z) = z$ is orthogonal to the range of C_φ

Therefore the range of C_φ is not dense in H^2

Thus C_φ is not cyclic. ■

We give the following theorem:

Theorem (2.2.12):

Let φ be a holomorphic self map of U that has a fixed point $p \in U$ with $\varphi'(p) = 0$, then C_φ is not a cyclic operator.

Proof:

Let $\psi = \alpha_p o \varphi o \alpha_p$, where α_p is the special automorphism mapping

Hence $C_\psi = C_{\alpha p} o C_\varphi o C_{\alpha p}$, that is C_φ and C_ψ are similar

Since $\psi'(0) = \varphi'(p) = 0$, then from proposition (2.2.11), C_ψ is not cyclic and hence C_φ is not cyclic. ■

We give the following corollary:

Corollary (2.2.13):

Let $\varphi(z) = z^n\psi(z)$ be analytic self map of U, $n \geq 2$, then C_φ is not cyclic operator.

Proof:

It is clear that $\varphi(0) = 0$, $\varphi'(0) = 0$ and hence from theorem (2.2.12) C_φ is not cyclic operator. ■

In general if x is cyclic vector for the operator T, then Tx may not be a cyclic vector for T as the following example shows:

Example (2.2.14):

Let H be a Hilbert space and $\{e_n\}$ be an orthonormal basis for H. Define the operator $U : H \longrightarrow H$, as follows:

$U(e_n) = e_{n+1}$, $n = 0, 1, \ldots$

It is clear that e_0 is a cyclic vector for the operator U while $Ue_0 = e_1$ is not a cyclic vector.

We prove the following theorem:

Theorem (2.2.15):

Let f be a cyclic (hypercyclic, supercyclic) vector for C_φ, then $C_\varphi f$ is cyclic (hypercyclic, supercyclic) vector for C_φ.

Before we prove this theorem, we give the following lemma:

Lemma (2.2.16):

Let H be a Hilbert space and T be an operator on H. If T has a dense range and M is a dense set in H, then T(M) is a dense set in H.

Proof:

We claim that $T(H) \subseteq \overline{T(M)}$, in fact if $T(h) \in T(H)$, where $h \in H$, then by the density of M there exists $m_n \in M$, such that the sequence $\{m_n\}$ converges to h.

Since T is a continuous operator, then the sequence $\{Tm_n\}$ converges to T(h), hence $T(h) \in \overline{T(M)}$, so that $T(H) \subseteq \overline{T(M)}$

Thus $\overline{T(H)} \subseteq \overline{T(M)}$

Since T has a dense range, then $\overline{T(H)} = H$ and hence $\overline{T(M)} = H$. ■

Proof of the Theorem:

We prove this theorem when f is cyclic vector for C_φ, the proofs for the other cases are similar.

Let $M = \text{span}\{C_{\varphi n}(f), n = 0, 1, \ldots\}$, since f is a cyclic vector for C_φ, then $\overline{M} = H^2$. It is clear that:

$$\begin{aligned}\text{span}\{C_{\varphi n}(C_\varphi f), n = 0, 1, \ldots\} &= \text{span}\{C_\varphi (C_{\varphi n} f), n = 0, 1, \ldots\}\\ &= C_\varphi \text{span}\{C_{\varphi n}(f), n = 0, 1, \ldots\}\\ &= C_\varphi (M)\end{aligned}$$

Since $\overline{M} = H^2$ and C_φ has a dense range, then from lemma (2.2.16), $\overline{C_\varphi(M)} = H^2$

Thus $C_\varphi f$ is a cyclic vector for C_φ. ■

We turn our attention to the adjoint of composition operators. It is well-known that if T is a hypercyclic operator, then T* (the adjoint of T) has no eigenvalues [11]. For a supercyclic operator T, the adjoint T* has at most one eigenvalue [11].

The following results for the cyclicity of the adjoint of composition operators are new to the best of our knowledge.

Proposition (2.2.17):

Let φ be a holomorphic self map of U, then C_{φ}^{*} is not a hypercyclic operator.

Proof:

It is clear that 1 is an eigenvalue of the operator C_{φ}, so that C_{φ}^{*} is not hypercyclic operator. ■

The proof of the following lemma is well-known, thus it is omitted.

Lemma (2.2.18) [2]:

If T is a bounded operator on H, then $\|T\| = \|T^*\|$, where T* is the adjoint of the operator T.

The proof of the following proposition appeared in [1, theorem 2.2].

Proposition (2.2.19):

Suppose that T is a bounded linear operator on the Banach space X having the following properties:

(a) T is supercyclic.

(b) There exists $\mu > 0$, such that $\|T^n\| \leq \mu$, for each positive n.

Then for each $x \in X$, $T^n x \longrightarrow 0$ as $n \longrightarrow \infty$.

We give the following theorem:

Theorem (2.2.20):

Let φ be a holomorphic self map of U that fixes a point $p \in U$, then C_{φ}^{*} is not a supercyclic operator.

Proof:

Let $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where α_p is the special automorphism mapping, therefore $\psi(0) = 0$ and $C_\psi = C_{\alpha p} C_\varphi C_{\alpha p}$, that is C_ψ is similar to C_φ and hence C^*_ψ is similar to C^*_φ.

The following well-known norm estimate for composition operator shows the sequence $\{C^n_\psi\} = \{C_{\psi n}\}$ is bounded (see corollary (1.3.3))

$$\|C_{\psi n}\| \le \left(\frac{1+|\psi_n(0)|}{1-|\psi_n(0)|}\right)^{1/2} = 1, \text{ for all positive integer n}$$

Since $\|C^{*n}_\psi\| = \|C^n_\psi\|$ (lemma (2.2.18)), then the sequence $\{C^{*n}_\psi\}$ is bounded.

We can easily show that C^{*n}_ψ (1) = 1, for every positive integer n, therefore proposition (2.2.19) shows that C^*_ψ cannot be a supercyclic operator and hence C^*_φ is not a supercyclic operator. ■

<u>Corollary (2.2.21):</u>

If C_φ is a compact operator, then C^*_φ is not a supercyclic operator.

Proof:

Since C_φ is compact, then φ has an interior fixed point (proposition (1.5.5))

Thus C^*_φ is not a supercyclic operator. ■

In corollary (2.1.28) we see that if C_φ is hyponormal, then C_φ is not supercyclic. The following corollary shows that C^*_φ is also not a supercyclic operator.

<u>Corollary (2.2.22):</u>

Let φ be a holomorphic self map of U. If C_φ is hyponormal, then C^*_φ is not a supercyclic operator.

Proof:

Since 0 is a fixed point for φ, then C^*_φ is not supercyclic (theorem (2.2.20)). ■

We remark that although, the adjoint of a composition operator induced by a mapping φ with fixed point in U can never be supercyclic, it can be cyclic (see theorem (2.2.24) below)

We recall that an operator T is said to be upper triangular operator if T has upper triangular matrix $A = (a_{ij})$, i.e., $a_{ij} = 0$, for all $i > j$.

The proof of the following proposition appeared in [12].

Proposition (2.2.23):

Let T be an upper triangular operator whose diagonal entries with respect to some orthonormal basis for H are distinct, then T is cyclic.

We give the following theorem:

Theorem (2.2.24):

Let φ be a holomorphic self map on U and $\varphi(0) = 0$, $\varphi'(0) \neq 0$, $(\varphi'(0))^n \neq 1$, $\forall\, n = 1, 2, \ldots$, then C^*_φ is cyclic.

Proof:

Since $\varphi(0) = 0$, $\varphi'(0) \neq 0$, $(\varphi'(0))^n \neq 1$, $\forall\, n = 1, 2, \ldots$

Then $\varphi(z) = a_0 z + a_1 z^2 + \ldots$, $a_0 \neq 0$, $a_0^n \neq 1$, for all positive integer n.

The matrix of C_φ with respect to the orthonormal basis $\{z^n\}$ is:

$$A = \begin{bmatrix} 1 & & & \\ 0 & a_0 & & \text{O} \\ \vdots & & a_0^2 & \\ & * & & \ddots \end{bmatrix}$$

Therefore the matrix of C^*_φ is an upper triangular matrix

$$\bar{A}^t = \begin{bmatrix} 1 & & * & \\ & \bar{a}_0 & & \\ & & \bar{a}_0^2 & \\ O & & & \ddots \end{bmatrix}$$

Since the diagonal entries are distinct, then by proposition (2.2.23), C_φ^* is cyclic. ■

Corollary (2.2.25):

If φ has an interior fixed point p with $\varphi'(p) \neq 0$, $(\varphi'(p))^n \neq 1$, for all positive integer n, then C^*_φ is cyclic.

Proof:

Let $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where α_p is the special automorphism mapping, therefore $\psi(0) = 0$, $\psi'(0) = \alpha'_p(\varphi \circ \alpha_p(0))\ \varphi'(\alpha_p(0))\alpha'_p(0) = \varphi'(p) \neq 0$ and $(\psi'(0))^n = (\varphi'(p))^n \neq 1$, for all n

So that from theorem (2.2.24), C_ψ^* is cyclic

Since C_φ, C_ψ are similar, then C_φ^*, C_ψ^* are similar

Thus C_φ^* is cyclic. ■

Corollary (2.2.26):

Let φ be a non-elliptic analytic self map of U (φ is not the identity). If φ has a fixed point $p \in U$, $\varphi'(p) \neq 0$, then C_φ^* is cyclic.

Proof:

Since φ is non-elliptic with fixed point $p \in U$, then from theorem (1.4.19), $|\varphi'(p)| < 1$

Therefore $(\varphi'(p))^n \neq 1$, for all positive integer n, and hence from corollary (2.2.25), C^*_φ is cyclic. ■

We give the following lemma:

Lemma (2.2.27):

If T is an operator that has an eigenvalue of multiplicity greater than one, then T* is not cyclic.

Proof:

Let λ be an eigenvalue of multiplicity greater than one, therefore dim $\ker(T - \lambda I) \geq 2$

Since $\ker(T - \lambda I) = [R(T - \lambda I)^*]^{\perp}$, then $\dim[R(T - \lambda I)^*]^{\perp} \geq 2$

Hence $(T - \lambda I)^*$ is not cyclic (part (1) of remark (2.1.16))

Since $(T - \lambda I)^* = T^* - \bar{\lambda} I$, then T* is not cyclic (part (2) of remark (2.1.16)). ■

It is shown from theorem (1.4.19) that if φ has a Denjoy Wolff point $a \in \partial U$, then $0 < \varphi'(a) \leq 1$.

C. C. Cown proved in [6] the following theorem:

Theorem (2.2.28):

Let φ be a holomorphic self map of U that has Denjoy-Wolff point $a \in \partial U$ with $\varphi'(a) < 1$. If $\lambda \in C$, such that $(\varphi'(a))^{1/2} < |\lambda| < (\varphi'(a))^{-1/2}$, then λ is an eigenvalue for C_φ of infinite multiplicity.

The following corollary follows from theorem (2.2.28) and lemma (2.2.27)

Corollary (2.2.29):

If $w \in \partial U$ is the Denjoy Wolff point for φ with $\varphi'(w) < 1$, then C_φ^* is not cyclic.

Definition (2.2.30):

A non-constant sequence $\{z_k\}$ is an F-sequence for φ if $\varphi(z_k) = z_{k+1}$, for all k.

The following theorem appeared in [6], the proof is long thus is omitted.

Theorem (2.2.31):

Let φ be a holomorphic self map of U, has Denjoy Wolff point a in ∂U with $\varphi'(a) = 1$. If there is an F-sequence $\{z_k\}$ for φ for which:

$$\inf\left\{\left|\frac{z_k - z_{k+1}}{1 - z_k \overline{z}_{k+1}}\right| : k = 0,1,\ldots\right\} > 0$$

Then each λ with $|\lambda| = 1$ is eigenvalue of C_φ of infinite multiplicity.

The following corollary follows from theorem (2.2.31) and lemma (2.2.27).

Corollary (2.2.32):

If the conditions of theorem (2.2.31) are satisfied then C^*_φ is not a cyclic operator.

Definition (2.2.33):

Let φ be a holomorphic self map of U. φ is called an inner function if $|\varphi(z)| = 1$ almost every where on ∂U.

The following theorem appeared in [6]:

Theorem (2.2.34):

Let φ be an inner function, not linear fractional transformation with Denjoy Wolff point $a \in \partial U$. If $|\lambda| < (\varphi'(a))^{1/2}$, then λ is an eigenvalue for C^*_φ of infinite multiplicity.

The following corollary follows from theorem (2.2.34) and lemma (2.2.27).

Corollary (2.2.35):

Let φ be an inner function, not linear fractional transformation with Denjoy-Wolff point $a \in \partial U$, then C_φ is not cyclic.

We illustrate corollary (2.2.35) by the following example:

Example (2.2.36):

Let $\varphi(z) = (\alpha_{-1/3}(z))^2$, where $\alpha_{-1/3}$ is the special automorphism mapping, i.e., $\alpha_{-1/3}(z) = \dfrac{-\frac{1}{3} - z}{1 + \frac{1}{3}z}$, $(z \in U)$

We showed in proposition (1.1.14) that $\alpha_{-1/3}$ is conformal automorphism of U and it takes ∂U onto ∂U. Therefore:

$|\varphi(z)| = |\alpha_{-1/3}(z)|\,|\alpha_{-1/3}(z)| = 1$, for each $z \in \partial U$

Hence φ is inner function

To find the fix points of φ, we put $\varphi(z) = z$, therefore $\left(\dfrac{-\frac{1}{3} - z}{1 + \frac{1}{3}z}\right)^2 = z$

We simplify this equation, we have $z^3 - 3z^2 + 3z - 1 = 0$, therefore $(z - 1)^3 = 0$, that is the mapping φ(z) has only one fixed point $z = 1 \in \partial U$

Corollary (2.2.35) shows that the composition operator C_φ is not cyclic.

CHAPTER THREE

LINEAR FRACTIONAL CYCLICITY

INTRODUCTION

Let φ be a holomorphic function defined on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as follows:

$$\varphi(z) = \frac{az+b}{cz+d}$$

where a, b, c and d are complex numbers, then φ is said to be linear fractional transformation. In chapter one, we studied the basic properties of linear fractional transformation. In this chapter, we study the cyclicity of the composition operator induced by a linear fractional transformation. P. S. Bourdon and J. H. Shapiro [3], proved several theorems (see Table I), we give the details of the proofs. We give some new results, to the best of our knowledge, for the cyclicity of the adjoint of the composition operator induced by the linear fractional transformation. Also, we study the cyclicity of the composition operator induced by the function $\varphi(z) = \frac{z}{c-z}$, where c is a complex number and $\psi(z) = \alpha z + \beta$, α and β are complex numbers.

This chapter consists of three sections, in section one we study the cyclicily of C_φ and C_φ^* where φ has interior fixed point, we prove that if φ is elliptic, then C_φ is cyclic if and only if C_φ^* is cyclic. If φ is not elliptic, but linear fractional transformation with interior fixed point, then we prove that C_φ^* is cyclic .Note that when φ has interior fixed point then C_φ and C_φ^* are not hypercyclic (supercyclic) operators (see chapter two). In section two, we study the cyclicity when $\varphi \in$ LFT (U) has no interior fixed point. In this case, we show that if φ is automorphism, then C_φ is hypercyclic. If φ is not automorphism, then the third and fourth rows of table I show these cases.

In section three, we give some definitions and remarks, for example, we show if C_φ is multicyclic operator, then C_φ is cyclic, see [3] for more details. In chapter four, we show for example, that in contrast to what happens in the first row of table I, there exists a holomorphic (but not linear fractional) self map φ of U with interior and boundary fixed points, such that C_φ is cyclic (see example (4.1.16)).

Table I

Cyclic behaviour of C_φ, φ is linear fractional, not an automorphism.

Fixed points of φ (relative to U)	***Cyclicity of C_φ***	***Examples***
Interior & boundary	Not cyclic	$\varphi(z) = \frac{z}{2-z}$
Interior & exterior	Cyclic, not hypercyclic	$\varphi(z) = \frac{-z}{2+z}$
Exterior & boundary (hyperbolic)	Hypercyclic	$\varphi(z) = \frac{1+z}{2}$
Boundary only (parabolic)	Cyclic, not hypercyclic	$\varphi(z) = \frac{1}{2-z}$

3.1 LINEAR FRACTIONAL SELF MAPS OF U WITH INTERIOR FIXED POINT

In this section, we discuss the cyclicity for C_φ, where φ is linear fractional self map of U with interior fixed point. We summarize this section by the following theorem:

<u>Theorem (3.1.1):</u>

Suppose that φ is (not the identity) linear fractional self map of U, which has interior fixed point, then:

1. The operators C_φ and C_φ^* are not hypercyclic (supercyclic).
2. If φ is elliptic, then C_φ is cyclic if and only if C_φ^* is cyclic.

3. If φ is non-elliptic, then:
   (i) The operator C_φ^* is cyclic operator.
   (ii) The cyclicity of the operator C_φ depends on the nature of the fixed point for φ, that is:
   - If φ has interior and exterior fixed points, then C_φ is cyclic.
   - If φ has interior and boundary fixed points, then C_φ is not cyclic.

Part (1) of this theorem is proved in chapter two.

In chapter one, we showed that if φ is loxodromic or elliptic, then it has interior fixed point, hence C_φ and C_φ^* are not hypercyclic (supercyclic). We remark that although composition operators induced by elliptic mappings are not supercyclic, they can be cyclic as the following theorem shows:

Theorem (3.1.2) [3]:

If $\varphi \in$ LFT(U) is elliptic, then C_φ is cyclic if and only if the argument of λ is irrational multiple of π, where $\lambda = \varphi'(p)$, p is the interior fixed point of φ.

Before the proof, we need some preliminaries.

Proposition (3.1.3)[15]:

Let f be a holomorphic map on U and $Z(f) = \{a \in U : f(a) = 0\}$.If Z(f) has a limit point in U then f is the zero function.

We prove the following lemma:

Lemma (3.1.4):

Let f be a holomorphic map on U. If f vanishes at infinitely many points on a circle in U, then f is zero function.

Proof:

It is clear that Z(f) is bounded set in the complex plane.

If Z(f) has no limit point, then Z(f) is closed and hence by Heine-Borel theorem, Z(f) is compact.

This contradicts every compact infinite set has a limit point

So that z(f) has limit point, hence from proposition (3.1.3), f is the zero function. ■

Proof of Theorem (3.1.2):

Note that any elliptic self map on U has to be an automorphism of U with interior fixed point p, and hence must be conjugate (by automorphisms) to a rotation about the origin. Specifically, if φ is elliptic, then it is automorphism with fixed point $p \in U$ (proposition (1.4.25)).

Hence by the special automorphism mapping α_p, we get $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where ψ is an automorphism fixes the origin, so that by proposition (1.2.8), $\psi(z) = \lambda z$, for all $z \in U$, where $|\lambda| = 1$, $\lambda = \psi'(0) = \varphi'(p)$.

If $\arg\lambda$ is a rational multiple of π, then C_φ fails to be cyclic because in this case the orbit of any function in H^2 under C_ψ is a finite set [3].

If, however, $\arg\lambda$ is irrational multiple of π, then C_ψ is cyclic and K_α is cyclic vector for all $0 \neq \alpha \in U$. To see this, let $\alpha \neq 0$ be a point in U and f is orthogonal to $\text{orb}(C_\psi, K_\alpha) = \{K_{\bar{\lambda}^n \alpha} : n = 0, 1 \ldots\}$, that is $\langle f, K_{\bar{\lambda}^n \alpha} \rangle = 0$, for all non-negative integer n.

Therefore, $f(\bar{\lambda}^n \alpha) = 0$, $n = 0, 1, \ldots$; hence the function f vanishes at infinitely many points on the circle $|z| = |\alpha|$, therefore by lemma (3.1.4), f is the zero function

Because C_φ is similar to C_ψ, then C_φ is cyclic. ■

We give the following theorem:

<u>Theorem (3.1.5):</u>

If $\varphi \in LFT(U)$ is elliptic self map on U, then C_φ is cyclic if and only if C_φ^* is cyclic.

Proof:

Since φ is elliptic, then it is automorphism with interior fixed point p. As the proof of theorem (3.1.2), φ is conjugate via the special automorphism mapping α_p to the automorphism mapping ψ(z) = λz, λ ∈ ∂U. It is easy to prove that $C^*_{\lambda z} = C_{\bar{\lambda} z}$, where $\bar{\lambda}$ is the complex conjugate of λ. Since, C_φ is similar to $C_{\lambda z}$, then C^*_φ is similar to $C_{\bar{\lambda} z}$.

Since $\arg(\bar{\lambda}) = -\arg(\lambda)$, then from theorem (3.1.2), $C_{\lambda z}$ is cyclic if and only if $C_{\bar{\lambda} z}$ is cyclic, hence C_φ is cyclic if and only if C^*_φ is cyclic. ■

The following theorem is new to the best of our knowledge.

<u>Theorem (3.1.6):</u>

Let φ be a non-elliptic linear fractional self map of U (φ is not the identity mapping) have a fixed point p in U, then C^*_φ is cyclic.

Proof:

Suppose $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where α_p is the special automorphism mapping, therefore ψ(0) = 0. Without loss of generality, we may assume that:

$$\psi(z) = \frac{z}{az + b},\ b \neq 0$$

It is clear that $\psi'(0) = \frac{1}{b}$, hence $\psi'(0) \neq 0$. Form theorem (1.4.19), we have $|\psi'(0)| < 1$, therefore $(\psi'(0))^n \neq 1$, n = 1, 2, … corollary (2.2.25) shows that C^*_ψ is cyclic and hence by the similarity C^*_φ is cyclic. ■

<u>Corollary (3.1.7):</u>

If φ ∈ LFT(U) is loxodromic, then C^*_φ is cyclic.

Proof:

If φ is loxodromic, then by theorem (1.4.23), φ has interior fixed point, therefore by theorem (3.1.6), C^*_φ is cyclic. ■

Corollary (3.1.8):

If $\varphi \in$ LFT(U), (φ is not the identity), which fixes both interior point and boundary point of U, then C_{φ}^{*} is cyclic.

Proof:

Since φ has interior and boundary fixed points, then φ is not elliptic (theorem (1.4.23), hence C_{φ}^{*} is cyclic (theorem (3.1.6)). ■

The following theorem appeared in [3], we give the details of its proof:

Theorem (3.1.9):

If $\varphi \in$ LFT(U) is non-elliptic has a fixed point p in U and fixed point q outside the closure of U, then the composition operator C_{φ} is cyclic on H^2.

Before we prove this theorem we need the following lemma:

Lemma (3.1.10):

Let φ be a linear fractional self map of U take the form:

$\varphi(z) = \dfrac{az+1}{az+1-bz}$,where a, b are complex numbers and $a - b \neq 0$

Then we can write φ in terms of a reproducing kernel:

$\varphi = \lambda + \alpha K_{\bar{b}-\bar{a}}$, where $\lambda = 1 + \dfrac{b}{a-b}$, $\alpha = \dfrac{b}{b-a}$ and $K_{\bar{b}-\bar{a}}(z) = \dfrac{1}{1-(b-a)z}$

Proof:

$$\varphi(z) = \frac{az+1}{az+1-bz} = \frac{az+1-bz+bz}{az+1-bz} = 1 + \frac{bz}{az+1-bz}$$

$$= 1 + \frac{b}{a-b}\left(\frac{z+\frac{1}{a-b}-\frac{1}{a-b}}{z+\frac{1}{a-b}}\right) = 1 + \frac{b}{a-b}\left(1-\frac{\frac{1}{a-b}}{z+\frac{1}{a-b}}\right)$$

$$= 1 + \frac{b}{a-b} - \frac{b}{a-b}\left(\frac{1}{1-(b-a)z}\right) = \lambda + \alpha K_{\bar{b}-\bar{a}}(z)$$

Where $\lambda = 1 + \dfrac{b}{a-b}$ and $\alpha = \dfrac{b}{b-a}$. ■

Proof of Theorem (3.1.9):

Let $\psi = \alpha_p \circ \varphi \circ \alpha_p$, where α_p is the special automorphism mapping. It is clear that $\psi(0) = 0$, hence $\psi(z) = \dfrac{z}{az+b}$, $b \neq 0$

Therefore, $\psi'(0) = \dfrac{1}{b}$. By theorem (1.4.19), $|\psi'(0)| < 1$, hence $|b| > 1$. The other fixed point is $\dfrac{1-b}{a}$ (if a = 0, then the fixed point is ∞). Since α_p is self inverse, then $\psi(\alpha_p(z)) = \alpha_p(\varphi(z))$, hence $\psi(\alpha_p(q)) = \alpha_p(q)$, that is $\alpha_p(q)$ is fixed point for the mapping ψ, therefore $\alpha_p(q) = \dfrac{1-b}{a}$

Since q is outside $\bar{U}$, then by proposition (1.1.14), $\alpha_p(q)$ is outside $\bar{U}$.

Thus $\left|\dfrac{1-b}{a}\right| > 1$ or equivalently

$$\left|\frac{a}{1-b}\right| < 1 \qquad \text{.......... (3.1)}$$

We claim that for any non-zero $\alpha \in U$, the reproducing kernel function $K_\alpha(z) = \dfrac{1}{1-\bar{\alpha}z}$ is cyclic vector for C_ψ

A straight forword induction argument shows that for any non-negative integer n:

$$K_\alpha \circ \psi_n(z) = \frac{as_n z + 1}{as_n z + 1 - \bar{\alpha} z b^{-n}}$$

Where $s_0 = 0$ and for positive n, $s_n = \sum_{k=1}^{n} \frac{1}{b^k}$

Now, fix a vector $g \in H^2$ that is orthogonal to the orbit $\{K_\alpha o\psi_n : n = 0, 1, \ldots\}$

We claim that g is the zero function. To see this, note that the sequence $K_\alpha o\psi_n$ converges to 1 in H^2, therefore:

$$0 = \lim_n \langle g, K_\alpha o\psi_n \rangle = g(0)$$

Recalling from lemma (3.1.10), we can write $K_\alpha o\psi_n$ in terms of a reproducing kernel:

$$K_\alpha o\psi_n = \lambda_n + \gamma_n K_{\beta n}$$

Where λ_n and γ_n are complex constants

$$\gamma_n = \frac{\overline{\alpha} b^{-n}}{\overline{\alpha} b^{-n} - aS_n} \neq 0 \text{ and } \beta_n = \frac{\alpha}{\overline{b}^n} - \overline{a}\overline{S}_n$$

Thus the orthogonality hypothesis on g yield:

$$0 = \langle g, K_\alpha o\psi_n \rangle = \overline{\lambda}_n g(0) + \overline{\gamma}_n g(\beta_n) = \overline{\gamma}_n g(\beta_n)$$

Thus g vanishes identically on the sequence $\{\beta_n\}$. Upon recalling that $|b| > 1$, we see from he definition that $\{S_n\} \longrightarrow \frac{1}{b-1}$, hence $\beta_n \longrightarrow \frac{-\overline{a}}{\overline{b}-1}$, where by inequality (3.1), this limit belongs to U.

Thus g vanishes on a sequence with limit point in U, hence from proposition (3.1.3), g is the zero function. This shows that K_α is cyclic for C_ψ

Since C_φ and C_ψ are similar, then C_φ is also cyclic. ■

<u>Corollary (3.1.11):</u>

If $\varphi \in$ LFT(U) is loxodromic, then C_φ is cyclic.

Proof:

Since φ is loxodromic, then φ has an interior fixed point and exterior fixed point

Hence C_φ is cyclic by theorem (3.1.9). ■

The proof of the following theorem is very long, thus is omitted.

Theorem (3.1.12):

Suppose that φ is a linear fractional self map of U which fixes both an interior and a boundary point of U. Then C_φ is not cyclic. In fact, the closed linear span of any orbit has infinite codimension in H^2.

We end this section by studying the cyclicity of the operator C_φ and its adjoint, where $\varphi(z) = \frac{z}{c-z}$, where c is a complex number.

Proposition (3.1.13):

The mapping $\varphi(z) = \frac{z}{c-z}$, where $|c| \geq 2$ is analytic self map of U.

Proof:

Since $|c| \geq 2$ and $|c - z| \geq |c| - |z|$, for all $z \in U$, then $|c - z| > 1$, so that

$|\varphi(z)| = \frac{|z|}{|c-z|} < 1$, for all $z \in U$

Thus φ is holomorphic self map of U. ■

Remark (3.1.14):

It is clear that φ fixes the origin, so that C_φ and C_φ^* are not supercyclic operators (theorem (2.1.18) and theorem (2.2.20))

We prove the following theorem:

Theorem (3.1.15):

Let $\varphi(z) = \frac{z}{c-z}$, $|c| \geq 2$, then C_φ^* is cyclic operator.

Proof:

It is clear that $\varphi(0) = 0$ and $\varphi'(0) = \frac{1}{c}$, so that $0 < |\varphi'(0)| < 1$

Thus C_φ^* is cyclic operator (theorem (2.2.24)). ■

Remark (3.1.16):

It is easy to prove that 0 and $c-1$ are the fixed points of the mapping $\varphi(z)=\dfrac{z}{c-z}$, $|c|\geq 2$.

We prove the following theorem:

Theorem (3.1.17):

Let $\varphi(z)=\dfrac{z}{c-z}$, $|c|\geq 2$

1. If c =2, then C_φ is not cyclic.
2. If $c \neq 2$, then C_φ is cyclic.

Proof:

1. It is clear that φ has 0 and 1 as fixed points. Thus C_φ is not cyclic (theorem (3.1.12)).
2. If $c \neq 2$, then φ has interior fixed point {0} and exterior fixed point $c-1$, hence C_φ is cyclic (theorem (3.1.9)). ■

3.2 LINEAR FRACTIONAL SELF MAPS OF U WITH NO INTERIOR FIXED POINT

In this section, we consider the linear fractional self map φ of U that has no interior fixed point. Three cases exhaust the possibilities:

- φ is an automorphism. In this case, we prove that C_φ is hypercyclic.
- φ is not an automorphism and not parabolic, so that it has two fixed points; the attractive one necessarily on ∂U, the other necessarily outside the closure of U. In this case, we show that C_φ is again hypercyclic.
- φ is parabolic, but not an automorphism. In this case, φ has only one fixed point, which necessarily lies on ∂U. We show that C_φ is strongly non-hypercyclic (theorem (3.2.20)), not supercyclic (theorem (3.2.27)) and cyclic (theorem (3.2.25)).

The following proposition appeared in [3], we give the details of the proof.

Proposition (3.2.1):

Let Z_w denote the collection of functions that are continuous on the closed unit disc, analytic on the interior, and which vanish at $w \in \partial U$. Then Z_w is dense in H^2.

Proof:

Suppose that:

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2$$

is orthogonal to Z_w, then for every non-negative integer n, the polynomial $z^{n+1} - wz^n$ belongs to Z_w, so it is orthogonal to f. Thus:

$$0 = \langle f, z^{n+1} - wz^n \rangle = \hat{f}(n+1) - \overline{w}\, \hat{f}(n)$$

From this, it follows that $\hat{f}(n) = \overline{w}^n \hat{f}(0)$, for all n. Since w is on the unit circle and $f \in H^2$, this forces $\hat{f}(0) = 0$ and therefore all the Taylor coefficients of f must vanish

Thus $f \equiv 0$, so Z_w is dense in H^2. ■

Remark (3.2.2):

As in the proof of proposition (3.2.1), we can prove that if $\alpha \notin U$, then the set of polynomials that vanish at α is dense in H^2.

Notation:

The statement $T^n \longrightarrow 0$ on a set X means that $\|T^n x\| \longrightarrow 0$, for every vector $x \in X$.

The proof of the following theorem appeared in [17].

Theorem (The Hypercyclic Criterion) (3.2.3):

Suppose that there is a dense subset X of a Hilbert space H on which $T^n \longrightarrow 0$ on X, and another dense set Y on which is defined a (possibly discontinuous) map $S : Y \longrightarrow Y$, such that:

(a) TS is the identity on Y.

(b) $S^n \longrightarrow 0$ on Y.

Then T is hypercyclic operator.

We showed in proposition (1.4.17) that if $\varphi \in$ LFT(U) has two fixed points α and β, then $\varphi'(\alpha) = \lambda$ and $\varphi'(\beta) = 1/\lambda$, where λ is the multiplier for φ. If φ is automorphism, then it is clear that φ^{-1} is also automorphism and fixes the same points α, β and $\varphi^{-1\prime}(\alpha) = 1/\lambda$, $\varphi^{-1\prime}(\beta) = \lambda$

We prove the following proposition:

Proposition (3.2.4):

Suppose that φ is hyperbolic self map of U, then φ is automorphism if and only if the two fixed points for φ lie on ∂U.

Proof:

If φ has two boundary fixed points, then from proposition (1.4.26), φ is an automorphism.

Conversely, if φ is an automorphism, then φ has no interior fixed point (otherwise φ is elliptic (see proposition (1.4.25))

Therefore, from theorem (1.4.19), φ has attractive fixed point $\alpha \in \partial U$ with $0 < \varphi'(\alpha) = \lambda < 1$, where λ is the multiplier for φ (if $\varphi'(\alpha) = \lambda = 1$, then φ is the identity mapping), the other fixed point β outside U (theorem (1.4.23))

It is clear that φ^{-1} is automorphism with no interior fixed point and:

$\varphi^{-1\prime}(\alpha) = 1/\lambda > 1$, $\varphi^{-1\prime}(\beta) = \lambda < 1$

Therefore, from theorem (1.4.19), β is attractive fixed point for φ^{-1} with $\beta \in \partial U$. ■

The following proposition shows that every automorphism mapping of U is linear fractional self map on U

Proposition (3.2.5) [20]:

If φ is automorphism mapping, then there exists $p \in U$ and $w \in \partial U$, such that $\varphi = w\alpha_p$, where α_p is the special automorphism self map of U.

Proof:

Since $0 \in U$ and φ is a conformal automorphism, then there exists $p \in U$, such that $\varphi(p) = 0$. Define $\psi = \varphi \circ \alpha_p$, where α_p is the special automorphism mapping.

It is clear that ψ fixes the origin, since φ and α_p are automorphisms, then ψ is automorphism

Hence by proposition (1.2.8), there exists $w \in \partial U$, such that $\psi(z) = \varphi(\alpha_p(z)) = wz$, for all $z \in U$

Since α_p is self inverse, then $\varphi(z) = w\alpha_p(z)$, for all $z \in U$. ■

We remark that if φ is automorphism mapping with no interior fixed point of U, then either φ is parabolic automorphism or hyperbolic automorphism.

If φ is parabolic automorphism, then φ fixes only one point $\alpha \in \partial U$ with $\varphi'(\alpha) = 1$, see chapter 1. In this case φ^{-1} is also parabolic and fixes the same point. Therefore, from proposition (1.4.22), α is attractive fixed point for φ and φ^{-1}.

If φ is hyperbolic automorphism, then from the proof of proposition (3.2.4), φ has two boundary fixed points α, β where α is attractive fixed point for φ and β is attractive fixed point for φ^{-1}.

The following theorem appeared in [19], we give the details of its proof.

Theorem (3.2.6):

Suppose that φ is a conformal automorphism of U with no fixed points in the interior of U., then C_φ is hypercyclic on H^2.

Proof:

We note from proposition (3.2.5) that φ is linear fractional automorphism mapping of U.

Since φ has no interior fixed point, then φ is not elliptic. If φ is parabolic, then φ and φ^{-1} have the same attractive fixed point a on ∂U. If φ is not parabolic, then φ has two fixed points a, b on ∂U, where a is attractive fixed point for φ and b is attractive fixed point for φ^{-1}.

In order to treat both cases simultaneously, we set a = b if φ is parabolic

Let Z_a be the set of functions that are continuous on the closed unit disc, analytic on the interior and which vanish at a, and define Z_b similarly.

According to proposition (3.2.1), these sets are dense in H^2

We claim first that $C_\varphi^n \longrightarrow 0$ on Z_a

For this, note that for every z ∈ ∂U\{b}, we have $\varphi_n(z) \longrightarrow a$ (see proposition (1.4.24)), hence if $f \in Z_a$, then $f(\varphi_n(z)) \longrightarrow f(a) = 0$.

Upon applying the elementary case of the boundary integral representation of the H^2 norm (proposition (1.2.4)), we obtain:

$$\|C_\varphi^n f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\varphi_n(e^{i\theta}))|^2 \, d\theta \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

To finish the proof let $S = C_\varphi^{-1} = C_{\varphi^{-1}}$. As noted above, φ^{-1} is also automorphism of U with attractive fixed point b.

So, if we take the set Z_b, then S maps Z_b into itself, and the previous argument applied to show that $S^n \longrightarrow 0$, on Z_b.

Since the set Z_b is dense in H^2 (proposition (3.2.1)), then the hypothesis of hypercyclicity criterion are therefore satisfied with $T = C_\varphi$, $S = C_{\varphi^{-1}}$, $X = Z_a$ and $Y = Z_b$

So C_φ is hypercyclic operator. ■

We give the following proposition:

Proposition (3.2.7):

Let φ be a holomorphic self map of U with no interior fixed point. If φ is automorphism of U, non parabolic, then C_φ^* is not cyclic.

Proof:

Since φ is automorphism of U, then φ is linear fractional self map of U (proposition (3.2.5)).

Since φ has no interior fixed point, then the Denjoy-Wolff point w for φ belongs to ∂U with $\varphi'(w) \le 1$ (theorem (1.4.19)). Since φ is not parabolic, then $\varphi'(w) \neq 1$ that is $\varphi'(w) < 1$

Therefore from corollary (2.2.29), C_φ^* is not cyclic. ■

We need the following useful theorem:

Theorem (Walsh's Theorem) (3.2.8) [19]:

Suppose that G is simply connected domain whose boundary is a Jordan curve (A Jordan curve is, by definition, a holomorphic image of the unit circle). Let the holomorphic function F map U univalently onto G, then the polynomials in F are dense in H^2.

Corollary (3.2.9):

If φ maps U onto the interior of a Jordan curve lying in U, then C_φ has a dense range.

Proof:

From Walsh's theorem, we have the set of all polynomials in φ is dense in H^2. Since this set is a subset of the range of C_φ, then the range of C_φ is dense in H^2. ■

Remark (3.2.10):

We shall see in the next chapter that if φ maps the unit disk onto the interior of a Jordan curve lying in U, then C_φ is cyclic.

We prove the following lemma:

Lemma (3.2.11):

Suppose φ ∈ LFT(U) that is not an automorphism and does not have an interior fixed point. If φ is not parabolic, then φ is conjugate by an appropriate disk automorphism to self map ψ(z) = az + 1 − a, where 0 <a < 1.

Proof:

By the observations made earlier in this section, φ has its attractive fixed point p on ∂U and the other fixed point q outside the closure of U.

We may assume without loss of generality that φ fixes p =1 and q in the outside $\overline{U}$ (if p ≠1, then $(-p\alpha_r)^{-1} \circ \varphi \circ (-p\alpha_r)$ fixes 1 and $z_0 = \alpha_r(-\frac{q}{p})$ outside $\overline{U}$, where r is real number 0 < r < 1 and α_r is the special automorphism mapping).

Let $T(z) = w\alpha_{\frac{1}{\overline{q}}}(z)$, for all z ∈ U, where $w = \dfrac{\overline{q}(q-1)}{q(1-\overline{q})}$

It is easy to prove that T is automorphism of U and T(1) = 1, T(q) = ∞.

Suppose $\psi = T \circ \varphi \circ T^{-1}$, hence ψ is self map of U and ψ(1) = 1, ψ(∞) = ∞, therefore ψ must have the form ψ(z) = az + 1 − a

We claim that 0 < a < 1

Since ψ has no interior fixed point, then ψ is not elliptic, not loxodromic. Since ψ fixes two points, then ψ is not parabolic, hence ψ must be hyperbolic

Since 1 is the Denjoy-Wolff point for ψ and ψ has no interior fixed point and non-elliptic, then from theorem (1.4.19), $0 < \psi'(1) = a < 1$. ■

Remark (3.2.12):

Let $\psi \in$ LFT(U) of the form $\psi(z) = az + 1 - a$, $0 < a < 1$, then it is easy to prove that ψ is an automorphism of the half plane $G = \{z : \text{Re } z < 1\}$.

The following theorem appeared in [3] we give the details of its proof.

Theorem (3.2.13):

Suppose φ is a linear fractional self map of U that is not automorphism and does not have interior fixed point. If φ is not parabolic, then C_φ is hypercyclic.

Proof:

By the observation made earlier in this section, φ has an attractive fixed point on ∂U and fixed point outside the closure of U. Now, by lemma (3.2.11), we may assume without loss of generality that $\varphi(z) = az + 1 - a$, where $0 < a < 1$, φ fixes 1 and ∞.

Note that by remark (3.2.12), φ is automorphism of the half plane $G = \{z : \text{Re } z < 1\}$

Hence, if σ is a linear fractional transformation mapping from G onto U, then:

$$\psi = \sigma \circ \varphi \circ \sigma^{-1} \qquad (3.2)$$

It is clear that ψ is automorphism of U. We claim that ψ does not have interior fixed point. In fact, if ψ fixes $z_0 \in$ U, then $z_0 = \psi(z_0) = \sigma\varphi\sigma^{-1}(z_0)$, that is $\varphi(\sigma^{-1}(z_0)) = \sigma^{-1}(z_0)$, therefore either $\sigma^{-1}(z_0) = 1$ or ∞. This contradict the range of $\sigma^{-1}$is G, hence ψ does not have an interior fixed point.

By theorem (3.2.6), C_ψ is hypercyclic. Because σ maps G onto U and $U \subset G$, then σ(U) must also be a subset of U, in fact σ(U) is Jordan domain; therefore by corollary (3.2.9), C_σ has dense range.

By (3.2), $C_\varphi o C_\sigma = C_\sigma o C_\psi$. Since C_ψ is hypercyclic and C_σ has dense range, then C_φ is hypercyclic (theorem (2.1.2)). ■

Corollary (3.2.14):

Let $\varphi \in$ LFT(U) has boundary and exterior fixed points, then C_φ is hypercyclic.

Proof:

Since φ has no interior fixed point, then φ is not elliptic, not loxodromic. Since φ has two fixed points, then φ is not parabolic, therefore φ is hyperbolic

Since φ has boundary and exterior fixed points, then φ is not automorphism, so that C_φ is hypercyclic. ■

If φ satisfies the conditions of theorem (3.2.13), then φ(z) = az + 1 − a, 0 < a < 1, hence it is easy to prove that $\{a^n\}_{n=0}^{\infty}$ are eigenvalues of C_φ. Thus, C_φ^* is not supercyclic.

The following theorem shows more:

Theorem (3.2.15):

If the conditions of theorem (3.2.13) are satisfied, then C_φ^* is not cyclic.

Proof:

From the proof of theorem (3.2.13), we have φ(z) = az + 1 − a, 0 < a < 1

Note that, φ fixes 1 and ∞ and φ′(1) = a < 1, hence C_φ^* is not cyclic (theorem (2.2.29)). ■

In the previous section, we studied the cyclicity of the composition operator induced by the holomorphic mapping $\varphi(z) = \dfrac{z}{c-z}$, (z ∈ U). Here, we discuss the mapping ψ(z) = az + b, a ≠ 0, b ≠ 0. It is easy to prove that ψ maps U onto the ball with center at b and radius |a|.

We prove the following proposition:

Proposition (3.2.16):

Suppose $\psi(z) = az + b$, $a \neq 0$, $b \neq 0$, then ψ is self map of U if and only if $|a| + |b| \leq 1$.

Proof:

If $|a| + |b| \leq 1$, then:

$|\psi(z)| = |\psi(z) - b + b| \leq |\psi(z) - b| + |b| < |a| + |b| \leq 1$, for all $z \in U$

Thus ψ is self map of U

Conversely, suppose ψ is self map of U and $|a| + |b| > 1$, let $\varepsilon > 0$ (sufficiently small), such that $|a| + |b| - \varepsilon > 1$

We claim that $\left(\frac{|a|-\varepsilon}{|b|}+1\right)b \in$ Range $\psi = B_{|a|}(b)$, where $B_{|a|}(b)$ is the ball of center at b and radius $|a|$.

$$\left|\left(\frac{|a|-\varepsilon}{|b|}+1\right)b - b\right| = |a| - \varepsilon < |a|$$

Therefore $\left(\frac{|a|-\varepsilon}{|b|}+1\right)b \in$ Rangeψ, that is there exists $w \in U$, such that $\psi(w) = \left(\frac{|a|-\varepsilon}{|b|}+1\right)b$, therefore $|\psi(w)| = \left|\left(\frac{|a|-\varepsilon}{|b|}+1\right)b\right| = \left(\frac{|a|-\varepsilon}{|b|}+1\right)|b| = |a| - \varepsilon + |b| > 1$

This contradicts the fact that ψ is self map of U, so that $|a| + |b| \leq 1$. ■

Remark (3.2.17):

One can show easily that if $\psi(z) = az + b$, $a \neq 0$, $b \neq 0$, $|a| + |b| \leq 1$, then ψ has two fixed points $p = \frac{b}{1-a}$ and ∞.

Theorem (3.2.18):

Let $\varphi(z) = az + b, a \neq 0, b \neq 0$ be a holomorphic self map of U

(i) If $|b| = 1 - a$, then C_φ is hypercyclic and C_φ^* is not cyclic.

(ii) If $|b| \neq 1 - a$, then C_φ and C_φ^* are cyclic, but not hypercyclic (supercyclic) operators.

Proof:

(i) If $|b| = 1 - a$, φ has boundary fixed point $p = \frac{b}{1-a}$ and exterior fixed point ∞, hence C_φ is hypercyclic (corollary (3.2.14)), and C_φ^* is not cyclic (theorem (3.2.15)).

(ii) If $|b| \neq 1 - a$, then $|p| = \frac{|b|}{|1-a|} < \frac{|b|}{1-|a|} \leq 1$, hence p is interior fixed point. Thus C_φ and C_φ^* are not supercyclic and C_φ is cyclic operator (see the previous section). Since $\varphi'(p) = a \neq 0$ and $|\varphi'(p)| = |a| < 1$, then C_φ^* is cyclic (corollary (2.2.25)). ■

Remark (3.2.19):

Theorem (3.2.18) says, for example that if $\varphi(z) = az + b, a \neq 0, b \neq 0$ is self map of U, then:

1. C_φ is cyclic and C_φ^* is not supercyclic operator.
2. If a is a complex number with Im $a \neq 0$, then C_φ and C_φ^* are cyclic but not supercyclic.

We conclude the linear fractional cyclicity by studying the parabolic linear fractional self maps of U. If $\varphi \in$ LFT(U) is parabolic automorphism, then by theorem (3.2.6), C_φ is hypercyclic operator.

The proof of the following theorem is very long, thus is omitted.

Theorem (3.2.20) [19]:

Let φ be a linear fractional self map of U. If φ is parabolic non-automorphism, then C_φ is strongly non-hypercyclic, in the sense that the only functions that can adhere to C_φ-orbits are constant functions.

Lemma (3.2.21):

If $\varphi \in$ LFT(U) fixes the point $p \in \partial U$, then φ is conjugate by an appropriate automorphism mapping to a mapping ψ that fixes 1.

Proof:

Define f(z) = pz ($z \in U$). It is clear that f is automorphism mapping of U and f(1) = p. If $\psi = f^{-1}\varphi f$, then ψ is self map of U and fixes the point 1. ■

Lemma (3.2.22):

If $\varphi \in$ LFT(U) is parabolic with fixed point at 1, then $\varphi(z) = \frac{(2-a)z+a}{-az+(2+a)}$, where Re(a) ≥ 0 and Re a = 0 if and only if φ is automorphism.

Proof:

Let $T(z) = \frac{1+z}{1-z}$ ($z \in U$). One can prove that T maps U onto $\Pi = \{z : \text{Re } z > 0\}$ and $T(1) = \infty$

Define $\psi = T \circ \varphi \circ T^{-1}$, it is clear that ψ takes Π into itself and fixes ∞, hence, $\psi(z) = \lambda z + a$, where the fact that ψ preserves the right half plane forces $\lambda > 0$ and Re a≥0. By the chain rule, $\psi' = T'(\varphi \circ T^{-1}).\varphi'(T^{-1}).T^{-1\prime}$, so that $\lambda = \psi'(\infty) = 1$

Thus ψ(z) =z + a, Re a ≥ 0, therefore:

$$\varphi(z) = T^{-1} \circ \psi \circ T(z) = \frac{(2-a)z+a}{-az+(2+a)}, \text{ where Re } a \geq 0$$

It is easy to show that Re a = 0 if and only if ψ is automorphism mapping on Π. Since $\varphi = T^{-1}\psi T$ and T is maps U onto Π, then Re a = 0 if and only if φ is automorphism of U. ■

Definition (3.2.23) [17]:

Let f be a holomorphic on U. The zero-sequence of f is the collection of its zeros, listed in order of increasing moduli with each zero written down as many times as its multiplicity.

The proof of the following theorem appeared in [17].

Theorem (The Zero Sequence Theorem)(3.2.24):

Suppose $\{a_n\}$ is the zero-sequence of a function $f \in H^2$ that is not identically zero, then;

$$\sum_{n=1}^{\infty}(1-|a_n|) < \infty$$

The following theorem appeared in [3], we give the details of its proof.

Theorem (3.2.25):

Every parabolic linear fractional self map of U induces a cyclic composition operator on H^2.

Before the proof, we need the following lemma:

Lemma (3.2.26):

Let a be a complex number, such that Re(a) > 0.If $\beta_n = \dfrac{na}{2+na}$, n = 1, 2, …; then $\sum_n (1-|\beta_n|)$ is a divergent series.

Proof:

It is clear that $|\beta_n| < 1$ and $1 - |\beta_n|^2 = \dfrac{4(1+\text{Re}(na))}{4+4\text{Re}(na)+|na|^2}$. Now:

$$\frac{1}{n}\left(\frac{1}{1-|\beta_n|^2}\right) = \frac{1}{n}\left(\frac{4(1+\text{Re}(na))+|na|^2}{4(1+\text{Re}(na))}\right) = \frac{1}{n}\left(1+\frac{|na|^2}{4(1+\text{Re}(na))}\right)$$

$$< \frac{1}{n}\left(1+\frac{n^2|a|^2}{4n\,\mathrm{Re}(a)}\right) = \frac{1}{n}+\frac{|a|^2}{4\,\mathrm{Re}(a)}$$

$$\le 1 + \frac{|a|^2}{4\,\mathrm{Re}(a)}$$

Therefore $1 - |\beta_n|^2 > \frac{1}{n}r$, where $r = \left(1+\frac{|a|^2}{4\,\mathrm{Re}(a)}\right)^{-1}$

Hence $\frac{1}{n}r < (1 - |\beta_n|)(1 + |\beta_n|)$

Since $|\beta_n| < 1$, then $\frac{1}{n}\frac{r}{2} < 1 - |\beta_n|$. Since $\sum \frac{1}{n}\frac{r}{2}$ diverges, then $\sum_n (1-|\beta_n|)$ is divergent series. ■

Proof of theorem (3.2.25):

We have proved in theorem (3.2.6) that parabolic automorphisms induce hypercyclic composition operators. So, we need only to consider parabolic self maps φ of U that are not automorphism, for such a φ, we will show that the identity map u, defined on U by $u(z) = z$ ($z \in U$) is a cyclic vector for C_φ .

By lemma (3.2.21), without loss of generality, we may assume that 1 is the fixed point of φ.

Recall from lemma (3.2.22), that φ is of the form:

$$\varphi(z) = \frac{(2-a)z+a}{-az+(2+a)},\ z \in U$$

For some complex number a with $\mathrm{Re}(a) > 0$ (the strict positivity of Re(a) reflecting the fact that φ is not an automorphism)

For our purpose, a more convenient expression for φ is:

$$\varphi = \bar{\gamma} + \bar{\alpha} K_\beta$$

Where $\bar{\gamma} = \frac{a-2}{a}$, $\bar{\alpha} = \frac{4}{a(a+2)}$, $\bar{\beta} = \frac{a}{2+a}$ and $K_\beta = (1 - \bar{\beta}z)^{-1}$.

The requirement that Re(a) > 0 insure that none of the denominators in the definitions of $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ is zero, as discussed in example (1.1.17), $<f, K_\beta> = f(\beta)$, for all $f \in H^2$.

Now,suppose $f\in H^2$ is orthogonal to the C_φ -orbit of u .that is,suppose $<f, \varphi_n> = 0$, for all n = 0, 1… where $\varphi_0 = u$. Since φ_n is pointwise convergent to 1, so

$$0 = \lim <f, \varphi_n> = <f, 1> = f(0)$$

Using this along with the orthogonality of φ and f, we have:

$$0 = <f, \varphi> = <f, \overline{\gamma} + \overline{\alpha}\ K_\beta> = \gamma<f, 1> + \alpha<f, K_\beta> = \alpha f(\beta)$$

So that f(β) = 0. But f is also orthogonal to φ_n, for each n and the formula for φ_n is obtained from that of φ by replacing a with na.

Thus the last calculation actually shows that the function f vanishes identically on the sequence of points:

$$\beta_n = \frac{na}{2+na},\ n = 1, 2, \ldots$$

From lemma (3.2.26) we have $\sum_n (1-|\beta_n|) = \infty$, hence by the zero sequence theorem f must vanish identically on U.

We have shown that only the zero vector can be orthogonal to the C_φ-orbits of u, therefore u is a cyclic vector for C_φ. ■

The following theorem completes the proof of cyclicity of C_φ, where $\varphi \in$ LFT(U).

Theorem (3.2.27) [18]:

If φ is a parabolic linear fractional self map of U that is not an automorphism, then C_φ is not supercyclic.

The proof is long, so is omitted.

We give the following proposition:

<u>Proposition (3.2.28):</u>

If φ is parabolic self map of U, then C_φ^* is not supercyclic operator.

Proof:

Without loss of generality, we may assume that 1 is the fixed point for φ. The author in [18] shows that for each t ≥ 0, $e_t(z) = \left\{-t\frac{1+z}{1-z}\right\}$, (z ∈ U) is an eigenvector of C_φ with corresponding eigenvalue e^{iat}, where a is the Translation parameter [18] .Thus C_φ^* is not supercyclic operator. ■

3.4 REMARKS

Herrero introduced the corresponding hypercyclic idea; an operator T on a Banach space is called multihypercyclic if there is a finite subset of the space, the union of whose orbits is dense.

The operator T is called muticyclic if there exists a finite subset of the space for which the smallest T-invariant subspace is the whole space.

It is clear that if T is hypercyclic, then T is multihypercyclic. The author in [14] shows that the converse is true, that is every multihypercyclic operator is hypercyclic.

This fact in general is not true when T is cyclic operator that is if T is multicyclic, then T is not necessarily cyclic as the following example shows:

Let H be a Hilbert space and $\{e_n\}$ is orthonormal basis for H. If U is the forward shift operator, i.e., $U(e_i) = e_{i+1}$, i = 1, 2, …, then it is clear that U^2 is multicyclic operator where the orbit of e_1 and e_2 has dense span but U^2 is not cyclic operator, because the codimension of the range of U^2 is two.

Paul S. Bourdon and Joel H. Shapiro proved in [3] that if φ ∈ LFT(U) and C_φ is not cyclic composition operator, then every finitely generated invariant subspace of such an operator has infinite codimension. Therefore, if C_φ is not cyclic operator, where φ ∈ LFT(U) then it is not multicyclic operator. In other words, if $\varphi \in$ LFT(U) and C_φ is multicyclic operator, then C_φ is cyclic operator.

The author in [3] shows the following:

If the operator T on X satisfies the hypothesis of the hypercyclicity criterion (theorem (3.2.3)), then for any subsequence $\{n_k\}$ of positive integers, there exists $f \in X$, for which the set $\{T^{n_k} f\}$ is dense in X.

Let us call operators for which the last conclusion is true strongly hypercyclic. Since we used the hypercyclicity criterion to establish hypercyclicity and since the linear fractional maps that do not satisfy its hypotheses are also not hypercyclic, our work actually shows:

Every hypercyclic composition operator C_φ where $\varphi \in$ LFT(U) is strongly hypercyclic.

CHAPTER FOUR

LINEAR FRACTIONAL MODELS

INTRODUCTION

In chapter one, we observed that the linear fractional self maps of U fall naturally into several categories, determined by position of, and behaviour at the Denjoy-Wolff point.

In this chapter, we classify the arbitrary holomorphic self maps of U into the following types (see [3] for more details).

- Dilation type, if the Denjoy-Wolff point is in U.
- Hyperbolic type, if the Denjoy-Wolff point is on ∂U, and has derivative < 1 there.
- Parabolic type, if the Denjoy-Wolff point is on ∂U, the derivative is $= 1$ there.

The linear fractional model theorem (4.1.3) tells us that every univalent self map φ of U can be represented as:

$$\varphi = \sigma^{-1} \circ \psi \circ \sigma$$

where σ is univalent map $\sigma : U \longrightarrow C$, and $\psi \in$ LFT(U) has the same type of φ, that is if φ is of dilation type then ψ has interior fixed point. If φ is of hyperbolic type, then ψ is hyperbolic linear fractional self map of U with Denjoy-Wolff point is on ∂U and has derivative < 1 there (in fact in this case ψ is hyperbolic automorphism). If φ is of parabolic type, then ψ is parabolic, has only one fixed point on ∂U with derivative 1 there.

This chapter consists of three sections, in section one we show that if the set of polynomials in σ is dense in H^2 then the cyclic behaviour of the linear fractional composition operator C_ψ transfers to C_φ. In chapter two, we showed that if C_φ is cyclic, then the set of polynomials in φ is dense in H^2 (corollary (2.2.9))

The converse is not true as the following example shows; If $\psi(z) = \frac{z}{2-z}$, $z \in U$, then the set of pynomials in ψ is dense in H^2 (proposition (2.2.5) and theorem (2.2.7))., However, C_ψ is not cyclic (theorem (3.1.17))

The author in [3] proves that if $\|\varphi\|_\infty < 1$, then the converse of this theorem is true that is C_φ is cyclic if and only if the set of polynomials in φ is dense set in H^2, equivalently; C_φ is cyclic if and only if C_φ has dense range.

In section two, we study briefly the cyclicity of the composition operator induced by a holomorphic self map φ of U that has its Denjoy-Wolff point on ∂U. These are of hyperbolic and parabolic types. The map φ is said to be regular provided it is univalent and continuous on the closure of U, has Denjoy-Wolff point p on ∂U and maps the closed disk into $U \cup \{p\}$, [3].

Since φ has the Denjoy-Wolff point p on ∂U, then theorem (1.4.19) insures that $0 < \varphi'(p) \le 1$ and we will see (theorem (4.2.6)) that whenever $\varphi'(p) = 1$, then $\mathrm{Re}(\varphi''(p)) \ge 0$. We summarize the results of this section in table II below.

In section three, we conclude this chapter by some open problems suggested by our work.

Table II

Cyclic behaviour of C_φ;Denjoy-Wolff point at 1 $\varphi \in C^4(1)$, regular and $\varphi''(1) \neq 0$.

Hypothesis on $\varphi'(1)$	**Hypothesis on $\varphi''(1)$**	**Cyclicity of C_φ**	**Type of φ (Model for φ)**
< 1	None	Hypercyclic theorem(4.2.12)	Hyperbolic theorem(4.2.3)
= 1	Pure imaginary ≠ 0	Hypercyclic theorem(4.2.11)	Parabolic automorphism theorem(4.2.6)
= 1	Real part > 0	Cyclic, not hypercyclic theorem(4.2.10)&(4.2.8)	Parabolic, non-automorphism theorem(4.2.6)

4.1 TRANSFERENCE PRINCIPLE

In chapter three, we studied the cyclicity of the composition operator induced by a linear fractional self map of U. In this section, we discuss the cyclicity problem for more general composition operators C_φ.

In the following definition, we classify the arbitrary holomorphic self maps of U into the following types:

Definition (Classification of Arbitrary Self-Maps) (4.1.1) [3]:

A holomorphic self-map φ of U is of:

- Dilation type, if it has a fixed point in U.
- Hyperbolic type, if it has no fixed point in U and has derivative < 1 at its Denjoy-Wolff point.
- Parabolic type, if it has no fixed point in U and has derivative 1 at its Denjoy-Wolff point.

From the definitions in chapter one, it is easy to prove the following remarks:

Remarks (4.1.2):

1. If $\varphi \in$ LFT(U) is of hyperbolic type, then φ is hyperbolic in the sense of definition (1.4.18).
2. If $\varphi \in$ LFT(U) is of parabolic type, then φ is parabolic in the sense of definition (1.4.14).

Te following important theorem appeared in [3], we give it without proof.

Theorem (The Linear-Fractional Model Theorem) (4.1.3):

Suppose φ is a univalent self-map of U. Then there exists a univalent map $\sigma : U \longrightarrow C$ on U, and a linear fractional map ψ such that $\psi(U) \subset U$, $\psi(\sigma(U)) \subset \sigma(U)$, and:

$\sigma \circ \varphi = \psi \circ \sigma$ .... (4.1)

Furthermore:

(a) ψ, viewed as a self-map of U, has the same type as φ.

(b) If φ is of hyperbolic type, then ψ may be taken to be a conformal automorphism of U.

(c) If φ is of either hyperbolic or parabolic automorphism type, then σ may be taken to be a self map of U.

Definition (4.1.4) [3, 17]:

If G = σ(U), then the pair (ψ, G), (or, equivalently, (ψ, σ)) is called a linear-fractional model for φ. If in addition G is a Jordan domain (the region interior to Jordan curve) it is said that (ψ, G) is a Jordan model.

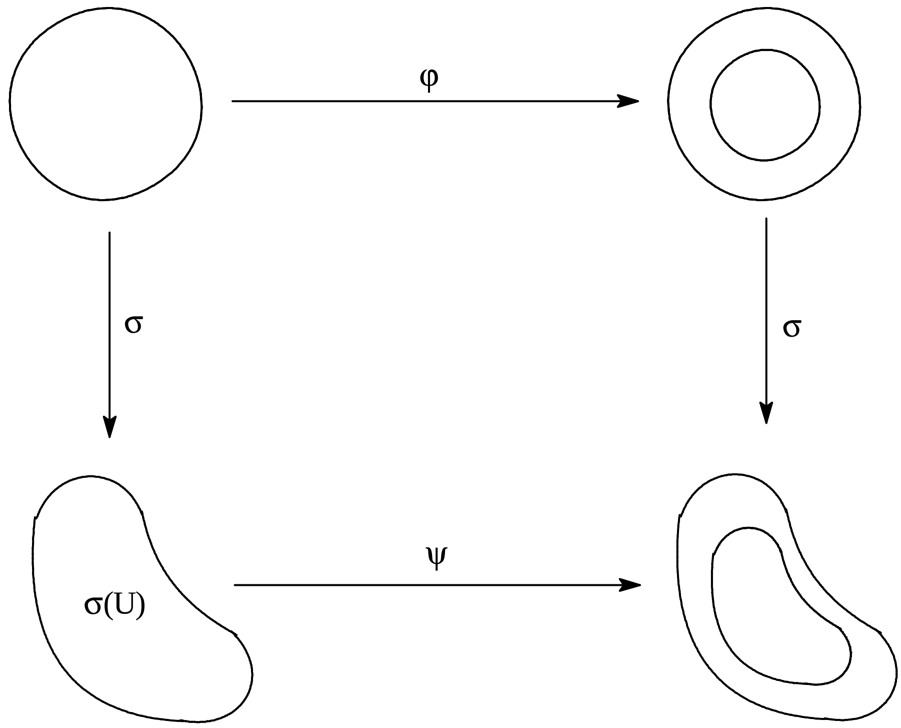

Figure 2: A linear-fractional model.

Remarks (4.1.5):

The linear fractional model theorem is the work of a number of authors, whose efforts stretch over nearly a century.

The dilation case is due to Koenigs in 1884. In this case equation (4.1) is $\sigma_o\varphi = \lambda\sigma$, where $\lambda = \varphi'(0)$ (see [17, chapter 6] for more details).

The hyperbolic case is due to Valiron. If one replaces the unit disc by the right half-plane, sending the Denjoy-Wolff point to ∞, then the resulting functional equation is again $\sigma_o\varphi = \lambda\sigma$ (see [3]).

Finally, the parabolic cases where established by Baker and Pommerenke (1979) and independently by Carl Cowen 1981. Once again the situation is best viewed in the right half plane, rather than the unit disc, with the Denjoy-Wolff point placed at ∞. Then the equation (4.1) is just $\sigma_o\varphi = \sigma + i$ in the automorphism case and $\sigma_o\varphi = \sigma + 1$ in the nonautomorphic case (see [3]).

The next result shows that C_φ inherits the hypercyclicity (respectively, cyclicity).

Theorem (4.1.6) [17]:

Suppose (ψ, G) is a Jordan model for φ, with $G \subset U$, and C_ψ hypercyclic (respectively, cyclic) on H^2, then C_φ is hypercyclic (respectively cyclic) on H^2.

The proof is very long, thus we omit .

The following important theorem appeared in [3], we give the details of its proof.

Theorem (Transference Principle) (4.1.7):

Suppose that φ is a univalent self map of U of either dilation hyperbolic, or parabolic automorphism type. Let σ be the intertwining map for φ promised by the linear fractional model theorem. Suppose further that the set of polynomials in σ is dense in H^2, then the cyclic behaviour of the linear fractional composition operator C_ψ transfers to C_φ. More precisely:

(i) If φ is of hyperbolic type or parabolic automorphism type, then C_φ is hypercyclic.

(ii) If φ is of dilation type (φ is not the identity and not elliptic linear fractional transformation) then C_φ. is cyclic, but not hypercyclic (supercyclic).

Proof:

(i) Suppose that φ is of hyperbolic type. Because φ has its Denjoy-Walff point on ∂U and has angular derivative < 1 at that point, the linear factional model theorem provides a univalent self map σ of U, and hyperbolic automorphism ψ, so that the functional equation $\sigma \circ \varphi = \psi \circ \sigma$ is satisfied. Because ψ is a non-elliptic, theorem (3.2.6) shows that C_ψ is hypercyclic. Let f be a hypercyclic vector for C_ψ. We claim that $f \circ \sigma$ is hypercyclic vector for C_φ. Applying the equation $\sigma \circ \varphi = \psi \circ \sigma$, we obtain:

$$C_\varphi^n (f \circ \sigma) = C_\sigma (f \circ \psi_n)$$

Hence, $orb(C_\varphi, f \circ \sigma) = C_\sigma(orb(C_\psi, f))$

Since the polynomials in σ are dense in H^2, then C_σ has dense range (the image of the composition operator C_σ contains the set of polynomials in σ). Since f is hypercyclic vector for C_ψ, then $orb(C_\psi, f)$ is dense in H^2. Therefore, by lemma (2.2.16)), $orb(C_\varphi, f \circ \sigma)$ is dense in H^2, that is $f \circ \sigma$ is hypercyclic vector for C_φ.

Now, suppose that φ is automorphism of parabolic type. Therefore, ψ is also parabolic automorphism, hence by theorem (3.2.6), C_ψ is hypercyclic operator. Since $\sigma \circ \varphi = \psi \circ \sigma$, then $C_\varphi \circ C_\sigma = C_\sigma \circ C_\psi$ (note

that by linear fractional model theorem, σ is self map of U, hence C_σ is an operator on H^2). Since C_ψ is hypercyclic and by our proof C_σ has dense range. Then C_φ is hypercyclic operator (theorem (2.1.2)).

(ii) Since φ is of dilation type, then φ has interior fixed point. Thus C_φ is not hypercyclic (supercyclic) operator. Without loss of generality, we assume that 0 is fixed point for φ (if $p \neq 0$ is fixed point for φ, then φ is conjugate by the special automorphism mapping α_p to a map, which has 0 as fixed point). From remarks (4.1.5), we have $\sigma \circ \varphi = \lambda \sigma$, where

$\lambda = \varphi'(0)$, hence from theorem (1.4.19), $0 \le |\lambda| = |\varphi'(0)| < 1$. If $\lambda = 0$, then $\sigma(\varphi(z)) = 0$, for all $z \in U$, which contradict with the univalent of $\sigma_o\varphi$, hence $0 < |\lambda| < 1$.

Let $\{a_n\}$ be a sequence of non-zero complex numbers chosen so that:

$$\sum_{n=0}^{\infty} |a_n| \, \|\sigma^n\| < \infty$$

Define $v = \sum_{n=0}^{\infty} a_n \sigma^n$. It is clear that v belongs to H^2.

Since $C_\varphi \sigma^n = \sigma^n(\varphi) = \sigma(\varphi)\sigma(\varphi) \ldots \sigma(\varphi) = \lambda\sigma\lambda\sigma \ldots \lambda\sigma = \lambda^n\sigma^n$, then σ^n is an eigenvector for C_φ corresponding to the eigenvalue λ^n, for all n. We claim that v is a cyclic vector for C_φ. Let $f \in H^2$ be arbitrary and suppose that $<C_\varphi^k v, f> = 0$, for k = 0, 1, …,; it suffices to show that f is the zero vector. We have for every non-negative integer k:

$$0 = <C_\varphi^k v, f> = <\sum_{n=0}^{\infty} a_n \lambda^{nk} \sigma^n, f> = \sum_{n=0}^{\infty} a_n <\sigma^n, f> (\lambda^k)^n$$

Hence if we define $h(z) = \sum_{n=0}^{\infty} a_n <\sigma^n, f> z^n$, then $h(\lambda^k) = 0$, for k = 0, 1, …; note that h is analytic on U. Since $|\lambda| < 1$, then the sequence $\{\lambda^k\}$ converges to the zero, that is, $\{\lambda^k\}$ has a limit point 0, hence $h \equiv 0$ on U (proposition (3.1.3)).

Because $a_n \neq 0$, for all n, we have $<\sigma^n, f> = 0$, for all n

Because by hypothesis, the polynomials in σ are dense in H^2 it follows that $f \equiv 0$. This completes the proof that C_φ is cyclic and with it, the proof of the transference theorem. ■

The transference technique introduced above requires the density of the polynomials, not in φ but in σ. Theorem (4.1.11) below shows that if $\|\varphi\|_\infty < 1$, then density of the polynomials in φ is equivalent to density of the polynomials in σ.

Before the theorem, we need some preliminaries.

Lemma (4.1.8):

Let φ be a holomorphic self map of U. If C_φ has dense range, then φ is univalent.

Proof:

Since C_φ has dense range and $z \in H^2$, then there is a sequence $\{f_n\}$ of functions in H^2 such that:

$$f_n(\varphi(z)) \longrightarrow z \quad \text{.......} (4.2)$$

Suppose $\varphi(a) = \varphi(b)$, where a, b $\in$ U; therefore $f_n(\varphi(a)) = f_n(\varphi(b))$, for all n, so that from (4.2), a = b, hence φ is univalent on U. ■

We prove the following proposition:

Proposition (4.1.9):

Let φ be a univalent self map of U. Let σ be the intertwining map for φ promised by the linear fractional model theorem. If $\|\varphi\|_\infty < 1$, then $\sigma(U)$ is bounded.

Proof:

Since $\|\varphi\|_\infty < 1$, then C_φ is compact (theorem (1.5.1)), so that φ has interior fixed point (proposition (1.5.5)), without loss of generality, we assume $\varphi(0) = 0$, let $\lambda = \varphi'(0)$. Remarks (4.1.5) shows:

$$\sigma \circ \varphi = \lambda\sigma \quad \text{.......} (4.3)$$

where $\lambda = \varphi'(0)$. We observe that $0 < |\lambda| \le 1$, where the first inequality follows from the univalence of $\sigma \circ \varphi$ (see the proof of theorem (4.1.7) part (ii)) and the second inequality follows from the Schwarz lemma.

Since $\|\varphi\|_\infty < 1$, then $\overline{\varphi(U)} \subset U$, hence by the Heine-Borel theorem $\overline{\varphi(U)}$ is compact. Since σ is continuous mapping, then $\sigma(\overline{\varphi(U)})$ is bounded. Note that equation (4.3) may be rewritten $\sigma = \frac{1}{\lambda}\sigma \circ \varphi$, so that $\sigma(U) = \frac{1}{\lambda}\sigma(\varphi(U)) \subseteq \frac{1}{\lambda}\sigma(\overline{\varphi(U)})$. Thus $\sigma(U)$ is bounded. ■

The following proposition appeared in [3] without proof, we give the proof.

Proposition (4.1.10):

Let φ be a holomorphic self map of U. If C_φ has dense range, then $C_{\varphi n}$ is also for all n.

Proof:

Since (range $C_\varphi)^- = H^2$, then 0 = (range $C_\varphi)^\perp = \ker C_\varphi^*$

If f ∈ (range $C_{\varphi n})^\perp$ = ker $C_{\varphi_n}^*$, then $C_{\varphi_n}^*$ f = 0, that is $C_\varphi^{*\,n}$ f = 0, therefore $C_\varphi^*(C_\varphi^{*\,n-1}$ f) = 0, hence $C_\varphi^{*\,n-1}$ f ∈ ker C_φ^*. But ker C_φ^* = 0, so that $C_\varphi^{*\,n-1}$ f = 0, we continuous until we have f = 0, that is (range $C_{\varphi n})^\perp$ = 0

Thus (range $C_{\varphi n})^- = H^2$. ■

The following theorem appeared in [3], we give the details of its proof.

Theorem (4.1.11):

Suppose that φ is analytic on U and $\|\varphi\|_\infty < 1$, then C_φ is cyclic if and only if the polynomials in φ are dense in H^2.

Proof:

We have seen that the density of the polynomials in φ is a necessary condition for cyclicity of C_φ (corollary (2.2.9)). Our goal is to prove the converse.

Suppose that the set of polynomials in φ is dense in H^2 (or equivalently that C_φ has dense range). Note that by lemma (4.1.8), φ must be univalent on U. Because $\|\varphi\|_\infty < 1$, then from chapter one, φ has interior fixed point (see also proof proposition (4.1.9)). Without loss of generality, we assume that φ(0) = 0. It is clear that φ is not elliptic (if φ is elliptic, then φ automorphism, hence $\|\varphi\|_\infty = 1$).

Let $\lambda = \varphi'(0)$ and observe from the proof of theorem (4.1.7) part (ii) that $0 < |\lambda| < 1$. The dilation model guarantees the existence of univalent map $\sigma : U \longrightarrow C$, such that $\sigma \circ \varphi = \lambda\sigma$

Note that from proposition (4.1.9) that $\sigma(U)$ is bounded subset of C. Because $\sigma(U)$ is bounded, we may choose a positive integer n, large enough, so that $\lambda^n\sigma$ maps U into itself. We claim that the range of the composition operator $C_{\lambda^n\sigma}$ is dense in H^2, or equivalently that the set of polynomials in $\lambda^n\sigma$ is dense. Since the set of polynomials in $\lambda^n\sigma$ equals the set of polynomials in σ, this will complete the proof of the theorem. Because $\sigma(U)$ is an open set containing 0, there is integer m such that the function v defined by $v(z) = \sigma^{-1}(\lambda^m z)$ is a self map of U. Therefore, $f \circ v \in H^2$, for all $f \in H^2$.

$$\text{Range } C_{\lambda^n\sigma} \supset \{(f \circ v) \circ (\lambda^n\sigma) : f \in H^2\}$$

$$= \{f \circ \sigma^{-1}\lambda^{m+n}\sigma : f \in H^2\}$$

$$= \{f \circ \varphi_{m+n} : f \in H^2\}$$

$$= \text{Range } C_{\varphi m+n.}$$

Since C_φ has dense range, then $C_{\varphi m+n}$ has dense range (proposition (4.1.10)). Thus the range of $C_{\lambda^n\sigma}$ contains a dense set and is therefore dense. ■

<u>Remarks (4.1.12):</u>

1. Since the density of the polynomials in φ is equivalent to the density of the range of C_φ (theorem (2.2.7)), therefore, if $\|\varphi\|_\infty < 1$, then C_φ is cyclic if and only if C_φ has dense range.
2. Density of the polynomials in φ does not in general imply cyclicity of C_φ (see the example in the introduction of this chapter).

<u>Corollary (4.1.13):</u>

If φ maps U univalently onto the interior of Jordon curve lying in U, then C_φ is cyclic.

Proof:

By Walsh's theorem (3.2.8), the polynomials in φ are dense in H^2, whenever $\varphi(U)$ is a Jordan domain. Since φ maps U univalently onto the interior of Jordan curve lying in U, then $\|\varphi\|_\infty < 1$, hence C_φ is cyclic (theorem (4.1.11)). ■

Corollary (4.1.14):

Suppose that φ is an analytic self map of U and that $\|\varphi_n\|_\infty < 1$, for some $n \geq 1$. If the set of polynomials in φ is dense in H^2, then C_φ is cyclic.

Proof:

We proved in proposition (4.1.10) that the density of the set of polynomials in φ implies density of the set of polynomials in φ_n, therefore $C_{\varphi n}$ is cyclic (theorem (4.1.11)). Furthermore, if f is cyclic vector for $C_{\varphi n}$, then clearly f is cyclic vector for C_φ. ■

The proof of the following theorem is similar to the proof of theorem (4.1.7) part (ii). Thus we omit.

Theorem (4.1.15) [3]:

Suppose that σ maps U univalently onto a domain $G \subset C$ and that there exists a complex number $\lambda \in U$, such that $\lambda G \subset G$. Suppose further that the polynomials in σ are dense in H^2. Let $\varphi = \sigma^{-1}\lambda\sigma$, then the composition operator C_φ is cyclic.

The following example shows how theorem (3.1.12) can fail for general self maps of φ.

Example (4.1.16) [3]:

The mapping $\sigma(z) = \log\left(\dfrac{1+z}{1-z}\right)$ is univalent on U. The holomorphic function defined by $\varphi(z) = \sigma^{-1}\left(\dfrac{\sigma(z)}{2}\right)$, maps U univlently onto the shaded region of figure (3), and fixes 0, 1 and -1.

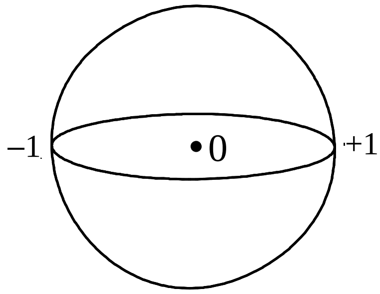

Figure (3).

Shapiro used theorem (4.1.15) and theorem (2.2.7) to prove that C_φ is cyclic [3].

We end this section by studying the power of cyclic operators and the set of cyclic vectors.

Recall that if T is hypercyclic on H then all positive powers of T are also hypercyclic [1] and the set of all hypercyclic vectors for T is dense set in H. Positive powers of supercyclic operators are also always supercyclic [1] and the set of supercyclic vectors is dense set in H. These results don't have analogues for cyclic operators. The forward shift S on the classical Hardy space defined by:

$$S(f(z)) = zf(z),\ f \in H^2$$

is cyclic. However, S^2 is not cyclic because the codimension of the range of S^2 is two. Moreover, the set of cyclic vectors for S is not dense in H^2, [1].

The following theorem shows that there is a connection between all powers of a cyclic operator and the density of the set of its cyclic vectors.

<u>Theorem (4.1.17) [1]:</u>

Suppose that T is a bounded linear operator on the Banach space X and that T^n is cyclic for each positive integer n, then the set of cyclic vectors for T is a dense subset of X.

<u>Remark (4.1.18):</u>

The converse of the preceding theorem is not true. Consider for example the operator T on $\mathbb{C}^2$ defined by:

$$T(z_1, z_2) = (z_2, 0)$$

Each vector (α, β) with both α and β non-zero will be cyclic for T, hence T has dense set of cyclic vectors. However, T^2 is the zero operator, and thus is not cyclic.

The following theorem appeared in [1].

Theorem (4.1.19):

If C_φ is cyclic operator, then all positive powers of C_φ are also cyclic, moreover the set of cyclic vectors is dense in H^2.

Proof:

Since the cyclicity of C_φ depends on the "type" of inducing map φ and since the n-th iterate of φ, φ_n, is of the same type as φ, hence if C_φ is cyclic, so is $C_\varphi^n = C_{\varphi_n}$, for every positive integer n. Moreover, theorem (4.1.17) shows that the set of cyclic vectors is dense in H^2. ■

4.2 THE HYPERBOLIC AND PARABOLIC MODELS

In this section, we turn our attention to the models that applied when a self map of U has its Denjoy-Wolff point on ∂U, these are the hyperbolic and parabolic cases of the linear fractional model theorem. Recall that the map $T(z) = \frac{1+z}{1-z}$ maps U onto Π = {z : Re(z) > 0} and takes the point 1 to ∞. Let φ be a self map of U, that has Denjoy-Wolff point on ∂U, so that φ is either of a hyperbolic or parabolic type. Without loss of generality (in terms of the cyclicity problem) we may assume that φ has Denjoy-Wolff point 1, so φ(1) = 1 and 0 < φ′(1) ≤ 1. We denote by Φ the self-map of the right half plane that corresponds to φ via T:

$$\Phi = T \circ \varphi \circ T^{-1}$$

Clearly, the sequence of Φ-iterates of any point in Π converges to ∞, so ∞ functions as the half plane analogue to Denjoy-Wolff point of Φ. We will also need to transfer to the right half plane the alternative characterization of the Denjoy-Wolff point in terms of angular limits and derivatives for 0 < α < π, let $S_\alpha = \{w : |\arg w| < \alpha/2\}$, (see figure (4)).

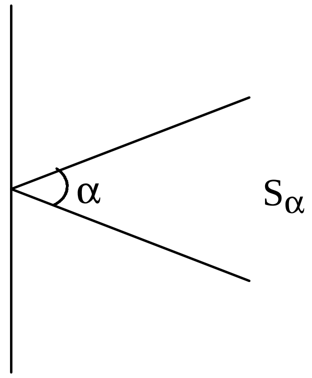

Figure (4).

Definition (4.2.1)[3]:

(1) We say that a function F defined on Π has angular limit L at ∞, and write $\angle\lim_{w\to\infty} F(w) = L$, provided that given any α with $0 < \alpha < \pi$, F(w) converges to L as w approaches ∞, through S_α.

(2) We say that a self map Φ of Π has angular derivative Q at ∞ (and write $\Phi'(\infty) = Q$) provided that Φ′ has angular limit Q at ∞.

Transferring information from U to Π via T, we have:

Theorem (4.2.2) [3]:

If Φ is a self-map of Π with Denjoy-Wolff point ∞, then Φ has angular limit ∞ at ∞, and has angular derivative at ∞ equals to $\frac{1}{\varphi'(1)}$, where $\varphi'(1)$ is the angular derivative of $T^{-1}\circ\Phi\circ T$ at 1. Thus, $\Phi'(\infty) \geq 1$.

For self maps of Π with Denjoy-Wolff point ∞, the hyperbolic and parabolic parts of the linear fractional model theorem have the following simple forms:

Theorem (4.2.3) [3] (Right Half-Plane Models):

Suppose Φ maps the right half plane into itself, and has Denjoy-Wolff point at ∞. Let $C = \Phi'(\infty)$ so that $C \geq 1$.

(a) The hyperbolic model: If $C > 1$, then there exists a non-constant analytic self-map V of Π, such that:

$$V_o\Phi = CV \quad (4.1)$$

In other words (ψ, V) is a linear fractional model for Φ, where ψ is automorphism of Π given by $\psi(w) = Cw$, for all $w \in \Pi$.

(b) The parabolic models: If $C = 1$, then there exists a non-zero complex number a with $Re(a) \geq 0$, and a non-constant analytic function V defined on Π such that:

$$V_o\Phi = V + a \quad (4.2)$$

Moreover, in equation (4.2): If $Re(a) = 0$ (the parabolic automorphism model) then V may be taken to be a self-map of Π.

Remarks (4.2.4):

(1) When $Re(a) \neq 0$ (the parabolic non-automorphism model), we can not assert that the intertwining map V may be taken to be a self-map of Π.

(2) In both models above, univalence of Φ implies univalence of V.

(3) To obtain further information about the natural of Φ, we assume that Φ has some smoothness near ∞ (i.e., that the original map φ has some smoothness near its Denjoy-Wolff point). This information will allow us to derive asymptotic representations of the intertwining maps V in the right half plane models of theorem (4.2.3).

We seek series representations for holomorphic self-map φ of U about its Denjoy-Wolff point, when that point lies on the boundary. We assume (without loss of generality) that φ has Denjoy-Wolff point 1. If $\alpha \in (0, \pi)$ and S_α is the angular approach region with angle α at 1 (see figure (4)) then by theorem (1.2.13), we can expand the mapping φ as follows:

$$\varphi(z) = 1 + \varphi'(1)(z-1) + \gamma(z) \quad (4.3)$$

where $\gamma(z) = O(|z-1|)$ as $z \longrightarrow 1$ in S_α ($\gamma(z) = O(|z-1|)$ means the growth of $\gamma(z)$ as $z \longrightarrow 1$ in S_α).

Definition (4.2.5) [3]:

If the expansion (3) holds with $\gamma(z) = O(|z - 1|)$ as $z \longrightarrow 1$ in the full disk, we say that $\varphi \in C^1(1)$. More generally, if $0 \le \varepsilon < 1$, we say that $\varphi \in C^{(n+\varepsilon)}(1)$ provided that φ has the expansion:

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(1)}{k!}(z-1)^k + \gamma(z) \quad \text{...} (4.4)$$

where $\gamma(z) = O(|z - 1|^{n+\varepsilon})$ as $z \longrightarrow 1$ in U.

Recall from section one in this chapter that in order to apply the transference principle , theorem (4.1.7), when the self-map φ of U has linear fractional model (ψ, σ), we must find conditions on φ that imply that the polynomials in σ are dense in H^2. Rather than work with φ and σ. The author in [3] works with their right half plane equivalents Φ and V, then transfer the information obtained back to the disk setting. In this section, we present briefly the theorems (without proofs) concern the holomorphic self map φ of U.

The proof of the following theorem appeared in [3]:

Theorem (4.2.6):

Suppose φ is a holomorphic self map of U that is of parabolic type, has Denjoy-Wolff point at 1, and that $\varphi \in C^2(1)$, then:

(a) $\mathrm{Re}(\varphi''(1)) \ge 0$.

(b) If either $\varphi''(1) = 0$ or $\mathrm{Re}(\varphi''(1)) > 0$, then φ is of non-automorphism type.

(c) Conversely, if $\varphi''(1)$ is non zero and pure imaginary and $\varphi \in C^{3+\varepsilon}(1)$, then φ is of automorphism type.

Remark (4.2.7):

Shapiro in [3] shows by an example that the third statement of theorem (4.2.6) is false for maps with less than C^3 at the Denjoy-Wolff point.

The proof of the following theorem is very long, thus we omit.

Theorem (4.2.8) [3]:

Suppose that φ is of parabolic type and has C^2-smoothness at Denjoy-Wolff point. If Re(φ′′) does not vanish at the Denjoy-Wolff point (so that, by theorem (4.2.6), φ is of non-automorphism type), then C_φ is not hypercyclic, in fact only constant functions may adhere to C_φ-orbits.

Definition (4.2.9):

We call a map φ regular provided it is univalent and continuous on the closure of U, has Dinjoy-Wolff point p on ∂U and maps the closed disk into U ∪ {p}.

The following theorem shows that if φ is a regular map of parabolic non-automorphism type that has $C^{3+\varepsilon}$-smoothness at the Dinjoy-Wolff point, then although C_φ is not hypercyclic (as we have just showed) it is nevertheless cyclic. This completes the proof of the statements made in the third row of table II of the introduction.

Theorem (4.2.10) [3]:

Suppose that φ is a regular self map of U of parabolic type with Dinjoy-Wolff point at 1, suppose further that $\varphi \in C^{3+\varepsilon}(1)$ with $\mathrm{Re}(\varphi''(1)) > 0$. Then C_φ is cyclic.

The following theorem proves the second row of table II of the introduction.

Theorem (4.2.11) [3]:

Suppose that φ is a regular self-map of U that is of parabolic type, has Dinjoy-Wolff point at 1, and has $C^{3+\varepsilon}$-smoothness at 1. If φ′′(1) is pure imaginary (and non-zero), then C_φ is hypercyclic.

We end this section by the following result, which is summarized in the first row of table II of the introduction.

Theorem (Hyperbolic Hypercyclicity) (4.2.12) [3]:

If φ is a regular self map of U that is of hyperbolic type and has $C^{1+\varepsilon}$-smoothness at its Dinjoy-Wolff point, then C_φ is hypercyclic.

4.3 OPEN PROBLEMS

In this section, we present some open problems suggested by our work. We see in theorem (2.1.27) if C_φ is hyponormal, then 0 is fixed point for the mapping φ, hence C_φ and C_φ^* are not hypercyclic (supercyclic) operators.

Question 1:

Are C_φ and C_φ^* cyclic operators, when C_φ is hyponormal operator?

We saw in chapter two that if C_φ is compact operator, then C_φ and C_φ^* are not hypercyclic (supercyclic) operators.

Question 2:

What is the relation between the compactness of the operator C_φ and the cyclicity of C_φ , C_φ^* ?

In chapter three, we studied the cyclicity of C_φ^* when $\varphi \in$ LFT(U). We proved that if φ is elliptic, then C_φ^* is cyclic if and only if C_φ is cyclic. If φ is non-elliptic with interior fixed point (φ is not the identity mapping) then C_φ^* is cyclic but not hypercyclic, not supercyclic. If φ has no interior fixed point, then either φ is parabolic or non-parabolic. If φ is non-parabolic, then C_φ^* is not cyclic (proposition (3.2.7), theorem (3.2.15)). If φ is parabolic, then we proved that C_φ^* is not supercyclic (proposition (3.2.28)).

The following question completes the study of cyclicity of C_φ^*, when $\varphi \in$ LFT(U).

Question 3:

Is C_φ^* cyclic when φ is parabolic?

We turn our attention to arbitrary holomorphic self map φ of U. We saw in chapter four that there is three cases: φ is dilation type, hyperbolic type and parabolic type.

Let φ be a univalent self-map of U of dilation type (except those trivial cases (the identity and elliptic mappings)). We assume without loss of generality that 0 is fixed point for φ, hence by the proof of theorem (4.1.7) part (ii) $0 < |\varphi'(0)| < 1$. Therefore, C_φ^* is cyclic (corollary (2.2.26)).

Note that C_φ^* is not hypercyclic (supercyclic) since φ has interior fixed point.

If φ is of hyperbolic type, then corollary (2.2.29) shows that C_φ^* is not cyclic.

If φ is of parabolic type, then theorem (2.2.31) shows that C_φ^* is not necessary cyclic.

Question 4:

If φ is of parabolic type, then what is the condition on φ to make C_φ^* cyclic?

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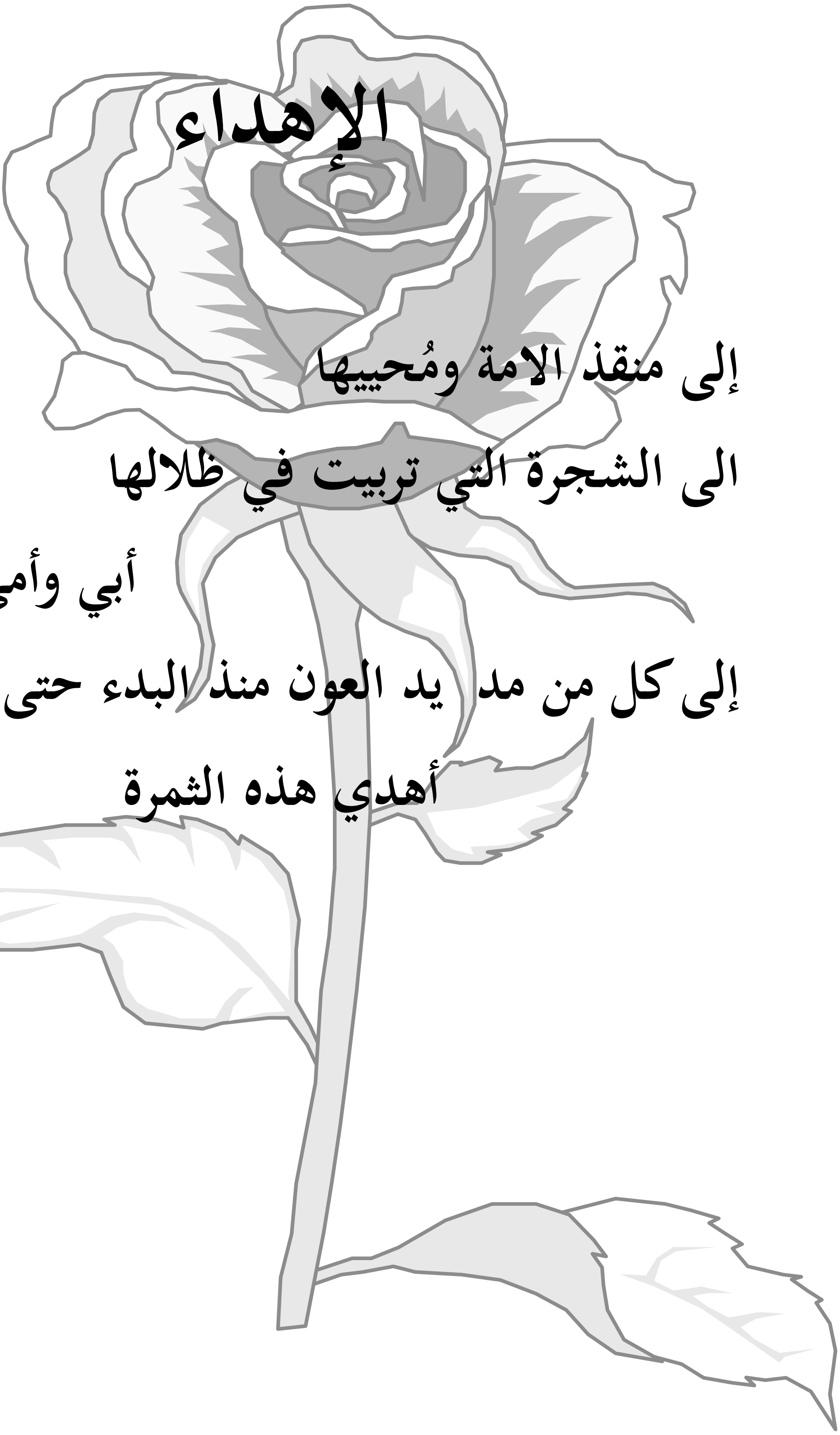

إلى منقذ الامة ومُحييها

الى الشجرة التي تربيت في ظلالها

أبي وأمي وأخوتي

إلى كل من مد يد العون منذ البدء حتى القطاف

أهدي هذه الثمرة

ليث

INTRODUCTION

Let H(U) be the set of all holomorphic functions on the unit ball U of the complex plane. If f belongs to H(U), then by Taylor theorem one can expand the function f about the origin as follows:

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \text{ , } (z \in U)$$

If the coefficients $\{\hat{f}(n)\}$ is a square summable sequence, i.e., $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$, then we say that the function f belongs to H^2 or $H^2(U)$. Therefore:

$H^2 = \{f \in H(U) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$. H^2 is called the Hardy space.

It is clear that if φ is a self map of U that belongs to H(U), then for any function f ∈ H(U), the composition $f \circ \varphi$ also belongs to H(U). The Littlewood's principle theorem [17] shows that for any f that belongs to H^2, the composition $f \circ \varphi$ also belongs to H^2. Thus the composition operators C_φ defined by:

$$C_\varphi f = f \circ \varphi \quad \text{(f holomorphic on U)}$$

takes the Hardy space H^2 into itself. Littlewood's principle also shows that C_φ is a bounded operator on H^2. Several authors have studied the properties of composition operators, for example, compactness, subnormality, and spectra of composition operators [8, 17]. Here is another direction for the composition operators: The study of cyclicity, which was followed by Shapiro, Bourdon and others [3, 17].

Recall that an operator T on a Hilbert space H is said to be cyclic if there is a vector x in H (called a cyclic vector for T) whose orbit, orb(T, x) = $\{T^n x : n = 0, 1, \ldots\}$ has dense linear span in H. The operator T is supercyclic, if there is a vector x in H (called a supercyclic vector of T), such that the set:

$$\{\lambda_n T^n x : \lambda_n \in \mathbb{C} \text{ , } n \in 0,1,\ldots\}$$

Is dense in H. It may happen that orbit, orb(T, x) is dense in H, in this case T is called hypercyclic and x is a hypercyclic vector [9, 10, 11].

Because the closed linear span of orb(T, x) is the smallest closed T-invariant subspace that contains the vector x, the concept of cyclicity is intimately connected with the study of invariant subspaces. Hypercyclicity has the same connection with invariant subsets.

One of our main concerns in this thesis was to give conditions that are necessary and (or) sufficient for the composition operator to be a cyclic operator.

This thesis contains some new results, to the best of our knowledge, on the cyclicity of the adjoint composition operators.

This thesis consists of four chapters. In chapter one, we recall the definition of Hardy space H^2 and the composition operator on H^2, also we give some information about the conformal automorphism mapping, specially when φ is a linear fractional transformation.

In chapter two, we recall the definitions of cyclic, supercyclic and hypercyclic for the composition operator. We give important properties and proved several theorems, also discussed the cyclicity of normal, hyponormal and isometric composition operator.

In chapter three, we study the cyclicity of the composition operator induced by a linear fractional transformation. Bourdon and Shapiro characterize the cyclic behaviour of the composition operators induced by a linear fractional mapping, see table I, page (58), see also [3, 17, 19]. We give the details of the proofs and other properties, discussed the cyclicity of the adjoint composition operator and investigate the cyclicity of the operators C_φ and C_ψ, where:

$$\varphi(z) = \frac{z}{c - z}, \ (z \in U) \quad , \quad \psi(z) = az + b, \ (z \in U)$$

In chapter four, we use the linear fractional model theorem to study the cyclicity of composition operators induced by more general mapping. Finally, we state some open problems suggested by our work.

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الصفة الدائرية للمؤثرات التركيبية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة دكتوراه فلسفة
في علوم الرياضيات

من قبل
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آب ٢٠٠٥ م

بسم الله الرحمن الرحيم

(نرفعُ درجاتٍ مَن نشاءُ وفَوقَ كُلِ ذِي عِلمٍ عَليم)

صدق الله العظيم

(سورة يوسف، جزء من الآية ٧٦)

المستخلص

لتكن $H(U)$ مجموعة كل الدوال التحليلية المعرفة على U، حيث U كرة الوحدة في المستوي العقدي. يُعرف فضاء هاردي H^2 بأنه مجموعة كل الدوال $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ التي تنتمي الى $H(U)$ والتي تحقق $\sum_{n=0}^{\infty}|\hat{f}(n)|^2 < \infty$. لتكن φ دالة تحليلية من U الى U، يُعرف المؤثر التركيبي C_φ على H^2 بالشكل التالي:

$$C_\varphi f = fo\varphi \; (\forall f \in H^2)$$

ليكن H فضاء هلبرت وليكن T مؤثراً معرفاً على H. اذا وجد متجه $x \in H$ بحيث أن $\text{span}\{T^n x : n = 0, 1, \ldots\}$ كثيف في H فان T يسمى مؤثر دائري ويقال أن المتجه x متجه دائري للمؤثر T. اذا وجد متجه $H \ni x$ بحيث أن المجموعة $\{\lambda_n T^n x : \lambda_n \in X, n = 0, 1, \ldots,\}$ كثيفة في H فأن T يسمى مؤثر دائري فائق. إذا وجد متجه $x \in H$ بحيث أن المجموعة $\{T^n x : n = 0, 1, \ldots,\}$ كثيفة في H فأن T يسمى مؤثر دائري فوقي.

درسنا في هذا البحث الشروط الكافية و (أو) الضرورية التي تجعل المؤثر التركيبي مؤثراً دائرياً، حيث أعطينا بعض النتائج المعروفة وحاولنا الحصول على نتائج أخرى، خصوصاً عندما تكون φ معرفة بالشكل:

$$\varphi(z) = \frac{az+b}{cz+d}, \; z \in U$$

حيث أن d, c, b, a أعداد عقدية.

كما أعطينا بعض الشروط الكافية و (أو) الضرورية التي تجعل المؤثر C_φ^* مؤثراً دائرياً، حيث C_φ^* هو المؤثر المرافق للمؤثر التركيبي C_φ.