Ban Ban Dotumｨｦb2ﾉb2ﾀ]ｸ2p^ｸ2 _ｸ2 ﾘ_ｸ2 倭ｸ2 Xaｸ2

Appendix A

```
program simulation;
        {uses crt;}
type
    arr=array [1..500] of real;
var
   f,g,t:array[1..50,1..50] of real;
   b00,b11,b0w,b1w,b0m,b1m:arr;
   sum,x,tt,z,z1:array[1..15] of real;
   n,m,i,j,k,k1,l,r,rr:integer;
sum1,sum2,sum3,sum4,sum5,sum6,sum7,sum8,sum9,sum10,sum11,sum12,
sum13,sum14,sum15,sum16,sum17,sum18,sum19,av,av1,v,v1,skew,skew1,
kur,kur1,b0,b1,newb0,sum20,sum21,sum22,sum23,sum24,sum25,sum26,
sum27,av4,av5,v4,v5,skew4,skew5,kur4,kur5,sumz1,sumxz1,av2,av3,v2,v3,
skew2,skew3,kur2,kur3,newb1,sumz,sumxz,sumsqrx,ps,lmda:real;
procedure sumall(b00,b11:arr;var
sum4,sum5,sum6,sum7,sum8,sum9,sum10,sum11:real);
begin
         sum4:=sum4+b00[l];
         sum5:=sum5+b11[l];
         sum6:=sum6+sqr(b00[l]);
         sum7:=sum7+sqr(b11[l]);
         sum8:=sum8+(sqr(b00[l]*b00[l]));
         sum9:=sum9+(sqr(b11[l]*b11[l]));
         sum10:=sum10+(sqr(b00[l]*sqr(b00[l])));
         sum11:=sum11+(sqr(b11[l]*sqr(b11[l])));
end;
```

```
procedure eval(sum4,sum5,sum6,sum7,sum8,sum9,sum10,sum11:real);
begin
   av:=sum4/500;
   av1:=sum5/500;
   writeln('mean of b0=',av:9:4);
   writeln('mean of b1=',av1:9:4);
   v:=(1/500)*(sum6-(500*sqr(av)));
   v1:=(1/500)*(sum7-(500*sqr(av1)));
   writeln('var of b0=',v:9:4);
   writeln('var of b1=',v1:9:4);
   skew:=(1/500)*(sum8-(3*av*sum6)+(2*av*sqr(av)))/sqrt(v*sqr(v));
   skew1:=(1/500)*(sum9-
(3*av1*sum7)+(2*av1*sqr(av1)))/sqrt(v1*sqr(v1));
   writeln('skew of b0=',skew:9:4);
   writeln('skew of b1=',skew1:9:4);
   kur:=((1/500)*(sum10-(4*sum8*av)+(6*sum6*sqr(av))-
(3*sqr(av)*sqr(av)))/sqr(v))-3;
   kur1:=((1/500)*(sum11-(4*sum9*av1)+(6*sum7*sqr(av1))
   -(3*sqr(av1)*sqr(av1)))/sqr(v1))-3;
   writeln('kurtosis of b0=',kur:9:4);
   writeln('kurtosis of b1=',kur1:9:4);
end;
begin
   {clrscr;}
   writeln('enter the no. of groups= m');
   readln(m);
   writeln('enter the no. of element in every groups= n');
   readln(n);
   writeln('enter beta0');
```

```
    readln(b0);
    writeln('enter beta1');
    readln(b1);
r:=1;
while (r<=20) do
    begin
    writeln('r=',r);
    ps:=ln(r)-(1/(2*r))-(1/(12*exp(2*ln(r))))+(1/(120*exp(4*ln(r))))
    -(1/(252*exp(6*ln(r))))+(1/(240*exp(8*ln(r))));
    sum4:=0; sum5:=0; sum6:=0; sum7:=0;
    sum8:=0; sum9:=0; sum10:=0; sum11:=0;
    sum12:=0; sum13:=0; sum14:=0; sum15:=0;
    sum16:=0; sum17:=0; sum18:=0; sum19:=0;
    sum20:=0; sum21:=0; sum22:=0; sum23:=0;
    sum24:=0; sum25:=0; sum26:=0; sum27:=0;
    randomize;
    for j:= 1 to m do
       x[j]:=j-((m+1)/2);
    for l:=1 to 500 do
       begin
           for i:=1 to n do
              for j:= 1 to m do
                  f[i,j]:=random;
             lmda:=0;
           for i:=1 to n do
             for j:= 1 to m do
                 begin
                 lmda:=exp(b0+b1*x[j]);
                 g[i,j]:=-lmda*ln(f[i,j]);
```

```
            end;
        for j:=1 to m do
            begin
                for i:=1 to n-1 do
                    for k:=i+1 to n do
                        if g[i,j]>g[k,j] then
                            begin
                                t[i,j]:=g[i,j];
                                g[i,j]:=g[k,j];
                                g[k,j]:=t[i,j];
                            end;
            end;
        for j:= 1 to m do
            begin
                sum[j]:=0;
                for i:= 1 to r do
                    sum[j]:=sum[j]+g[i,j];
            end;
        for j:=1 to m do
            tt[j]:=1/r*(sum[j]+(n-r)*g[r,j]);
{********************** Estimation by ML***********************}
        rr:=m*r;
        repeat
                b0:=newb0;   b1:=newb1;
                sum1:=0;  sum2:=0;  sum3:=0;
                for j:= 1 to m do
                    begin
                     sum1:=sum1-(r*tt[j]*exp(-(b0+(b1*x[j]))));
                     sum2:=sum2-(r*x[j]*tt[j]*exp(-(b0+(b1*x[j]))));
```

```
                 sum3:=sum3-(r*sqr(x[j])*tt[j]*exp(-(b0+(b1*x[j]))));
                end;
            newb0:=b0+1-((sum3*rr)/(sqr(sum2)-(sum1*sum3)));
            newb1:=b1+((sum2*rr)/(sqr(sum2)-(sum1*sum3)));
   until (abs(newb0-b0)<=0.00001) or (abs(newb1-b1)<=0.00001);
   b00[l]:=newb0;
   b11[l]:=newb1;
   sumall(b00,b11,sum4,sum5,sum6,sum7,sum8,sum9,sum10,sum11);
{********************** Estimation by WLS**********************}
   sumz:=0; sumxz:=0; sumsqrx:=0; z[j]:=0;
   for j:=1 to m do
      begin
        sumsqrx:=sumsqrx+sqr(x[j]);
        z[j]:=ln(tt[j])-ps+ln(r);
        sumz:=sumz+z[j];
        sumxz:=sumxz+(x[j]*z[j]);
      end;
   b0w[l]:=sumz/m;
   b1w[l]:=sumxz/sumsqrx;
 sumall(b0w,b1w,sum12,sum13,sum14,sum15,sum16,sum17,sum18,sum19);
{********************* Estimation by SWLS*********************}
  sumxz1:=0; sumz1:=0; sumsqrx:=0; z1[j]:=0;
  for j:=1 to m do
   begin
        sumsqrx:=sumsqrx+sqr(x[j]);
        z1[j]:=ln(tt[j])+exp(-1*ln(2*r-(1/3)+(1/(16*r))));
        sumz1:=sumz1+z1[j];
        sumxz1:=sumxz1+(x[j]*z1[j]);
   end;
```

```
        b0m[l]:=sumz1/m;
        b1m[l]:=sumxz1/sumsqrx;
  sumall(b0m,b1m,sum20,sum21,sum22,sum23,sum24,sum25,sum26,sum27);
  end; { end of l loop}
    readln;
    writeln('ESTIMATION b0 & b1 BY ML');
    writeln;
    eval(sum4,sum5,sum6,sum7,sum8,sum9,sum10,sum11);
    readln;
    writeln('ESTIMATION b0 & b1 BY WLS');
    writeln;
    eval(sum12,sum13,sum14,sum15,sum16,sum17,sum18,sum19);
    readln;
    writeln('ESTIMATION b0 & b1 BY SWLS');
    writeln;
    eval(sum20,sum21,sum22,sum23,sum24,sum25,sum26,sum27);
    readln;
if r<10 then r:=r+1 else r:=r+2;
end;
end.
```

Chapter One

Censored Data of the Exponential Distribution

1.1 Introduction

In this chapter, we shall introduce some basic concepts of right censored time to failure data, their types, associated distributions, censoring mechanism, maximum likelihood estimation, and some statistical transformation results.

1.2 Right Censored Time to Failure Data

In survival data investigation, it is quite common to find some units have not failed when observation is terminated. Their failure times are therefore unknown but known to be exceeding their survival times measured at the end of the investigation. Such failure times are said to be right censored.

This censoring mechanism may occur due to the need for early termination of the investigation or removal of units from use before failure, or failures of units occurring because of causes unrelated to the application of the operating conditions, etc, and records of survival times cannot subsequently be obtained. The following two examples illustrate the objective of collecting right censored data.

Example 1.2.1

Bartholomew in 1957 [6] considers the situation in which pieces of equipment are installed at different time. At a later data some of the pieces will have the failed and the rest will still be in use. The aim is to study the lifetime distributed of this type of equipment and to estimate quantities such as the proportion of the equipment that will fail within a specified time. Bartholomew gives the data in Table 1.1 showing the results for 10 pieces of equipment. The life test in question was terminated on August 31. At that time three items (numbers 2,4 and 10) has still not failed, and their failure timed are therefore censored; we know for these items only that their failure times exceed 72, 60 and 21 days, respectively.

Table 1.1 Operating Times for 10 Pieces of Equipment

Item Number	Date of Installation	Date of Failure	Lifetime (days)
1	11 June	13 June	2
2	21 June	_	≥ 72
3	22 June	12 August	51
4	2 July	_	≥ 60
5	21 July	23 August	33
6	31 July	27 August	27
7	31 July	14 August	14
8	1 August	25 August	24
9	2 August	6 August	4
10	10 August	_	≥ 21

Example 1.2.2

Nelson in 1972 [28] describes the results of a life test experiment in which specimens of a type of electrical insulating fluid were subjected to a constant voltage stress. The length of time to failure or breakdown of specimens was observed. Table 1.2 gives results for seven groups of specimens, tested at different voltages level.

Table 1.2 Times to Breakdown (in Minutes) at Each Seven Voltage Levels

Voltage Level (kV)	n_i	Breakdown Times
26	5	5.79,1579.52,2323.7,2450.5*,3442.3*
28	6	68.85,426.07,110.29,108.29,1067.6,2447.38*
30	14	17.05,22.66,21.02,175.88,139.07,144.12,20.46, 43.40,194.90,47.30,7.74,199.4*,184.4*,233.55*
32	15	0.40,82.85,9.88,89.29,215.10,2.75,0.79,15.93, 3.91,0.27,0.69,100.58,27.80,13.95,53.53
34	21	0.96,4.15,0.19,0.78,8.01,31.75,7.35,6.50,8.27, 33.91,32.52,3.16,4.85,2.78,4.67,1.31,12.06,36.71, 72.89,84.18*,96.78*
36	20	1.97,0.59,2.58,1.69,2.71,25.50,0.35,0.99,3.99, 3.67,2.07,0.96,5.35,2.90,13.77,14.9*,16.2*,14.3*, 19.43*,16.45*
38	12	0.47,0.73,1.40,0.74,0.39,1.13,0.09,2.38,3.19*, 3.23*,4.12*,4.9*

[a]Starred quantities denote censored observations.

1.3 Types of Right Censored Data

Censored data are said to have **Type I censoring** if censored observations occur only at specified values of the dependent variables. In this type of censoring, the censoring values are fixed and the number censored of observations is random. For instance, in Example 1.2.1, the experiments which the 10 pieces are installed at 10 different specified time and terminated at August 31. We see that 7 pieces have failed within the specified time and 3 pieces are censored at 31 August.

Censored data are said to have **Type II censoring** if the number censored of the observations is specified and their censored values are random, for example, in life testing when all units are put on test at the same time and the test terminated when a specified number have failed. For instance, in Example 1.2.2, seven groups of specimens are tested under seven different voltage levels. The breakdown or failure in each group is recorded. In this experiment observations are taken after a specified number of specimens have been failed at different voltage level. We see that 2, 1, 3, 0, 5, 2, and 4 specimens at still in use at voltage 26, 28, 30, 32, 34, 36, and 38.

More details discussion on censoring left-right mechanism and type (I, II), for signally, doubly, and multiply censored data is provided by Nelson et al [26] and Kalbfleisch et al [19].

We note that the type II censoring procedure is adopted in this thesis for two reasons:

1. Mathematically inference procedures are simpler for type II than type I [29].
2. Type II censoring usually does not allow an upper bounded to be placed on the total duration of the study.

1.4 Continuous Failure Time Distributions

In most applications when the mechanisms lead to censoring related to survival data, the assumption that the underlying distribution is normal is not realistic because the standard technique for least squares methods based on an additive regression model is not appropriates. A number of the parametric regression models can be found throughout the literature based on underlying distribution, such as, exponential, Weibull, gamma, lognormal, logistic, and extreme value distributions which have been widely used in life testing and survival problem.

We turn now to the mathematical representation of failure time distribution where we consider the case of an independent sample from a homogeneous population (no explanatory variables). We let non-negative random variable X represent the failure time of an individual selected randomly from population and x represent the specific value for X .

Let $f(x)$ be the p.d.f of X . The probabilities of an individual surviving until time x is given by the survivor function:

$$\left.\begin{array}{l} s(x) = pr(X \geq x) = \displaystyle\int_{x=0}^{\infty} f(t)dt \\ \text{Where} \qquad s(0)=1, \text{ and } s(\infty)=0 \end{array}\right\} \tag{1.1}$$

The hazard failure rate $h(t)$ defined as:

$$h(x) = \lim_{\Delta x \to 0} \frac{pr\left(x \leq X \leq x + \Delta x \mid X \geq x\right)}{\Delta x} = \frac{f(x)}{s(x)} \tag{1.2}$$

The hazard function specifics the instantaneous rate of failure at time x, given that the individual survived until time x. The functions $f(x)$, $s(x)$ and $h(x)$ satisfy the following well known relations [23]:

$$f(x) = h(x)\exp\left\{-\int_0^x h(t)dt\right\} \tag{1.3}$$

$$s(x) = \exp\left\{-\int_0^x h(t)dt\right\} \tag{1.4}$$

For the purpose of our later discussion, we shall give a brief discussion of the properties and the theoretical basis of the exponential and gamma distributions. These distributions have been discussed in details by Mann et al [25], and Al-Faris [4].

1.4.1 The Exponential Distribution

The exponential distribution has been widely used as a model in areas ranging from the studies on the lifetime of the manufactured items e.g. Epstein [11], to research involving survival times in chronic diseases e.g. Fiegl and Zelen [12]. The one parameter exponential is obtained by taken the hazard failure rate:

$$h(t) = \frac{1}{\lambda} \qquad x > 0,\ \lambda > 0 \tag{1.5}$$

The p.d.f and the c.d.f are found from (1.3) and (1.4) to be:

$$f(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda} & x > 0 \\ 0 & e.w. \end{cases} \tag{1.6}$$

and

$$F(x) = 1 - s(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x/\lambda} & 0 < x < \infty \\ 1 & x = \infty \end{cases} \tag{1.7}$$

The distribution where $\lambda = 1$ is called the standard exponential distribution which have a graphical representation of Figure 1.1.

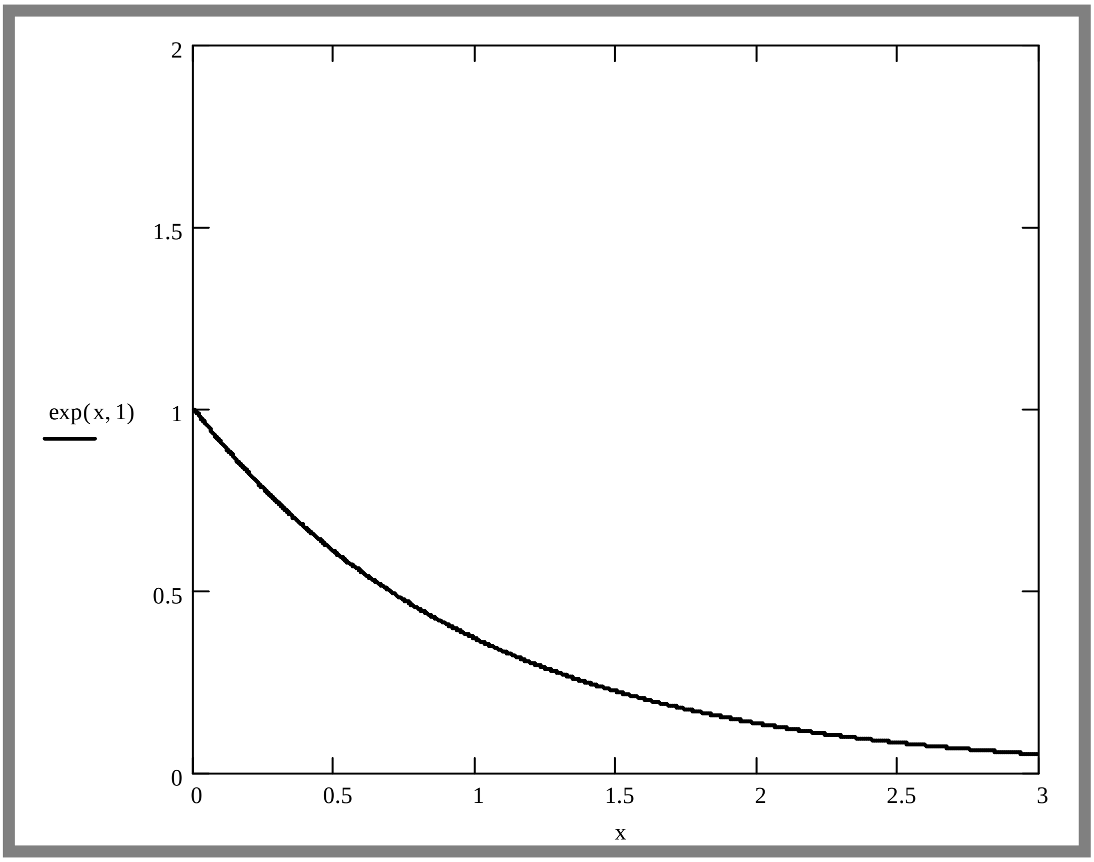

Figure 1.1 The standard exponential p.d.f.

The moment generating function of r.v. $X \sim Exp(\lambda)$ is:

$$M(t) = E[e^{tX}] = \int_t e^{tx} f(x) dx = \int_0^\infty e^{xt} \frac{1}{\lambda} e^{-x/\lambda} dx = \int_0^\infty \frac{1}{\lambda} e^{-\frac{1}{\lambda}(1-\lambda t)x} dx = (1 - \lambda t)^{-1}$$

The rth distribution moment about origin can be obtained by differentiating $M(t)$ r times at $t = 0$. Thus:

$$\mu'_r = E[X^r] = \left.\frac{d^r M(t)}{dt^r}\right|_{t=0} = r!\lambda^r \tag{1.8}$$

In particular $r = 1,2$, the mean and the variance of the distribution are λ and λ^2 respectively.

1.4.2 The Gamma Distribution

The gamma distribution arise as a model from statistical studies of interval between events occurring in time or space [15], specifically when the interest in the waiting time from the occurrence of one event until r further events have occurred in a Poisson process with constant rate λ. This distn sometimes referred to as a special Erlangian distn after the Swedish scientist who used the distn in early studies of queuing problem. The gamma distn has important applications in the study of life time models, such as stops of the machine, failure or breakdown of equipment (e.g. electronic computer), air or road accidents, coal mining disasters, telephone calls, etc., are examples of such events that occur in a real time and have properties exported for gamma case.

A r.v. X is said to have a gamma distn, denoted by $X \sim G(\alpha, \beta)$, if X has p.d.f:

$$f(x) = \begin{cases} \dfrac{\beta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & 0 < x < \infty \\ 0 & e.w. \end{cases} \tag{1.9}$$

Where $\alpha > 0$ and $\beta > 0$ are respectively the shape and scale parameters. This distn include the exponential distn as a special case $\alpha = 1$.

The survivor and hazard functions involve the incomplete gamma function. Integrating (1.9), we have:

$$s(x)=1-F(x) \qquad \text{and} \qquad h(x)=\frac{f(x)}{s(x)}=\frac{\dfrac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}}{1-\displaystyle\int_0^x \frac{\beta^{-\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-t/\beta}dt} \tag{1.10}$$

$$\text{where} \qquad F(x)=\begin{cases} 0 & x\le 0 \\ \displaystyle\int_0^x \frac{\beta^{-\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-t/\beta}dt & 0<x<\infty \\ 1 & x=\infty \end{cases}$$

The gamma distribution is used as a lifetime model; thought is not nearly as much as the exponential and Weibull distributions, partly because the survivor and hazard functions are not expressible in a simple closed form [23].

The moment generating function of r.v. $X \sim G(\alpha,\beta)$ is:

$$M(t)=E[e^{tX}]=\int_0^\infty e^{tx}f(x)dx=\int_0^\infty e^{tx}\frac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}dx$$

$$=\int_0^\infty \frac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{\frac{-1}{\beta}(1-\beta t)x}dx$$

Set $\qquad v=(1-\beta t)x \quad \Rightarrow \quad dv=(1-\beta t)dx \quad \Rightarrow \quad dx=\dfrac{dv}{1-\beta t}$

We have:

$$M(t)=\int_0^\infty \frac{\beta^{-\alpha}}{\Gamma(\alpha)}\left(\frac{v}{1-\beta t}\right)^{\alpha-1}e^{-v/\beta}\frac{dv}{(1-\beta t)}=\frac{1}{(1-\beta t)^{\alpha}}\int_0^\infty \frac{\beta^{-\alpha}}{\Gamma(\alpha)}v^{\alpha-1}e^{-v/\beta}dv$$

$$=(1-\beta t)^{-\alpha}$$

The r$^{\text{th}}$ distn moment about the origin can be obtained by differentiating $M(t)$ r times at $t=0$. Thus:

$$\mu_r' = E[X^r] = \left.\frac{d^r M(t)}{dt^r}\right|_{t=0} = \alpha(\alpha+1)(\alpha+2)...(\alpha+r-1)\beta^r, r = 1,2,... \quad (1.12)$$

In particular $r = 1,2$, the mean and variance of the distn are $\alpha\beta$ and $\alpha\beta^2$ respectively. Figure 1.2 show the p.d.f.'s for a few gamma distns.

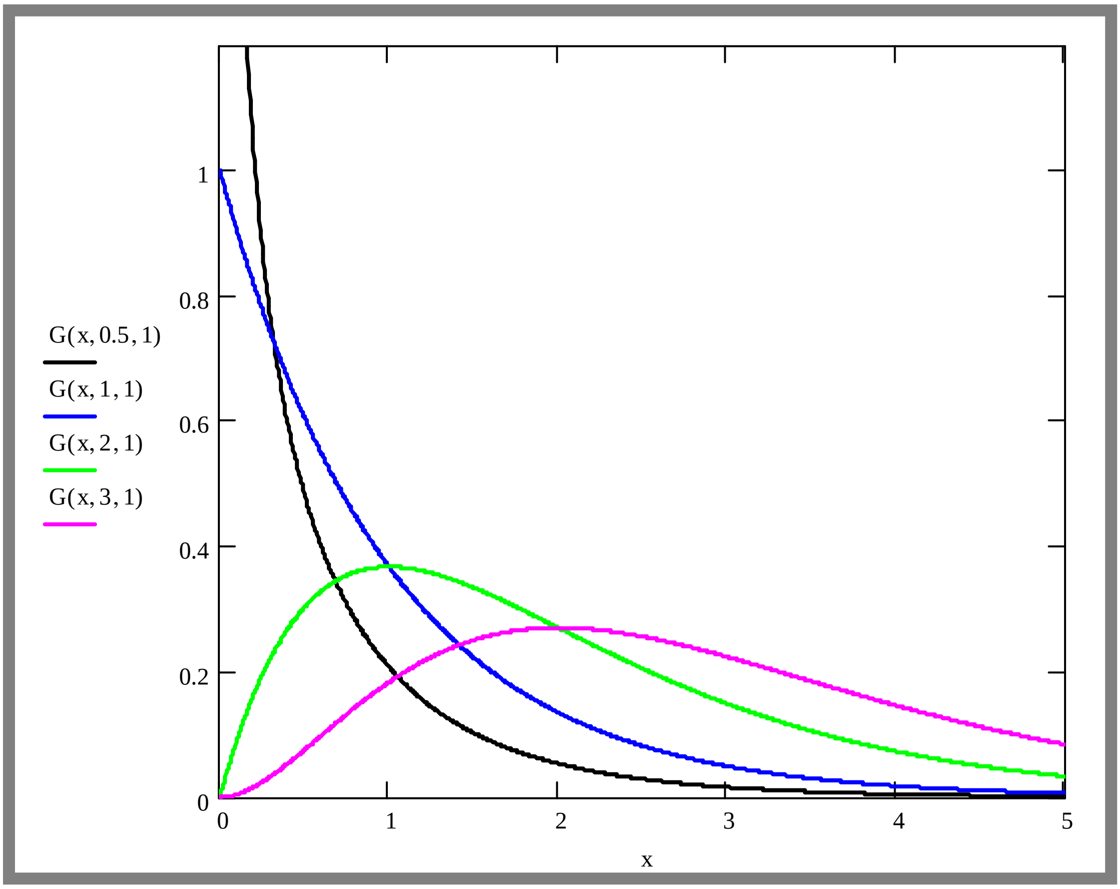

Figure 1.2 Gamma p.d.f.'s with $\beta = 1$ and $\alpha = 0.5, 1, 2 \text{and} 3$.

We note that, the gamma distribution also arise mathematically in the same situations in which the exponential distn is being used, specifically, if $X_1, X_2, ..., X_n$ is a r.s. of size n from $Exp(\beta)$ then the r.v. $Y = \sum_{i=1}^{n} X_i \sim G(n, \beta)$.

1.5 Censoring Mechanism for Exponential Distribution

Consider a life test involving a random sample from size n from $Exp(\lambda)$ and testing is terminated as soon as r failure has occurred.

Mathematically speaking let $X_1, X_2, ..., X_n$ be a random sample of size n from $Exp(\lambda)$ with p.d.f. *f(x)* and c.d.f. $F(x)$ are given respectively in equations (1.6) and (1.7).

Let $Y_1, Y_2, ..., Y_n$ be random variables representing the arrangement of the sample set $\{X_i\}$ in ascending order of magnitude then from order statistics theory, the joint p.d.f. of the set $\{Y_i\}$ is:

$$g(y_1, y_2, ..., y_n) = \begin{cases} n!\prod_{i=1}^{n} f(y_i) & 0 < y_1 < y_2 < ... < y_n < \infty \\ 0 & e.w. \end{cases} \tag{1.13}$$

Now if testing is terminated as soon as r failures have occurred. Then the joint p.d.f. $Y_1, Y_2, ..., Y_r$ is:

$$g^*(y_1, y_2, ..., y_r) = \int_{y_{r+1}}^{\infty} \int_{y_{r+2}}^{\infty} ... \int_{y_n}^{\infty} g(y_1, y_2, ..., y_n) dy_{r+1} dy_{r+2} ... dy_n$$

$$= n!\prod_{i=1}^{r} f(y_i) \int_{y_{r+1}=y_r}^{\infty} \int_{y_{r+2}=y_{r+1}}^{\infty} ... \int_{y_{n-1}=y_{n-2}}^{\infty} \int_{y_n=y_{n-1}}^{\infty} f(y_{r+1}) \cdot f(y_{r+2}) ... f(y_n) dy_{r+1} dy_{r+2} ... dy_n$$

Since

$$\int_{y_n=y_{n-1}}^{\infty} f(y_n) dy_n = F(y_n)\Big|_{y_{n-1}}^{\infty} = F(\infty) - F(y_{n-1}) = 1 - F(y_{n-1})$$

also

$$\int_{y_{n-1}=y_{n-2}}^{\infty} [1-F(y_{n-1})]f(y_{n-1})dy_{n-1} = \frac{-1}{2}[1-F(y_{n-1})]^2\Big|_{y_{n-2}}^{\infty}$$

$$= \frac{[1-F(y_{n-2})]^2}{2!}$$

If the successive integrations on $y_{n-2}, y_{n-3}, ..., y_{r+1}$ are made we have:

$$g^*(y_1, y_2, ..., y_r) = \begin{cases} \dfrac{n!}{(n-r)!}\displaystyle\prod_{i=1}^{r} f(y_i)[1-F(y_r)]^{n-r} & 0 < y_1 < y_2 < ... < y_r < \infty \\ 0 & e.w. \end{cases} \quad (1.14)$$

For exponential case with $f(t)$ and $F(t)$ in (1.6) and (1.7) we have:

$$g^*(y_1, y_2, ..., y_r) = \begin{cases} \dfrac{n!}{(n-r)!}\lambda^{-r}\exp[\dfrac{-1}{\lambda}\displaystyle\sum_{i=1}^{r} y_i + (n-r)y_r] & 0 < y_1 < y_2 < ... < y_r < \infty \\ 0 & e.w. \end{cases} \quad (1.15)$$

1.6 Maximum Likelihood Estimation for the Parameter λ

The likelihood function is:

$$L = L(\lambda, y_1, y_2, ..., y_r) = g^*(y_1, y_2, ..., y_r)$$

The natural logarithm of the likely function is:

$$\ln L = \ln[\frac{n!}{(n-r)!}] - r\ln\lambda - \frac{1}{\lambda}[\sum_{i=1}^{r} y_i - (n-r)y_r]$$

$$\frac{\partial \ln L}{\partial\lambda} = \frac{-r}{\lambda} + \frac{1}{\lambda^2}[\sum_{i=1}^{r} y_i - (n-r)y_r]$$

The ML estimator for λ is the solution of the equation $\frac{\partial \ln L(\lambda)}{\partial \lambda}=0$ at $\lambda=\hat{\lambda}$.

So $\left.\frac{\partial \ln L(\lambda)}{\partial \lambda}\right|_{\lambda=\hat{\lambda}}=0$ implies that $\frac{-r}{\hat{\lambda}}+\frac{1}{\hat{\lambda}^2}[\sum_{i=1}^{r} y_i+(n-r)y_r]$

Then the ML estimator of λ is:

$$\hat{\lambda}=\frac{1}{r}[\sum_{i=1}^{r} y_i+(n-r)y_r] \qquad (1.16)$$

1.7 Distribution of the ML Estimator of λ

In this section we consider the distribution of $\hat{\lambda}$, by taking the transformation, which appears briefly in [23]:

$V_i=(n-i+1)(Y_i-Y_{i-1})$, $i=1,2,...,r$, $Y_0=0$

That is:

$$V_1=nY_1$$
$$V_2=(n-1)(Y_2-Y_1)$$
$$V_3=(n-2)(Y_3-Y_1)$$
.
.
.
$$V_r=(n-r+1)(Y_r-Y_{r-1})$$

This transformation is one-to-one that maps the space $\mathrm{A}=\{(y_1,y_2,...,y_r):0<y_1<y_2<...<y_r<\infty\}$ onto the space $\mathrm{B}=\{(v_1,v_2,...,v_r):0<v_i<\infty, i=1,2,...,r\}$, with inverse transform:

$$Y_1 = \frac{V_1}{n}$$

$$Y_2 = \frac{V_1}{n} + \frac{V_2}{n-1}$$

$$Y_3 = \frac{V_1}{n} + \frac{V_2}{n-1} + \frac{V_3}{n-2}$$

.

.

.

$$Y_r = \frac{V_1}{n} + \frac{V_2}{n-1} + \frac{V_3}{n-2} + \ldots + \frac{V_r}{n-r+1}$$

and the Jacobian of the transformation is:

$$J = \frac{\partial(y_1, y_2, \ldots, y_r)}{\partial(v_1, v_2, \ldots, v_r)} = \begin{vmatrix} \frac{\partial y_1}{\partial v_1} & \frac{\partial y_1}{\partial v_2} & \frac{\partial y_1}{\partial v_3} & . & . & \frac{\partial y_1}{\partial v_r} \\ \frac{\partial y_2}{\partial v_1} & \frac{\partial y_2}{\partial v_2} & \frac{\partial y_2}{\partial v_3} & . & . & \frac{\partial y_2}{\partial v_r} \\ \frac{\partial y_3}{\partial v_1} & \frac{\partial y_3}{\partial v_2} & \frac{\partial y_3}{\partial v_3} & . & . & \frac{\partial y_3}{\partial v_r} \\ . & . & & . & & . \\ . & . & & & . & . \\ \frac{\partial y_r}{\partial v_1} & \frac{\partial y_r}{\partial v_2} & \frac{\partial y_r}{\partial v_3} & . & . & \frac{\partial y_r}{\partial v_r} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{n} & 0 & 0 & . & . & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & . & . & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & 0 & . & 0 \\ . & . & . & . & & . \\ . & . & . & & . & . \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & . & . & \frac{1}{n-r+1} \end{vmatrix}$$

$$= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdot \ldots \cdot \frac{1}{n-r+1} = \frac{(n-r)!}{n!}$$

also

$$\sum_{i=1}^{r} y_i + (n-r)y_r = \frac{rv_1}{n} + \frac{(r-1)v_2}{n-1} + \frac{(r-2)v_3}{n-2} + \ldots + \frac{v_r}{n-r+1} + (n-r)(\frac{v_1}{n} +$$

$$\frac{v_2}{n-1} + \frac{v_3}{n-2} + \ldots + \frac{v_r}{n-r+1}) = v_1 + v_2 + \ldots + v_r = \sum_{i=1}^{r} v_i$$

Therefore the joint p.d.f. of $V_1, V_2, \ldots, V_r$ is:

$$h(v_1, v_2, \ldots, v_r) = g(v_1, v_2, \ldots, v_r)|J| = \begin{cases} \frac{1}{\lambda^r} e^{\frac{-1}{\lambda}\sum_{i=1}^{r} v_i} & 0 < v_i < \infty \\ 0 & \text{e.w.} \end{cases}$$

Thus the set $\{V_i\}$ represent a sample of size r from $Exp(\lambda)$.

That is:

$$V_i \sim Exp(\lambda) \qquad \forall i = 1, 2, \ldots, r$$

Now

$$\hat{\lambda} = \frac{1}{r}\left[\sum_{i=1}^{r} y_i + (n-r)y_r\right] = \frac{1}{r}\sum_{i=1}^{r} v_i$$

but

$$\sum_{i=1}^{r} v_i \sim G(r, \lambda)$$

hence $\hat{\lambda} = \frac{1}{r}\sum_{i=1}^{r} v_i \sim G(r, \frac{\lambda}{r})$ (1.17)

1.8 Some Statistical Transformation Results

Theorem 1.8.1

Let $X_1, X_2, ..., X_r$ be a r.s. of size r from $Exp(\lambda)$ then the r.v. $Y = \frac{2r\overline{X}}{\lambda} \sim \chi^2(2r)$ where $\overline{X} = \frac{1}{r}\sum_{i=1}^{r} X_i$

Proof

Using m.g.f. technique:

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t\frac{2r\overline{X}}{\lambda}}\right] = E\left[e^{\frac{2}{\lambda}t\sum_{i=1}^{r} X_i}\right] = E\left[\prod_{i=1}^{r} e^{\frac{2}{\lambda}tX_i}\right] = \prod_{i=1}^{r} E\left[e^{\frac{2}{\lambda}tX_i}\right] =$$

$$\prod_{i=1}^{r} M_X\left(\frac{2t}{\lambda}\right) = \prod_{i=1}^{r} \frac{1}{(1-2t)} = \left(\frac{1}{1-2t}\right)^r = \frac{1}{(1-2t)^{2r/2}}$$

Which is the m.g.f. of $\chi^2(2r)$.

Theorem 1.8.2

If the r.v. $X \sim G(\alpha, \beta)$, then the r.v. $Y = \ln X$ distributed as log-gamma distn with p.d.f.

$$g(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \exp\left[\alpha y - e^{y/\beta}\right] & -\infty < y < \infty \\ 0 & e.w. \end{cases}$$

With mean and variance of Y are respectively:

$E[Y] = E[\ln X] = \ln\beta + \psi(\alpha)$

$\text{var}[Y] = \text{var}[\ln X] = \psi'(\alpha)$

Proof

The p.d.f. of r.v. $X \sim G(\alpha,\beta)$ is given by (1.9) as:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & 0 < x < \infty \\ 0 & e.w. \end{cases}$$

The function $y = \ln x$ define one-to-one transformation that maps the space $\text{A} = \{x : 0 < x < \infty\}$ onto the space $\text{B} = \{y : -\infty < y < \infty\}$, with inverse $x = e^{y}$ and the Jacobian of this transformation is: $J = \dfrac{dx}{dy} = e^{y}$.

Then the p.d.f. of Y is:

$$g(y) = f(e^{y})|J| = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(e^{y}\right)^{\alpha-1} e^{-e^{y}/\beta} e^{y} = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\alpha y} e^{\frac{-e^{y}}{\beta}}, -\infty < y < \infty$$

To find the mean and the variance of Y, we consider the moment generating function of Y as follows:

$$M_Y(t) = E\left[e^{tY}\right] = \int_{y} e^{ty} g(y) dy = \int_{-\infty}^{\infty} e^{ty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\alpha y} e^{\frac{-e^{y}}{\beta}} dy$$

$$= \frac{\Gamma(\alpha+t)\beta^{t}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha+t)\beta^{\alpha+t}} e^{(\alpha+t)y} e^{\frac{-e^{y}}{\beta}} dy \qquad (1.18)$$

Since the integral side of eq. (1.18) is unity, then

$$M_Y(t) = \frac{\Gamma(\alpha+t)\beta^{t}}{\Gamma(\alpha)} \qquad (1.19)$$

Set

$$\phi(t) = \ln M(t) = \ln \Gamma(\alpha + t) + t \ln \beta - \ln \Gamma(\alpha)$$

$$\phi'(t) = \psi(\alpha + t) + \ln \beta \tag{1.20}$$

$$\phi''(t) = \psi'(\alpha + t) \tag{1.21}$$

Where

$$\psi(w) = \frac{d \ln \Gamma(w)}{dw} \quad \text{and} \quad \psi'(w) = \frac{d^2 \ln \Gamma(w)}{dw^2}$$

Are known as digamma and trigamma functions. Setting $t = 0$ the mean and variance of Y are:

$$\left.\begin{array}{l} E[Y] = \phi'(0) = \psi(\alpha) + \ln \beta \\ \operatorname{var}[Y] = \phi''(0) = \psi'(\alpha) \end{array}\right\} \tag{1.22}$$

We note that the digamma function $\psi(\alpha)$, and trigamma function $\psi'(\alpha)$ are tabulated [2].

Also, we note that, the following is an important well-known result can be deduced from theorem 1.8.1.

If the r.v. $\frac{2r\overline{X}}{\lambda} \sim \chi^2(2r)$, then the r.v. $Y = \frac{\lambda}{2r}\chi^2(2r)$ (1.23)

Where the distn is known as a non central chi-square distn with non centrality $\frac{\lambda}{2r}$.

Chapter Three

Monte Carlo Investigation Results

3.1 Introduction

In this chapter we present the results of the large scale Monte Carlo investigation to assess the approximations to the biases of ML and SWLS estimators. Moment properties of the three methods of estimation are tabulated and make comparisons with them. Mean square efficiencies of ML estimators relative to WLS and SWLS are assessed for the case of a single explanatory variable, but before that we consider a procedure for generating random variates from exponential distribution.

A computer program in Appendix (A) for evaluating the estimator's values of the regression coefficients for the three methods of estimation is mode by using Monte Carlo simulation.

3.2 Random Variates Generation [35]

Many methods and procedures are proposed in the literature for generating random variates from exponential distribution. We shall adopt the most well known method, namely the inverse transform method.

3.2.1 Inverse Transform Method

This method is the most common used in for generating random variates from the exponential distribution, which can be describe as follows:

The p.d.f. and c.d.f. of exponential distributed random variates are given by the equations (1.6) and (1.7) respectively.

Setting $U = F(x)$ where $U \in (0,1)$

$\Rightarrow U = 1 - e^{-y/\lambda}$, that implies

$Y = -\lambda \ln(1-U)$

Since the random variable $(1-U)$ is distributed in the same way as U, we may let:

$Y = -\lambda \ln U$

The procedure for generating random variate from $Exp(\lambda)$ can be described as in the following algorithm:

The algorithm IT

1- Read λ.

2- Generate U from $U(0,1)$.

3- Set $Y = -\lambda \ln U$.

4- Deliver Y as a random variate generating from $Exp(\lambda)$.

For a single explanatory variable with equally spaced values has been examined, the regression model for the mean that given in (2.2) becomes:

$$\left.\begin{array}{ll} \mu_i = e^{(\beta_0+\beta_1 x_i)}, & \\ \text{where} \quad x_i = i - \dfrac{g+1}{2} & i = 1,2,\dots,g \end{array}\right\} \qquad (3.1)$$

Equal numbers of censored exponential observations within the groups where taken with $r_i = r = 1(1)10(2)20$, $g = 5{,}10$, with initial values of $\underset{\sim}{\beta}$ equal to zero.

A simulation run size 500 was used.

3.3 Moment Properties of the ML Estimators

For a single explanatory variable $(k = 1)$ the approximation to the bias of ML estimator given by the eq. (2.44) takes the form:

$$b_0 = -\tfrac{1}{R} \quad \text{and} \quad b_1 = -\tfrac{1}{2}\frac{\sum\limits_i r_i x_i^3}{\left(\sum\limits_i r_i x_i^2\right)^2} \tag{3.2}$$

Also the information matrix of (2.14) becomes:

$$I = \begin{pmatrix} \sum\limits_i r_i & \sum\limits_i r_i x_i \\ \sum\limits_i r_i x_i & \sum\limits_i r_i x_i^2 \end{pmatrix}$$

Let $D = \begin{vmatrix} \sum\limits_i r_i & \sum\limits_i r_i x_i \\ \sum\limits_i r_i x_i & \sum\limits_i r_i x_i^2 \end{vmatrix} = (\sum\limits_i r_i)(\sum\limits_i r_i x_i^2) - (\sum\limits_i r_i x_i)^2$

$$I^{-1} = \frac{1}{D}\begin{pmatrix} \sum\limits_i r_i x_i^2 & -\sum\limits_i r_i x_i \\ -\sum\limits_i r_i x_i & \sum\limits_i r_i \end{pmatrix}$$

Hence, the asymptotic variances and covariance of ML estimators is:

$$\left.\begin{array}{l} \operatorname{var}(\hat{\beta}_0)_{asy} = D^{-1}\sum_i r_i x_i^2, \operatorname{var}(\hat{\beta}_1)_{asy} = D^{-1}\sum_i r_i \\ \quad \text{and} \quad \operatorname{cov}(\hat{\beta}_0, \hat{\beta}_1)_{asy} = -D^{-1}\sum_i r_i x_i \end{array}\right\} \tag{3.3}$$

We have

$$\mu_i = e^{\beta_0 + \beta_1 x_i}, \qquad x_i = 1 \quad i = 1, 2, \ldots, g$$

Let $\underset{\sim}{\hat{\beta}} = \left(\hat{\beta}_0, \hat{\beta}_1\right)$

From (2.15), we have:

$$\left.\begin{array}{l} f_0 = f_0\left(\hat{\beta}_0, \hat{\beta}_1\right) = \sum_i r_i \left(t_i e^{-\left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)} - 1\right) = 0 \\ f_1 = f_1\left(\hat{\beta}_0, \hat{\beta}_1\right) = \sum_i r_i x_i \left(t_i e^{-\left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)} - 1\right) = 0 \end{array}\right\} \tag{3.4}$$

And from (2.16)

$$\hat{\beta}_0^{(s+1)} = \hat{\beta}_0^{(s)} + \delta_0, \; \hat{\beta}_1^{(s+1)} = \hat{\beta}_1^{(s)} + \delta_1 \tag{3.5}$$

Where $\underset{\sim}{\delta} = -\underset{\sim}{A}^{-1} \underset{\sim}{f}(\underset{\sim}{\hat{\beta}}), \; \underset{\sim}{\delta} = \begin{pmatrix} \delta_0 \\ \delta_1 \end{pmatrix}$ (3.6)

and $\underset{\sim}{A} = \begin{pmatrix} \dfrac{\partial f_0(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_0} & \dfrac{\partial f_0(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_1} \\ \dfrac{\partial f_1(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_0} & \dfrac{\partial f_1(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_1} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Where

$$\frac{\partial f_0(\hat{\beta}_0,\hat{\beta}_1)}{\partial\hat{\beta}_0}=-\sum_i r_i t_i e^{-\left(\hat{\beta}_0+\hat{\beta}_1 x_i\right)}=a$$

$$\frac{\partial f_0(\hat{\beta}_0,\hat{\beta}_1)}{\partial\hat{\beta}_1}=-\sum_i r_i x_i t_i e^{-\left(\hat{\beta}_0+\hat{\beta}_1 x_i\right)}=b$$

$$\frac{\partial f_1(\hat{\beta}_0,\hat{\beta}_1)}{\partial\hat{\beta}_1}=-\sum_i r_i x_i^2 t_i e^{-\left(\hat{\beta}_0+\hat{\beta}_1 x_i\right)}=c$$

$$\underset{\sim}{A}^{-1}=\frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

Then (3.6) becomes:

$$\begin{pmatrix}\delta_0\\ \delta_1\end{pmatrix}=\frac{1}{b^2-ac}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} f_0 \\ f_1\end{pmatrix}=\frac{1}{b^2-ac}\begin{pmatrix} cf_0-bf_1 \\ -bf_0+af_1\end{pmatrix}=\begin{pmatrix}\dfrac{cf_0-bf_1}{b^2-ac}\\ \dfrac{-bf_0+af_1}{b^2-ac}\end{pmatrix}$$

$$\delta_0=\frac{cf_0-bf_1}{b^2-ac},\ \delta_1=\frac{-bf_0+af_1}{b^2-ac} \qquad (3.7)$$

Substitute (3.7) in (3.5) we have:

$$\left.\begin{aligned}\hat{\beta}_0^{(s+1)}&=\hat{\beta}_0^{(s)}+\delta_0=\hat{\beta}_0^{(s)}+\frac{cf_0-bf_1}{b^2-ac}=\hat{\beta}_0^{(s)}+\frac{b\sum_i r_i x_i-c\sum_i r_i}{b^2-ac}+1\\ \hat{\beta}_1^{(s+1)}&=\hat{\beta}_1^{(s)}+\delta_1=\hat{\beta}_1^{(s)}+\frac{-bf_0-af_1}{b^2-ac}=\hat{\beta}_1^{(s)}+\frac{b\sum_i r_i-a\sum_i r_i x_i}{b^2-ac}\end{aligned}\right\} \qquad (3.8)$$

In practice, we take $r_i = r$, then

$$\sum_{i=1}^{g} r_i = \sum_{i=1}^{g} r = rg = R \text{ (say)}$$

$$\sum_{i=1}^{g} r_i x_i = r\sum_{i}\left(i - \frac{g+1}{2}\right) = r\left[\frac{g(g+1)}{2} - \frac{g(g+1)}{2}\right] = 0$$

$$\sum_{i=1}^{g} r_i x_i^2 = r\sum_{i}\left(i - \frac{g+1}{2}\right)^2 = r\sum_{i}\left[i^2 - (g+1)i + \left(\frac{g+1}{2}\right)^2\right]$$

$$= r\left[\frac{g(g+1)(2g+1)}{6} - \frac{(g+1)g(g+1)}{2} + \frac{g(g+1)^2}{4}\right]$$

$$= \frac{rg(g+1)}{12}[4g + 2 - 6g - 6 + 3g + 3] = \frac{rg(g+1)(g-1)}{12} = \frac{rg\left(g^2 - 1\right)}{12}$$

Then the approximate solution of ML estimators of (3.8) at stage $(s+1)$ becomes:

$$\left.\begin{aligned} \hat{\beta}_0^{(s+1)} &= \hat{\beta}_0^{(s)} - \frac{cR}{b^2 - ac} + 1 \\ \hat{\beta}_1^{(s+1)} &= \hat{\beta}_1^{(s+1)} + \frac{bR}{b^2 - ac} \end{aligned}\right\} \tag{3.9}$$

This process is repeated until the difference between new β_0 and β_1 estimators and the old β_0 and β_1 estimators less than the same specific bounded which is (10^{-6}).

Without loss of generality, we may assume that the x-values are centered so that $\sum_{i} r_i x_i = 0$.

In this case, we have from (3.2) and (3.3) the following results:

$$b_0 = -R^{-1} \quad \text{and} \quad b_1 = 0 \tag{3.10}$$

$$\left.\begin{aligned}
\operatorname{var}(\hat{\beta}_0)_{asy} &= \left(\sum_{i=1}^{g} r\right)^{-1} = (rg)^{-1} = R^{-1} \\
\operatorname{var}(\hat{\beta}_1)_{asy} &= \left(\sum_{i=1}^{g} r x_i^2\right)^{-1} = \frac{12}{rg\left(g^2-1\right)} = \frac{12}{R\left(g^2-1\right)} \\
\text{and} \quad \operatorname{cov}(\hat{\beta}_0, \hat{\beta}_1) &= 0
\end{aligned}\right\} \tag{3.11}$$

In Table 3.1 values of the biases of the ML estimators are shown for the cases $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$ values of the approximate biases $b_0 = -R^{-1}$ for $\hat{\beta}_0$ are shown in parentheses, the bias of $\hat{\beta}_1$ being zero to order R^{-1}.

The results of Table 3.1 show that there is a good agreement between the biases of $\hat{\beta}_0$ obtained by simulation and the approximate values $-R^{-1}$ even for r as small as one. The biases for $\hat{\beta}_1$ are close to zero which agree with the approximation given in eq. (3.10).

Table 3.2 presents the values of the variances, skewness, and kurtosis of the ML estimators are shown for the cases $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$. Values of the large sample variances given by eq. (3.11) for $\hat{\beta}_0$ and $\hat{\beta}_1$ are shown in parentheses.

The result of Table 3.2 shows that there is very good adequate between the large sample variances of eq. (3.11) and simulation value for all values of r.

And we find that for all values of r, the estimators show a small skewness and some negative kurtosis which indicating that normal approximation to the distribution of the estimators will be effective for $r \geq 5$.

Table 3.1 Values of Biases for ML Estimators

r	g=5		g=10	
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
1	-0.02221(-0.2000)	-0.0304	-0.1120(-0.1000)	0.0066
2	-0.1019(-0.1000)	-0.0012	-0.0665(-0.0500)	-0.0007
3	-0.0814(-0.0667)	-0.0157	-0.0327(-0.0333)	-0.0028
4	-0.0642(-0.0500)	-0.0182	-0.0315(-0.0250)	0.0003
5	-0.0402(-0.0400)	-0.0023	-0.0225(-0.0200)	-0.0009
6	-0.0426(-0.0333)	0.0026	-0.0116(-0.0167)	0.0012
7	-0.0313(-0.0285)	-0.0050	-0.0153(-0.0143)	-0.0025
8	-0.0333(-0.0250)	-0.0096	-0.0236(-0.0125)	0.0008
9	-0.0252(-0.0222)	0.0033	-0.0186(-0.0111)	-0.0021
10	-0.0239(-0.0200)	-0.0007	-0.0081(-0.0100)	0.0001
12	-0.0172(-0.0167)	0.0060	-0.0051(-0.0083)	0.0009
14	-0.0110(-0.0143)	-0.0030	-0.0096(-0.0071)	-0.0004
16	-0.0119(-0.0125)	0.0015	-0.0029(-0.0063)	-0.0011
18	-0.0051(-0.0111)	0.0004	-0.0077(-0.0056)	-0.0009
20	-0.0170(-0.0100)	0.0003	-0.0028(-0.0050)	0.0015

Table 3.2 Values of Variances, Skewness, and Kurtosis for ML Estimators

i. $(\hat{\underset{\sim}{\beta}}_0)$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.2380(0.2000)	4.1625	11.8366	0.1046(0.1000)	2.4325	1.7417
2	0.1156(0.1000)	2.7243	5.7039	0.0493(0.0500)	1.7637	-0.5829
3	0.0616(0.0667)	2.0973	0.6827	0.0348(0.0333)	1.2339	-1.3994
4	0.0515(0.0500)	1.8417	-0.0506	0.0264(0.0250)	1.1296	-1.9464
5	0.0404(0.0400)	1.2270	-1.6496	0.0199(0.0200)	0.8890	-2.3553
6	0.0350(0.0333)	1.3885	-1.3279	0.0173(0.0167)	0.6700	-2.5649
7	0.0244(0.0286)	1.1619	-1.8334	0.0156(0.0143)	0.7881	-2.3325
8	0.0238(0.0250)	1.2809	-1.5433	0.0136(0.0125)	1.0652	-2.0702
9	0.0222(0.0222)	1.0034	-2.1215	00101(0.0111)	0.9119	-2.3531
10	0.0239(0.0200)	1.0064	-0.0244	0.0086(0.0100)	0.5420	-2.7449
12	0.0186(0.0167)	0.8430	-2.3155	0.0076(0.0083)	0.4338	-2.8225
14	0.0119(0.0143)	0.6426	-2.6096	0.0068(0.0071)	0.6306	-2.6525
16	0.0121(0.0125)	0.6383	-2.6506	0.0063(0.0063)	0.3192	-2.8950
18	0.0110(0.0111)	0.4629	-2.7603	0.0057(0.0056)	0.5567	-2.7288
20	0.0102(0.0100)	0.8143	-2.5061	00052(0.0050)	0.3187	-2.8988

Table 3.2 Continued

ii. $(\hat{\underset{\sim}{\beta}}_1)$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.1365(0.1000)	1.5841	0.4420	0.0143(0.0121)	0.2715	-2.7399
2	0.0571(0.0500)	0.8351	-1.8490	0.0074(0.0061)	0.2945	-2.8719
3	0.0417(0.0333)	0.3642	-2.5951	0.0045(0.0040)	0.3521	-2.8574
4	0.0279(0.0250)	0.9078	-2.1065	0.0032(0.0030)	0.1573	-2.9559
5	0.0217(0.0200)	0.5472	-2.5451	0.0025(0.0024)	0.2067	-2.9458
6	0.0188(0.0167)	0.3339	-2.7962	0.0019(0.0020)	0.0515	-2.9820
7	0.0165(0.0143)	0.5112	-2.6569	0.0018(0.0017)	0.3125	-2.9191
8	0.0127(0.0125)	0.6462	-2.5558	0.0015(0.0015)	0.0692	-2.9783
9	0.0113(0.0111)	0.2449	-2.8479	0.0015(0.0013)	0.2879	-2.9294
10	0.0103(0.0100)	0.3086	-2.8615	0.0013(0.0012)	0.0981	-2.9822
12	0.0082(0.0083)	0.0600	-2.9364	0.0010(0.0010)	0.0036	-2.9919
14	0.0075(0.0071)	0.3518	-2.8636	0.0010(0.0009)	0.1550	-2.9706
16	0.0062(0.0063)	0.1955	-2.9024	0.0007(0.0008)	0.2003	-2.9689
18	0.0054(0.0056)	0.1666	-2.9571	0.0007(0.0007)	0.1880	-2.9694
20	0.0051(0.0050)	0.2057	-2.9292	0.0006(0.0006)	-0.0984	-2.9857

3.4 Moment Properties of WLS Estimators

From (2.28), the WLS estimators under our model that given in (3.1) when $r_i = r$ are:

$$\left.\begin{aligned} \hat{\beta}_{0w} &= \frac{\sum_i z_i \{\psi'(r_i)\}^{-1}}{\sum_i \{\psi'(r_i)\}^{-1}} = \frac{\sum_i z_i \{\psi'(r)\}^{-1}}{\sum_i \{\psi'(r)\}^{-1}} = \frac{\sum_i z_i}{g} \\ ,\hat{\beta}_{1w} &= \frac{\sum_i x_i z_i \{\psi'(r_i)\}^{-1}}{\sum_i x_i^2 \{\psi'(r_i)\}^{-1}} = \frac{\sum_i x_i z_i \{\psi'(r)\}^{-1}}{\sum_i x_i^2 \{\psi'(r)\}^{-1}} = \frac{\sum_i x_i z_i}{\sum_i x_i^2} = 0 \end{aligned}\right\} \quad (3.12)$$

And from eq. (2.29), the variances and covariance matrix is:

$$\operatorname{cov}\left(\hat{\underset{\sim}{\beta}}_w\right) = (\underset{\sim}{X}' \underset{\sim}{W} \underset{\sim}{X})' = \frac{1}{D_w}\begin{pmatrix} \sum_i x_i^2 \{\psi'(r_i)\}^{-1} & -\sum_i x_i \{\psi'(r_i)\}^{-1} \\ -\sum_i x_i \{\psi'(r_i)\}^{-1} & \sum_i \{\psi'(r_i)\}^{-1} \end{pmatrix}$$

Where $D_w = \left[-\sum_i \{\psi'(r_i)\}^{-1}\right]\left[\sum_i x_i^2 \{\psi'(r_i)\}^{-1}\right] - \left[\sum_i x_i \{\psi'(r_i)\}^{-1}\right]^2$

Hence, the asymptotic variances and covariance of WLS estimators when $r_i = r$ are:

$$\left.\begin{aligned} \operatorname{var}(\hat{\beta}_{0w})_{asy} &= \sum_i \psi'(r_i) = g\psi'(r) \\ \operatorname{var}(\hat{\beta}_{1w})_{asy} &= \frac{g}{\sum_i x_i^2 \{\psi'(r_i)\}^{-1}} = \frac{12 g \psi'(r)}{\left(g^2 - 1\right)} \\ \operatorname{cov}(\hat{\beta}_{0w}, \hat{\beta}_{1w})_{asy} &= 0 \end{aligned}\right\} \quad (3.13)$$

Table 3.3 presents the values of the simulated biases of WLS estimators for the case $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$. The WLS estimator is an unbiased.

And Table 3.4 gives the values of the variances, skewness, and kurtosis of the WLS estimators are shown for the cases $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$. Values of the large sample variances given by eq. (3.13) for $\hat{\beta}_{0w}$ and $\hat{\beta}_{1w}$ are shown in parentheses.

The result of Table 3.4 shows that there is a good adequate between the large sample variances of eq. (3.13) and simulation value for all values of r.

And we notice that for all values of r, the estimators show a small skewness and some negative kurtosis which indicating that normal approximation to the distribution of the estimators will be effective even for small values of r.

Table 3.3 Values of Biases for WLS Estimators

r	g=5		g=10	
	$\hat{\beta}_{0w}$	$\hat{\beta}_{1w}$	$\hat{\beta}_{0w}$	$\hat{\beta}_{1w}$
1	0.0128	-0.0392	-0.0059	0.0082
2	0.0045	0.0029	-0.0185	-0.0003
3	-0.0161	0.0151	0.0043	-0.0046
4	-0.0127	-0.0185	-0.0063	0.0006
5	0.0044	-0.0036	-0.0029	0.0004
6	-0.0084	0.0042	0.0066	0.0010
7	-0.0019	-0.0054	-0.0013	-0.0018
8	-0.0065	-0.0099	-0.0103	0.0007
9	-0.0039	0.0039	-0.0058	-0.0018
10	-0.0040	-0.0011	0.0023	0.0000
12	0.0002	0.0055	0.0030	0.0009
14	0.0038	-0.0028	-0.0029	-0.0006
16	0.0005	0.0010	0.0022	-0.0012
18	0.0063	0.0002	-0.0019	-0.0011
20	-0.0073	0.0002	0.0015	0.0015

Table 3.4 Values of Variances, Skewness, and Kurtosis for WLS Estimators

i. $(\hat{\underset{\sim}{\beta}}_{0w})$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.2919(0.2414)	1.9268	5.4036	0.1632(0.1207)	1.1679	-0.8730
2	0.1295(0.1065)	1.2721	0.4790	0.0598(0.0532)	1.0691	-1.5133
3	0.0698(0.0689)	1.1421	-0.7887	0.0390(0.0345)	0.5922	-2.2570
4	0.0541(0.0511)	0.9071	-1.8863	0.0286(0.0256)	0.6488	-2.4538
5	0.0442(0.0407)	0.4888	-2.5740	0.0218(0.0203)	0.4568	-2.7413
6	0.0364(0.0338)	0.7293	-2.5757	0.0185(0.0169)	0. 2554	-2.8175
7	0.0261(0.0289)	0.5223	-2.5816	0.0166(0.0144)	0.4664	-2.5338
8	0.0255(0.0252)	0.6641	-2.3939	0.0143 (0.0126)	0.7669	-2.3931
9	0.0228(0.0224)	0.5413	-2.6041	0.0106(0.0112)	0.4908	-2.7410
10	0.0245(0.0201)	0.5735	-2.5157	0.0091(0.0101)	0.2016	-2.9152
12	0.0187(0.0168)	0.4156	-2.7053	0.0080(0.0084)	0.1503	-2.9379
14	0.0122(0.0143)	0.2180	-2.8701	0.0070(0.0072)	0. 3645	-2.8315
16	0.0126(0.0125)	0.2894	-2.8671	0.0064(0.0063)	0.1367	-2.9483
18	0.0111(0.0111)	0.1254	-2.9078	0.0058(0.0056)	0.3065	-2.8824
20	0.0104(0.0100)	0.4967	-2.7789	0.0052(0.0050)	0.1377	-2.9536

Table 3.4 Continued

ii. $(\hat{\underset{\sim}{\beta}}_{1w})$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.1558(0.1207)	1.5830	0.0773	0.0190 (0.0146)	0.3311	-2.6471
2	0.0639(0.0532)	0.8057	-1.8617	0.0086 (0.0065)	0.2958	-2.8585
3	0.0432(0.0345)	0.3283	-2.7158	0.0049(0.0042)	0.4319	-2.8125
4	0.0283(0.0256)	0.8900	-2.1886	0.0035(0.0031)	0.1544	-2.9491
5	0.0225(0.0203)	0.5944	-2.4723	0.0027(0.0025)	0.1401	-2.9604
6	0.0198(0.0169)	0.3288	-2.7491	0.0019(0.0020)	0.0697	-2.9781
7	0.0174(0.0144)	0.5176	-2.6595	0.0019(0.0017)	0.2575	-2.9398
8	0.0128(0.0126)	0.6504	-2.5684	0.0015(0.0015)	0.0699	-2.9792
9	0.0115(0.0112)	0.2364	-2.8402	0.0016(0.0014)	0.2704	-2.9364
10	0.0104(0.0107)	0.3118	-2.8627	0.0013(0.0012)	0.1090	-2.9809
12	0.0082(0.0084)	0.0791	--2.9333	0.0010(0.0010)	0.0087	-2.9919
14	0.0076(0.0072)	0.3527	-2.8549	0.0010(0.0009)	0.1665	-2.9682
16	0.0063(0.0063)	0.2108	-2.9081	0.0008(0.0008)	0.2026	-2.9678
18	0.0055(0.0056)	0.1757	-2.9538	0.0007(0.0007)	0.2157	-2.9615
20	0.0051(0.0050)	0.2071	-2.9295	0.0006(0.0006)	-0.0912	-2.9855

3.5 Moment Properties of SWLS Estimators

For the SWLS estimators expression (2.30), (2.31) apply with $\psi'(r_i)$ replaced $\left(r_i - \frac{1}{2} + \frac{1}{10r_i}\right)^{-1}$, eq. (3.12) gives the SWLS estimators, and eq. (3.13) gives approximate variances which are:

$$\left.\begin{aligned} \operatorname{var}(\hat{\beta}_{0m})_{asy} &= \left(R - \tfrac{1}{2}g + \tfrac{1}{10}R^{-1}g^2\right)^{-1} \\ \operatorname{var}(\hat{\beta}_{1m})_{asy} &= 12\left(g^2 - 1\right)^{-1}\left(R - \tfrac{1}{2}g + \tfrac{1}{10}R^{-1}g^2\right)^{-1} \end{aligned}\right\} \tag{3.14}$$

Table 3.5 shows the values of the simulated biases of SWLS estimators for the cases $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$. The true values of biases of SWLS given by eq. (2.42) where $b_{*0} = \frac{\sum_i d_i}{g}$ for $\hat{\beta}_{0m}$ are shown between parentheses the biases of $\hat{\beta}_{1m}$ which is $b_{*1} = \frac{\sum_i x_i d_i}{\sum_i x_i}$ being zero.

The results of Table 3.5 show that there is an expected agreement between the biases of $\hat{\beta}_{0m}$ obtained by simulation and the approximate values $\frac{\sum_i d_i}{g}$ for all values of r. The biases for $\hat{\beta}_{1m}$ are close to zero which agree with the approximate values $\frac{\sum_i x_i d_i}{\sum_i x_i}$ that being zero.

And Table 3.6 gives the values of the variances, skewness, and kurtosis of the SWLS estimators are shown for the cases $r_i = r = 1(1)10(2)20$; $i = 1,2,...g$, $g = 5,10$. Values of the large sample variances given by eq. (3.14) for $\hat{\beta}_{0m}$ and $\hat{\beta}_{1m}$ are shown in parentheses.

The result of Table 3.6 shows that there is a good adequate between the large sample variances of eq. (3.14) and simulation value for all values of r.

And we notice that for all values of r, the estimators show a small skewness and some negative kurtosis which indicating that normal approximation to the distribution of the estimators will be effective even for small values of r.

Table 3.5 Values of Biases for SWLS Estimators

R	g=5		g=10	
	$\hat{\beta}_{0m}$	$\hat{\beta}_{1m}$	$\hat{\beta}_{0m}$	$\hat{\beta}_{1m}$
1	0.0164(0.0035)	-0.0392	-0.0024(0.0035)	0.0082
2	0.0046(-0.0066)	0.0029	-0.0184(-0.0066)	-0.0003
3	-0.0161(-0.0050)	0.0151	0.0042(-0.0050)	-0.0046
4	-0.0127(-0.0039)	-0.0185	-0.0063(-0.0039)	0.0006
5	0.0044(-0.0031)	-0.0036	-0.0029(-0.0031)	0.0004
6	-0.0084(-0.0026)	0.0042	0.0066(-0.0026)	0.0010
7	-0.0019(-0.0022)	-0.0054	-0.0013(-0.0022)	-0.0018
8	-0.0065(-0.0019)	-0.0099	-0.0103(-0.0019)	0.0007
9	-0.0039(-0.0017)	0.0039	-0.0058(-0.0017)	-0.0018
10	-0.0040(-0.0016)	-0.0011	0.0023(-0.0016)	0.0000
12	0.0002(-0.0013)	0.0055	0.0030(-0.0013)	0.0009
14	0.0038(-0.0011)	-0.0028	-0.0029(-0.0011)	-0.0006
16	0.0005(-0.0010)	0.0010	0.0022(-0.0010)	-0.0012
18	0.0063(-0.0009)	0.0002	-0.0019(-0.0009)	-0.0011
20	-0.0073(-0.0008)	0.0002	0.0015(-0.0008)	0.0015

Table 3.6 Values of Variances, Skewness, and Kurtosis for SWLS Estimators

i. $(\hat{\underset{\sim}{\beta}}_{0m})$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.292(0.3333)	1.8999	5.2782	0.1635(0.1667)	1.1365	-0.9371
2	0.1311(0.1290)	1.2715	0.4774	0.0599(0.0645)	1.0683	-1.5147
3	0.0699(0.0789)	1.1421	-0.7886	0.0398(0.0395)	0.5923	-2.2570
4	0.054(0.0567)	0.9072	-1.8862	0.0287(0.0284)	0.6490	-2.4537
5	0.0449(0.0442)	0.4889	-2.5739	0.0219(0.0221)	0.4568	-2.7413
6	0.0365(0.0363)	0.7293	-2.5756	0.0185 (0.0181)	0.2555	-2.8175
7	0.0261(0.0307)	0.5223	-2.5816	0.0166 (0.0154)	0.4665	-2.5338
8	0.0255(0.0266)	0.6641	-2.3938	0.0143(0.0133)	0.7670	-2.3930
9	0.0228(0.0235)	0.5414	-2.6041	0.0106 (0.0117)	0.4908	-2.7410
10	0.0245(0.0210)	0.5735	-2.5157	0.0091 (0.0105)	0.2016	-2.9151
12	0.0187(0.0174)	0.4156	-2.7053	0.0080 (0.0087)	0.1503	-2.9379
14	0.0122(0.0148)	0.2180	-2.8701	0.0070 (0.0074)	0. 3645	-2.8315
16	0.0126(0.0129)	0.2894	-2.8671	0.0064 (0.0064)	0.1367	-2.9483
18	0.0111(0.0114)	0.1254	-2.9078	0.0058 (0.0057)	0.3065	-2.8824
20	0.0104(0.0102)	0.4967	-2.7789	0.0052 (0.0051)	0.1377	-2.9536

Table 3.6 Continued

ii. $(\hat{\underset{\sim}{\beta}}_{1m})$

r	g=5			g=10		
	var.	skew.	kurt.	var.	skew.	kurt.
1	0.1563(0.1667)	1.5830	0.0773	0.0193(0.0202)	0.3311	-2.6471
2	0.0642(0.0645)	0.8057	-1.8617	0.0087(0.0078)	0.2958	-2.8585
3	0.0434(0.0395)	0.3283	-2.7158	0.0050(0.0048)	0.4319	-2.8125
4	0.0285(0.0284)	0.8900	-2.1886	0.0035(0.0034)	0.1544	-2.9491
5	0.0229(0.0221)	0.5944	-2.4723	0.0027(0.0027)	0.1401	-2.9604
6	0.0198(0.0181)	0.3288	-2.7491	0.0019(0.0022)	0.0697	-2.9781
7	0.0174(0.0154)	0.5176	-2.6595	0.0019(0.0019)	0.2575	-2.9398
8	0.0128(0.0133)	0.6504	-2.5684	0.0015(0.0016)	0.0699	-2.9792
9	0.0115(0.0117)	0.2364	-2.8402	0.0016(0.0014)	0.2704	-2.9364
10	0.0104(0.0105)	0.3118	-2.8627	0.0013(0.0013)	0.1090	-2.9809
12	0.0082(0.0087)	0.0791	-2.9333	0.0010(0.0011)	0.0087	-2.9919
14	0.0076(0.0074)	0.3527	-2.8549	0.0010(0.0009)	0.1665	-2.9682
16	0.0063(0.0064)	0.2108	-2.9081	0.0008(0.0008)	0.2026	-2.9678
18	0.0055(0.0057)	0.1757	-2.9538	0.0007(0.0007)	0.2157	-2.9615
20	0.0051(0.0051)	0.2071	-2.9295	0.0006(0.0006)	-0.0912	-2.9855

Table 3.7 Variance Efficiencies of WLS Relative to ML

r	g=5		g=10		$E_0^* = E_1^*$
	E_0	E_1	E_0	E_1	
1	0.8153	0.8761	0.6409	0.7526	0.608
2	0.8927	0.8936	0.8244	0.8605	0.7752
3	0.8825	0.9653	0.8923	0.9184	0.8440
4	0.9519	0.9859	0.9231	0.9143	0.8808
5	0.9140	0.9644	0.9128	0.9259	0.9037
6	0.9615	0.9495	0.9351	1	0.9192
7	0.9349	0.9483	0.9398	0.9474	0.9304
8	0.9333	0.9922	0.9510	1	0.9389
9	0.9737	0.9826	0.9528	0.9375	0.9455
10	0.9755	0.9904	0.9451	1	0.9509
12	0.9947	1	0.9500	1	0.9589
14	0.9754	0.9868	0.9714	1	0.9647
16	0.9603	0.9841	0.9844	0.855	0.9690
18	0.9910	0.9818	0.9828	1	0.9724
20	0.9808	1	1	1	0.9752

Table 3.8 Variance Efficiencies of SWLS Relative to ML

r	g=5		g=10		$E_0^* = E_1^*$
	E_0	E_1	E_0	E_1	
1	0.8151	0.8733	0.6398	0.7409	0.6
2	0.8817	0.8894	0.8230	0.8506	0.775
3	0.8813	0.9608	0.8744	0.9000	0.8444
4	0.9537	0.9789	0.9209	0.9143	0.881
5	0.9000	0.9476	0.9087	0.9259	0.904
6	0.9590	0.9495	0.9351	1	0.9194
7	0.9349	0.9483	0.9398	0.9474	0.9306
8	0.9333	0.9922	0.9510	1	0.9391
9	0.9737	0.9826	0.9528	0.9375	0.9457
10	0.9755	0.9904	0.9451	1	0.951
12	0.9947	1	0.9500	1	0.9590
14	0.9754	0.9868	0.9714	1	0.9648
16	0.9603	0.9841	0.9844	0.855	0.9691
18	0.9910	0.9818	0.9828	1	0.9725
20	0.9808	1	1	1	0.9752

3.7 Bias Reduction Estimators

The bias approximation given by eq. (3.2) for the ML estimators work effectively for all values of the $\{r_i\}$ and the biases are of the same order of magnitude as the variances, we are led to consider bias reduction estimators given by:

$$\left.\begin{aligned}\hat{\hat{\beta}}_0 &= \hat{\beta}_0 + R^{-1} \\ \hat{\hat{\beta}}_1 &= \hat{\beta}_1 + \tfrac{1}{2}\frac{\sum_i r_i x_i^3}{\left(\sum_i r_i x_i^2\right)^2}\end{aligned}\right\} \tag{3.19}$$

In order to reduce the mean square errors of estimation, the mean square errors of the new estimators are given by:

$$mse\left(\hat{\hat{\beta}}_j\right) = mse\left(\hat{\beta}_j\right) + b_j^2 - 2b_j B_j, \qquad j = 1,2 \tag{3.20}$$

Where β_j denoted the true bias of $\hat{\beta}_j$.

The mean square efficiencies of the ML, WLS, and SWLS estimators relative to the bias reduction estimators are defined by:

$$\left.\begin{aligned}E_{1j} &= \frac{mse\left(\hat{\hat{\beta}}_j\right)}{mse\left(\hat{\beta}_j\right)} \\ E_{2j} &= \frac{mse\left(\hat{\hat{\beta}}_j\right)}{mse\left(\hat{\beta}_{jw}\right)} \qquad j = 0,1 \\ E_{3j} &= \frac{mse\left(\hat{\hat{\beta}}_j\right)}{mse\left(\hat{\beta}_{jm}\right)}\end{aligned}\right\} \tag{3.21}$$

Estimates of the efficiencies are obtained by simulation are shown in Table 3.9 for $r_i = r = 1(1)10(2)20$ and $g = 5,10$.

For the x-values under consideration, we have $b_1 = 0$ so the estimators $\hat{\hat{\beta}}_1$ and $\hat{\beta}_1$ are equivalent, therefore $E_{11} = 0$.

For estimation of β_0, if we use the approximation $\mathrm{var}(\hat{\beta}_0) = B_0 = R^{-1}$ in eq. (3.20), we obtain:

$$\frac{mse\left(\hat{\hat{\beta}}_0\right)}{mse\left(\hat{\beta}_0\right)} \approx \frac{R}{R+1} \tag{3.22}$$

Table 3.9
Mean Square Error Efficiencies 0f ML, WLS, and SWLS Estimators Relative to the Bias Reduction Estimators

r	g=5					g=10				
	E_{10}	E_{20}	E_{30}	E_{21}	E_{31}	E_{10}	E_{20}	E_{30}	E_{21}	E_{31}
1	0.8561	0.8153	0.8147	0.8761	0.8733	0.9127	0.6409	0.6398	0.7526	0.7409
2	0.9204	0.8927	0.8919	0.8936	0.8894	0.9517	0.8244	0.8249	0.8605	0.8506
3	0.9326	0.8825	0.8822	0.9653	0.9608	0.9691	0.8923	0.8926	0.9184	0.9000
4	0.9976	0.9519	0.9510	0.9859	0.9789	0.9769	0.9231	0.9231	0.9143	0.9143
5	0.9984	0.9140	0.9140	0.9644	0.9476	0.9803	0.9128	0.9128	0.9259	0.9259
6	0.9989	0.9615	0.9615	0.9495	0.9495	0.9841	0.9351	0.9359	1	1
7	0.9678	0.9349	0.9349	0.9483	0.9483	0.9871	0.9398	0.9398	0.9474	0.9474
8	0.9993	0.9333	0.9333	0.9922	0.9922	0.9886	0.9510	0.9510	1	1
9	0.9783	0.9737	0.9737	0.9826	0.9826	0.9879	0.9528	0.9528	0.9375	0.9375
10	0.9835	0.9755	0.9755	0.9904	0.9904	0.9885	0.9451	0.9451	1	1
12	0.9853	0.9947	0.9947	1	1	0.9910	0.9500	0.9500	1	1
14	0.9831	0.9754	0.9754	0.9868	0.9868	0.9926	0.9714	0.9714	1	1
16	0.9873	0.9603	0.9603	0.9841	0.9841	0.9937	0.9844	0.9844	0.855	0.855
18	0.9890	0.9910	0.9910	0.9818	0.9818	0.9945	0.9828	0.9828	1	1
20	0.9903	0.9808	0.9808	1	1	0.9952	1	1	1	1

Chapter Two

Methods of Estimation

2.1 Introduction

Suppose that we have g groups of individuals or units, where being n_i individuals in the i^th^ group, where $\sum_{i=1}^{g} n_i = N$. The response variable of interest is time to failure and we let Y_{ij} be a r.v. representing the failure time for the j^th^ individual in the i^th^ group. When failure occurs because of random causes and ageing has no effect, the distn of time failure is exponential will be assumed for statistically independent $\{Y_{ij}\}$. Also we shall assume that the measurements a variable for k explanatory variables $x_1, x_2, ..., x_k$, where all individuals in the same group having the same values of the explanatory variables. For the i^th^ group the explanatory variables are denoted by $x_{i1}, x_{i2}, ..., x_{ik}$.

The exponential p.d.f. of the r.v. Y_{ij} written as:

$$f(y_{ij}) = \begin{cases} \dfrac{1}{\mu_i} e^{-\frac{1}{\mu_i} y_{ij}} & 0 < y_{ij} < \infty \\ 0 & e.w. \end{cases} \qquad (2.1)$$

We shall assume that the dependence of μ_i on the explanatory variables is given by the model:

$$\mu_i = e^{\underset{\sim}{x}_i' \underset{\sim}{\beta}}, \qquad i = 1, 2, ..., g \tag{2.2}$$

Where $\underset{\sim}{x}_i' = (1, x_{i1}, x_{i2}, ..., x_{ik})'$ and $\underset{\sim}{\beta} = (\beta_0, \beta_1, \beta_2, ..., \beta_k)$

Equation (2-2) shows that the mean μ_i being adjusted by the multiplicative exponential factor to allow for the influence of the explanatory variables. Other models have been proposed in the literature when the response variable is exponentially distributed, for instance, Feigl and Zelen [12] consider the model:

$$\mu_i = \underset{\sim}{x}_i' \underset{\sim}{\beta}, \qquad i = 1, 2, ..., g \tag{2.3}$$

While Greenberg et al [13] use the reciprocal model:

$$\mu_i = (\underset{\sim}{x}_i' \underset{\sim}{\beta})^{-1}, \qquad i = 1, 2, ..., g \tag{2.4}$$

The linear and reciprocal models given by (2-3) and (2-4) are suffer from disadvantages that the values of $\underset{\sim}{\beta}$ must be restricted to guarantee that $\mu_i > 0$ for all values of $\underset{\sim}{x}_i'$, while the model in (2-2) has the advantage that it has positive values for all values of $\underset{\sim}{x}_i'$ and $\underset{\sim}{\beta}$. Further more the model (2-2) has been used widely in the analysis of survival data, for example, Prentice [32], Lawless [22] and Kahn [18] used the (2-2) model extensively for machines accelerated life tests.

In the analysis of time to failure data, right censoring of the data often occurs because of the need for early termination of the investigation. Many censoring schemes are of course possible, but we shall employee type II censoring within groups in which observation within a given group ceases after the occurrence of the given order statistic within the group. For the ith group, we assume that r_i smallest observations $Y_{i(1)} < Y_{i(2)} < ... < Y_{ir_i}$ are observed and the remanding $n_i - r_i$ observations being right censored at the value $Y_{i(r_i)}$.

Let $R = \sum_{i=1}^{g} r_i$ denote the total number of uncensored observations.

2.2 Estimation of $\underset{\sim}{\beta}$ by Maximum Likelihood Method

The most important, widely used formal and has robust of the parameter estimation techniques is the method of maximum likelihood. Estimation by ML is a general technique that may be applied when the underlying distn of the data is specified.

Using eq. (1-15), the joint p.d.f. of the observed order statistics $Y_{i(1)} < Y_{i(2)} < ... < Y_{i(r_i)}$ into the ith group is:

$$\frac{n_i\,!}{(n_i - r_i)!}\frac{1}{\mu_i^{r_i}}\exp\left[-\frac{1}{\mu_i}\left\{\sum_{j=1}^{r_i} y_{i(j)} - (n_i - r_i)y_{i(r_i)}\right\}\right], \quad i = 1,2,...,g \qquad (2.5)$$

and the log-likelihood over all groups is:

$$L = \sum_{i=1}^{g}\left[\ln\left\{\frac{n_i!}{(n_i - r_i)!}\right\} - r_i \ln\mu_i - \frac{1}{\mu_i}\left\{\sum_{j=1}^{r_i} y_{i(j)} - (n_i - r_i)y_{i(r_i)}\right\}\right] \qquad (2.6)$$

Set

$$t_i = \frac{1}{r_i}\left\{\sum_{j=1}^{r_i} y_{i(j)} - (n_i - r_i)y_{i(r_i)}\right\}, \qquad i = 1,2,..., g \qquad (2.7)$$

Then eq. (2-6) becomes:

$$L = \sum_{i=1}^{g}\left[\ln\left\{\frac{n_i!}{(n_i - r_i)!}\right\} - r_i \ln\mu_i - \frac{r_i t_i}{\mu_i}\right] \tag{2.8}$$

It has been shown in eq. (1-16) that, if no model is imposed on the $\{\mu_i\}$, the statistics:

$$T_i = r_i^{-1}\left\{\sum_{j=1}^{r_i} Y_{i(j)} + (n_i - r_i)Y_{i(r_j)}\right\}, \qquad i = 1,2,\ldots,g \tag{2.9}$$

are the ML estimators of the $\{\mu_i\}$. It easily one can show that these estimators are independent, unbiased, sufficient, and having minimum variance, [7], [29] and [39], as well as, we show the expression of eq. (1-17) that each

$T_i \sim G\left(r_i, \dfrac{\mu_i}{r_i}\right)$ with p.d.f.

$$f_i(t) = \begin{cases} \dfrac{1}{\Gamma(r_i)}\left(\dfrac{r_i}{\mu_i}\right)^{r_i} t^{r_i - 1} e^{-\frac{r_i t}{\mu_i}} & 0 < t < \infty \\ 0 & e.w. \end{cases} \tag{2.10}$$

With mean μ_i and variance μ_i^2 / r_i.

Under the regression model given by (2-2) we have:

$$L = \sum_{i=1}^{g}\left[\ln\left\{\frac{n_i!}{(n_i - r_i)!}\right\} - r_i \underset{\sim}{x}_i' \underset{\sim}{\beta} - r_i t_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}\right]$$

And hence

$$\frac{\partial L}{\partial \beta_r} = \sum_{i=1}^{g} r_i x_{ir}\left(t_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}} - 1\right), \qquad r = 0,1,2,\ldots,k \tag{2.11}$$

and

$$\frac{\partial^2 L}{\partial\beta_r \partial\beta_s} = -\sum_{i=1}^{g} r_i x_{ir} x_{is} t_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}, \qquad r,s = 0,1,2,...,k \tag{2.12}$$

The ML estimate of $\underset{\sim}{\beta}$ is therefore given by the solution of the $(k+1)$ likelihood equations of eq. (2-11)

$$\sum_{i=1}^{g} r_i x_{ir} t_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}} = \sum_{i=1}^{g} r_i x_{ir}, \qquad r = 0,1,2,...,k \tag{2.13}$$

and the information matrix is:

$$\underset{\sim}{I} = \left(\left(E \left[\frac{-\partial^2 L}{\partial\beta_r \partial\beta_s} \right] \right) \right) = \left(\left(\sum_{i=1}^{g} r_i x_{ir} x_{is} \right) \right)_{(k+1)\times(k+1)} \tag{2.14}$$

The solution of likelihood equations in (2-11) can be done iteratively by using Newton-Raphson method for solving the nonlinear equations as follows:

Let

$$f_r(\underset{\sim}{\hat{\beta}}) = \sum_{i=1}^{g} r_i x_{ir} \left(t_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\hat{\beta}}} - 1 \right) = 0, \qquad r = 0,1,2,...,k \tag{2.15}$$

Suppose that $\underset{\sim}{\hat{\beta}}^{(s)} = (\hat{\beta}_0^{(s)}, \hat{\beta}_1^{(s)}, ..., \hat{\beta}_k^{(s)})$ represent the approximate solution of the equations (2-15) at stage s . Then the approximate solution at the stage $(s+1)$ for $\hat{\beta}_r$ is:

$$\hat{\beta}_r^{(s+1)} = \hat{\beta}_r^{(s)} + \delta_r, \qquad r = 0,1,2,...,k \tag{2.16}$$

Set $\underset{\sim}{\delta} = -\underset{\sim}{A}^{-1} \underset{\sim}{f}(\underset{\sim}{\hat{\beta}}^{(s)})$

Where

$$\underset{\sim}{\delta} = \{\delta_0, \delta_1, ..., \delta_k\}, \quad \underset{\sim}{f}(\underset{\sim}{\hat{\beta}}^{(s)}) = \{f_0(\underset{\sim}{\hat{\beta}}^{(s)}), f_1(\underset{\sim}{\hat{\beta}}^{(s)}), ..., f_k(\underset{\sim}{\hat{\beta}}^{(s)})\}\text{, and}$$

$$A = \begin{pmatrix} \frac{\partial f_0(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_0} & \frac{\partial f_0(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_1} & \cdot\ \cdot\ \cdot & \frac{\partial f_0(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_k} \\ \frac{\partial f_1(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_0} & \frac{\partial f_1(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_1} & \cdot\ \cdot\ \cdot & \frac{\partial f_1(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_k(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_0} & \frac{\partial f_k(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_1} & \cdot\ \cdot\ \cdot & \frac{\partial f_k(\underset{\sim}{\hat{\beta}}^{(s)})}{\partial \hat{\beta}_k} \end{pmatrix}$$

2.3 Estimation of $\underset{\sim}{\beta}$ by weighted least squares method

An alternative method of estimation which provides a non iterative solution uses the weighted least squares [29] applied to the logarithmic values of the $\{t_i\}$ that obtained by the eq. (2-7). This approach utilizes the well-known results for the log chi-square distn given by eq. (1-22) with $\alpha = r$ and $\beta = 2$. In this case:

$$E\left[\ln \chi^2_{(2r)}\right] = \psi(r) + \ln 2 \quad \text{and} \quad \text{var}\left[\ln \chi^2_{(2r)}\right] = \psi'(r) \tag{2.17}$$

Now, according to the results of (2-10) and Theorem 1.8.1 the r.v.'s $T_i \sim G\left(r_i, \frac{\lambda_i}{r_i}\right)$ and $\frac{2r_i T_i}{\lambda_i} \sim \chi^2_{(2r_i)}$ that implies:

$$T_i \sim \frac{\lambda_i}{2r_i}\chi^2_{(2r_i)}, \qquad i = 1, 2, ..., g \tag{2.18}$$

And $\ln T_i \sim \ln \lambda_i - \ln(2r_i) + \ln(\chi^2_{(2r_i)}), \qquad i = 1,2,\ldots,g \qquad (2.19)$

Thus, if we set:

$$Z_i = \ln T_i - \psi(r_i) + \ln r_i, \qquad i = 1,2,\ldots,g \qquad (2.20)$$

Then $Z_i \sim \underset{\sim}{x}_i' \underset{\sim}{\beta} - \psi(r_i) - \ln 2 + \ln \chi^2_{(2r_i)}, \qquad i = 1,2,\ldots,g \qquad (2.21)$

With $E[Z_i] = \underset{\sim}{x}_i' \underset{\sim}{\beta}$ and $\operatorname{var}[Z_i] = \psi'(r_i) \qquad (2.22)$

Set $\varepsilon_i = \ln \chi^2_{(2r_i)} - \ln 2 - \psi(r_i), \qquad i = 1,2,\ldots,g \qquad (2.23)$

Then $Z_i = \underset{\sim}{x}_i' \underset{\sim}{\beta} + \varepsilon_i, \qquad i = 1,2,\ldots,g \qquad (2.24)$

With $E[\varepsilon_i] = 0, \quad \operatorname{var}[\varepsilon_i] = \psi'(r_i), \qquad i = 1,2,\ldots,g$

$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = 0, \qquad i \neq j = 1,2,\ldots,g$

In matrix form, we write eq. (2.24) as a linear model:

$$\underset{\sim}{Z} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{\varepsilon} \qquad (2.25)$$

Where

$$\underset{\sim}{Z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ . \\ . \\ . \\ z_g \end{pmatrix}, \quad \underset{\sim}{X} = \begin{pmatrix} 1 & x_{11} & x_{12} & . & . & . & x_{1k} \\ 1 & x_{21} & x_{22} & . & . & . & x_{2k} \\ 1 & x_{31} & x_{32} & . & . & . & x_{3k} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 1 & x_{g1} & x_{g2} & . & . & . & x_{gk} \end{pmatrix}_{g\times(k+1)}, \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \end{pmatrix}_{(k+1)\times 1}$$

and $\underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ . \\ \varepsilon_g \end{pmatrix}_{g \times 1}$

With $E\left[\underset{\sim}{\varepsilon}\right] = \underset{\sim}{0}$ and $\operatorname{cov}[\underset{\sim}{\varepsilon}] = \left(\left(W_{ij}\right)\right)$ (2.26)

Where $W_{ij} = \begin{cases} \psi'(r_i) & i = j \\ 0 & i \neq j \end{cases}$ (2.27)

Using the standard technique for generalized least square, then the estimates of the model parameters of the eq. (2-24) can be found by minimizing the sum of squares of the error set $\{\varepsilon_i\}$ that is, we minimize:

$$\Omega(\underset{\sim}{\beta}) = \sum_{i=1}^{g} \frac{1}{\psi'(r_i)} \varepsilon_i^2 = \sum_{i=1}^{g} \frac{1}{\psi'(r_i)} \left(\underset{\sim}{Z}_i - \underset{\sim}{X}_i' \underset{\sim}{\beta}\right)^2 = \left(\underset{\sim}{Z} - \underset{\sim}{X}\,\underset{\sim}{\beta}\right)' \underset{\sim}{W}^{-1} \left(\underset{\sim}{Z} - \underset{\sim}{X}\,\underset{\sim}{\beta}\right)$$

$$= \underset{\sim}{Z}' \underset{\sim}{W}^{-1} \underset{\sim}{Z} - \underset{\sim}{Z}' \underset{\sim}{W}^{-1} \underset{\sim}{X}\,\underset{\sim}{\beta} - \underset{\sim}{\beta}' \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{Z} + \underset{\sim}{\beta}' \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X}\,\underset{\sim}{\beta}$$

We differentiate partially $\Omega(\underset{\sim}{\beta})$ with respect to $\underset{\sim}{\beta}$, we have:

$$\frac{\partial \Omega(\underset{\sim}{\beta})}{\partial \underset{\sim}{\beta}} = -2 \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{Z} + 2 \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X}\,\underset{\sim}{\beta}$$

For minimum, we set:

$$\left.\frac{\partial \Omega(\underset{\sim}{\beta})}{\partial \underset{\sim}{\beta}}\right|_{\underset{\sim}{\beta} = \underset{\sim}{\hat{\beta}}} = \underset{\sim}{0}$$

Thus, the WLS estimator $\underset{\sim}{\hat{\beta}}_w$ is:

$$\underset{\sim}{\hat{\beta}}_w = \left(\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{Z} \quad (2.28)$$

The WLS $\hat{\underset{\sim}{\beta}}_w$ is unbiased estimator for $\underset{\sim}{\beta}$ what ever the values of r_i and n_i . Viz

$$E\left[\hat{\underset{\sim}{\beta}}_w\right]=E\left[\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{Z}\right]=\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}E\left[\underset{\sim}{Z}\right]$$
$$=\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\,\underset{\sim}{\beta}=\underset{\sim}{\beta}$$

With variance-covariance matrix is:

$$\operatorname{cov}\left[\hat{\underset{\sim}{\beta}}_w\right]=\operatorname{cov}\left[\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{Z}\right]$$
$$=\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\operatorname{cov}[\underset{\sim}{Z}]\underset{\sim}{W}^{-1}\underset{\sim}{X}\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}$$
$$=\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{W}\underset{\sim}{W}^{-1}\underset{\sim}{X}\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1} \tag{2.29}$$
$$=\left(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}\right)^{-1}$$

Where $\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}=\left(\left(\sum_{i=1}^{g}x_{ir}x_{is}/\psi'(r_i)\right)\right),\qquad r=s=0,1,\ldots,k$

2.4 Estimation of $\underset{\sim}{\beta}$ by Suggest Weighted Least Squares Method

The weighted least squares estimator for $\underset{\sim}{\beta}$ is given by eq. (2-28) need tables of the digamma and trigamma tables for evaluation the numerical values of $\hat{\underset{\sim}{\beta}}_w$.

If tables of the digamma and trigamma are not available, good approximations [9] are given by:

$$\psi(\alpha)\approx\ln\alpha-\left(2\alpha-\frac{1}{3}+\frac{1}{16\alpha}\right)^{-1} \tag{2.30}$$

$$\psi'(\alpha) \approx \left(\alpha - \frac{1}{2} + \frac{1}{10\alpha}\right)^{-1} \tag{2.31}$$

Using these approximations, the SWLS estimator for $\underset{\sim}{\beta}$ is:

$$\hat{\underset{\sim}{\beta}}_m = \left(\underset{\sim}{X}' \underset{\sim}{W}_*^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}' \underset{\sim}{W}_*^{-1} \underset{\sim}{Z}^* \tag{2.32}$$

Where

$$Z_i^* = \ln T_i + (2r_i - \tfrac{1}{3} + \tfrac{1}{16r_i})^{-1}, \qquad i = 1, 2, \ldots, g \tag{2.33}$$

$$W_{ij}^* = \begin{cases} \left(r_i - \frac{1}{2} + \frac{1}{10r_i}\right)^{-1} & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \tag{2.34}$$

In practice, we find that the estimators obtained by the three methods of estimation (ML, WLS, SWLS), for instant, the ML estimator $\hat{\underset{\sim}{\beta}}$ has an important property that $\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta}$ is distributed independently of $\underset{\sim}{\beta}$, similar properties applying to WLS and SWLS estimators.

This property for ML estimator can be deduced from the likelihood equation given by (2-13) which can be written as:

$$\sum_i r_i x_{ir} t_i^* e^{-x_i'\left(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta}\right)} = \sum_i r_i x_{ir}, \qquad r = 0, 1, \ldots, k \tag{2.35}$$

Where

$$t_i^* = t_i e^{-x_i' \hat{\underset{\sim}{\beta}}}, \qquad i = 1, 2, \ldots, g \tag{2.36}$$

The set of observation $\{t_i^*\}$ represented a sample for independent standardized gamma r.v.'s with p.d.f.'s:

$$f_i(t_i^*) = \begin{cases} \frac{1}{\Gamma(r_i)} r_i \left(r_i t_i^*\right)^{r_i - 1} e^{-r_i t_i^*} & , \quad 0 < t_i^* < \infty \\ & \quad\ i = 1, 2, \ldots, g \\ 0 & , \quad e.w. \end{cases} \tag{2.37}$$

Since these p.d.f. do not dependent of $\underset{\sim}{\beta}$, the property follows.

We further have:

$$\underset{\sim}{\hat{\beta}} - \underset{\sim}{\beta} \overset{d}{\sim} \underset{\sim}{\hat{\beta}}^{(0)} \tag{2.38}$$

Where $\underset{\sim}{\hat{\beta}}^{(0)}$ denoted the ML estimator of $\underset{\sim}{\beta}$ when the true model has $\underset{\sim}{\beta} = \underset{\sim}{0}$. Similarly

$$\underset{\sim}{\hat{\beta}}_w - \underset{\sim}{\beta} \overset{d}{\sim} \underset{\sim}{\hat{\beta}}_w^{(0)} \text{ and } \underset{\sim}{\hat{\beta}}_m - \underset{\sim}{\beta} \overset{d}{\sim} \underset{\sim}{\hat{\beta}}_m^{(0)} \tag{2.39}$$

Where $\underset{\sim}{\hat{\beta}}_w^{(0)}$ and $\underset{\sim}{\hat{\beta}}_m^{(0)}$ denoted respectively the WLS and SWLS estimators of $\underset{\sim}{\beta}$ when $\underset{\sim}{\beta} = \underset{\sim}{0}$.

These results show that all central moments and moment properties such as bias, variance, mean square error, skewness and kurtosis are independent of $\underset{\sim}{\beta}$ for the three estimators. So without loss of generality, we may therefore take $\underset{\sim}{\beta} = \underset{\sim}{0}$.

2.5 Bias Approximation for the Methods of Estimation

The WLS estimator:

$$\underset{\sim}{\hat{\beta}}_w = \left(\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{Z}$$

is shown in section 2.3 an unbiased for $\underset{\sim}{\beta}$ for all r_i and n_i, $i = 1, 2, \ldots, g$.

The SWLS estimator:

$$\underset{\sim}{\hat{\beta}}_m = \left(\underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \underset{\sim}{Z}^{\,*}$$

of section 2.4 can be Written as:

$$\underset{\sim}{\hat{\beta}}_m = \left(\underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \left(\underset{\sim}{Z} + \underset{\sim}{d}\right) \qquad (2.40)$$

Where

$$d_i = \psi(r_i) + \left(2r_i - \tfrac{1}{3} + \tfrac{1}{16r_i}\right)^{-1} - \ln r_i\,, \qquad i = 1,2,\ldots,g \qquad (2.41)$$

So the bias of the SWLS is:

$$\underset{\sim}{b}_* = \left(\underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}'\, \underset{\sim}{W}_*^{-1} \underset{\sim}{d} \qquad (2.42)$$

Approximation to the bias of ML estimator is developed by AL-Abood [1] corrected to order R^{-1} with the condition the x-values satisfy the orthogonality conditions:

$$\sum_i r_i x_{ir} x_{is} = 0, \qquad \forall r \neq s = 0,1,\ldots,k \qquad (2.43)$$

In this case the biases take the form:

$$\left.\begin{aligned} & b_0 = \frac{k+1}{2R} \qquad \text{and} \\ & b_r = \frac{-1}{2\sum_i r_i x_i^2} \sum_{s=1}^{k} \left\{ \frac{\sum_i r_i x_{ir} x_{is}^2}{\sum_i r_i x_{is}^2} \right\} \qquad \text{where} \quad r = 1,2,\ldots,k \end{aligned}\right\} \qquad (2.44)$$

Conclusions

1. The values of the simulated biases of SWLS method for β_0 are very close to those of WLS when $r \geq 2$, while the values of the simulated biases of ML are over estimate for all values of r. Furthermore the values of β_1 obtained by the three methods are adequate for $r \geq 2$. We notes that the values of the biases in the three methods are decreases when r increases and these result become better as the number of groups increase.
2. The ML values of the simulated variances of β_0 and β_1 are generally have smaller values than these obtained by WLS and SWLS when $r \leq 7$ and become close for $r \geq 8$. For WLS and SWLS, the values of the variances become identical when $r \geq 6$.
3. The results of skewness of WLS and SWLS are generally smaller than the skewness of ML which indicating that normal approximation to the distribution of the estimators will be effective even for small values of r. Generally the skewness values decreases as r increases.
4. In the three methods of estimation, most of the values of kurtosis negative which indicate that the data is concentrated to the top of the distribution curve.
5. The results of Table 3.9 show that there is good agreement between the approximating efficiencies of formula (3.22) and the values of E_{10}. The results show that ML estimators have superior mean square errors performance than the WLS estimators for small values of r, but as expected the differences in efficiencies of WLS and SWLS estimators are negligible for all values of r.

6. In all methods of estimation we find the results of estimates values of β_1 are very near to the reality from the estimate values of β_0 because of the effective of the explanatory variables.
7. The results of SWLS in general are satisfaction in compare with the results of WLS but less accurate because of the bias existence.
8. The important result obtained from Monte Carlo simulation is that the distribution of $\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta}$ is independent of $\hat{\underset{\sim}{\beta}}^{(0)}$.

Introduction

In recent years there is an increasing availability in the use of various types of regression models for the analysis of what so called lifetime, survival time or failure time data that have an end point the time until the failure or certain event occurs. The major areas of applications of such models appear in biomedical, industrial life testing, and reliability terminology.

We assume that observations are available on the failure time of n independent individuals. The main problem in this work examined the developing methods for assessing the dependence of failure time on the explanatory variables. The second problem involves the estimation and specifying models to represent lifetime distribution and of making inference based on these models.

We shall illustrate some applications of such models in present of the explanatory variables or without it, for instance, in medical field with a curable illness, the survival time might represent the time form initial diagnosis of illness to complete recovery, while the case of serious illness such as cancer, the survival time would represent time to death, and the explanatory variables might refer to various attributes of the patients such as age, sex, initial severity of illness, months from diagnosis ,prior therapy, etc. in an industrial context, survival time might represent lives of components subject to failure and the explanatory variables may refer to different operating condations for the component being tested, such as temperature, pressure, friction. Economists in the world of employments, and demographers study the length of time that people are in work force Kpedekpo [20], where employers are concerned with length of time employees work before changing jobs. Hoadley [16] studied the length of time telephones remain disconnected in vacant quarters in order to determine which telephones to remove for use else where and which to leave in for the next customer. Gross and Clark [14] study the success of medical treatments for certain diseases in measured by the length of patients survival. Prince [33] evaluates the life of TV programs, The Association for Advancement of Medical Instrumentation [5] has a proposed standard with methods for estimating the life of heart pacemakers, etc.

More details on models representation and associated statistical analysis are given by:

Feigl and **Zelen** [12] applied the exponential model on leukemia survival data using different type of explanatory variables on each patient, and the parameters are estimated by the method of maximum likelihood.

Pike [31] investigated the applications of the carcinogenesis on two groups of rates to examine two types of pretreatment regimens.

Hoel [17] assumed that the functional from of each age distribution is known up to the point of unknown parameters. These parameters plus the unknown net probabilities are estimated from the data by the maximum likelihood methods.

Nelson and **Hahn** [26] dealt with the estimation problems of the regression parameters when censored data are collected from accelerated life data of motorettes.

Prentice [32] suggested an exponential survivor function with censoring, and a number of explanatory variables describing patient etiology, general medical status and clinical stage of disease are recorded when a patient is taken on the study, when such concomitant variables seem important, an exponential relation between failure rate and explanatory variable is suggested and the result are extended to include a weibull component in the hazard.

Lawless [22] discussed estimation and predication procedures for model which is commonly used in reliability and life testing work, the so called inverse law model, which exponential time to fail data, and described confidence interval estimation procedure for this model, so this procedures do not involve the use to any asymptotic approximations and so all distributions given exact for any sample size.

Crowely and **Hu** [9] considered a number of analyses to asses the effects of various covariates on the survival of patients in the Stanford Heart Transplantation program.

Noure and **Readt** [30] considered an approach to the proportional hazards analysis of the survival data with covariates by parametric modeling of piece wise distributions. MLS using GLIM and an iterative method is straight forward are used with two applications, based on the weibull distributions are described and some possible generalizations are indicated.

Kronborg and **Aaby** [21] considered the problem of comparing survival functions or equivalently comparing baseline hazards in the stratified proportional hazards model, and proposed test is a direct of identity of survival functions in the sense that it is nonparametric and is sensitive against a broader class of alternatives than the proportional hazards alternative.

Lin and **Wei** [24] were interested in estimating the cumulative hazard function and survival function under the Cox proportional hazard model.

Razooq [34] discussed two methods of estimation namely maximum (ML) and weighted least square (WLS) for the estimation of the regression coefficient related to Weibull type II censored data. Goodness of fit test for assumed model is given and testing is mode on the any subset of non effective regresses.

Sigeo Aki and **Katuomi Hirano** [36] extracted formula for the lifetime distn for k-out-of-n in: F system where the observations are ordered according to three well-know distributions, exponential, Weibull, Pareto and they obtain feasible estimators by using moment method.

Sparling, Y. H. et al [38] consider a parametric family of the regression models for evaluating interval-censored event time (survival) data, where the employed Newton-Raphson method when the underling distributions are Weibull, negative binomial, and log-logistic distributions as a special cases.

The aim of this thesis is to find the best methods for estimating the regression parameters related to exponential of type II censoring data. One method is developed and the observed estimators are compared with the other methods of estimation taken place by Monte Carlo simulation.

This thesis consists of three chapters:

In chapter one, we give a brief summary of survival type II right censored data, regression models representation, exponential and gamma as a lifetime distributions, and derivation to some transform results.

In chapter two, we present three methods of estimation for regression coefficients of type II censored exponentially distributed data, where the regression model impose on the mean as:

$$\mu = \exp(\underset{\sim}{x}\,\underset{\sim}{\beta})$$

The utilized methods of estimation are maximum likelihood (ML), weighted least squares (WLS) and suggest weighted least squares (SWLS). We present moment properties of the estimators such as bias, variance, and mean square errors are presented theoretically.

In chapter three, we consider the results of Monte Carlo investigation of the biases of ML and SWLS estimators, moment properties of the three methods of estimation, a new bias reduction estimator for the ML estimators is develop and show a higher efficiency in mean square error with respect to the ML, WLS, and SWLS, and the Tables of these results.

References

[1] ***AL-Abood, A., M. (1997),*** Bias Reduction for a Gamma Regression Models, Journal of Saddam University, 1, No. 2, 49-61.

[2] ***Abramowitz, M., & Stegun, I. A. (1964),*** Handbook of Mathematical Tables, New York, 253-266.

[3] ***AL-Aggedy, J. S. A. (2000),*** Estimation of a Regression Parameters Related to Exponential Survival Data of Type II Censoring, M .Sc. Thesis, University of Baghdad.

[4] ***AL-Faris, R. Q. Z. (1995),*** Approximation to the Incomplete Gamma Integral with Random Variates Procedures Simulation. M. Sc. Thesis, Dept. of Math and comp. Applications, Al-Nahrain University.

[5] ***Association for the Advancement of Medical Instrumentation (1975),*** "AAMI Implantable Pacemaker Standard (Proposed)," Document AAMI IP-P 10/75.

[6] ***Bartholomew, D. J. (1957),*** A Problem in Life Testing. J. Am. Stat. Assoc., 52, 350-355.

[7] ***Cordeiro, G., M. (1983),*** Improved Likelihood Ratio Statistics for Generalized Linear Models, J. R. Stat. B, 45, No. 3, 404-413.

[8] ***Cox, D., R., and Lewis, P., A. (1966),*** The Statistics Analysis of Series of Events, New York and London.

[9] ***Crowley, J., and Hu, M. (1977),*** Covariance Analysis of Heart Transplant Survival Data, J. Am. Stat. Assoc. 72, 27-36.

[10] ***Dahiya, R. C., and Gurland, J. (1972),*** Goodness of Fit Tests for the Gamma and Exponential Distributions, Technometrics, 14, No. 3, 791-801.

[11] ***Epstein, B. (1958),*** The Exponential Distribution and its Role in Life-testing. Ind. Qual. Control, 15, 2-7.

[12] ***Feigle, P. and M. Zelen (1965),*** Estimation of Exponential Survival Probabilities with Concomitant Information, Biometrics, 21, 826-838.

[13] ***Greenberg, R., S. Bayard, and D. Byar (1974),*** Goodness of Fit of an Exponential Survival Model. Biometrics, 30, 601-608.

[14] ***Gross, A. J., and Clark, V. A. (1975),*** Survival Distributions: Reliability Applications in the Biometrical Sciences, John Wiley, New York.

[15] ***Gupta, S. S., and Groll, F. A. (1961),*** Gamma Distribution in Acceptance Sampling Based on Life Test, J. Am. Stat. Assoc., 56, 942-970.

[16] ***Hoadley, B. (1970),*** "Strategies for Removing Telephones When Service is Disconnected," Presented at the Joint Statistical Meeting, Detroit, MI.

[17] ***Hoel, D. G. (1972),*** A Representation of Mortality Data by Computing Risks, Biometrics, 28, 475-488.

[18] ***Kahn, H. D. (1979),*** Least Square estimation for the Inverse Power Law for Accelerated Life Tests, Appl. Stat., 28, No. 1, 40-46.

[19] ***Kalbfleisch, J. D., and Prentice, R. L. (1980),*** The Statistical Analysis of Failure Time Data, John Wiley, New York.

[20] ***Kpedekpo, D. D. (1969),*** "Working Life Tables for Males in Ghana, 1960," J. Am. Stat. Assoc. 64, 102-110.

[21] ***Kronborg, D., and Aaby, P. (1990),*** Piecewise Comparison of Survival Functions in Stratified Proportional Hazards Models, Biometrics, 46, 375-380.

[22] ***Lawless, J. F. (1976),*** Confidence Interval Estimation in Inverse Power Law Model, Appl. Stat., 25, No. 2, 128-338.

[23] ***Lawless, J. F. (1982),*** Statistical Models and Methods for Life Time Data, John Wiley, New York.

[24] ***Lin, D. Y., Fleming, T. R., and Wei, L. J. (1994),*** Confidence Bands for Survival Curves under the Proportional Hazards Model, Biometrika, 81, 1, 73-81.

[25] ***Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974),*** Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

[26] ***Nelson, W., and Hahn, G. L. (1972),*** Linear Estimation of a Regression Relationship and their Application, Technometrics, 14, No. 2, 247-276.

[27] ***Nelson, W. (1982),*** Applied Life Data Analysis, John Wiley, New York.

[28] ***Nelson, W. B. (1972),*** Graphical Analysis of Accelerated Life Test Data with the Inverse Power Law Model. IEEE Trans. Reliab., R21, 2-11.

[29] ***Neter, J., Wasserman, W., and Kunter, M. H. (1985),*** Applied Linear Statistical Models, 2nd – edition.

[30] ***Noura A. A. , Read, K. L. Q. (1990),*** Proportional Hazards Change Point Models in Survival Analysis, Appl., Stat., 39, No.2, 241-253.

[31] ***Pike, M. C. (1966),*** A Model of Analysis of a Certain Class of Experiments in Carcinogenesis, Biometrics, 22, 142-161.

[32] ***Prentice, R. L. (1973),*** Exponential Survivals with Censoring and Explanatory Variables, Biometrika, 60, 2, 279-288.

[33] ***Prince, M. (1967),*** "Life Table Analysis of Prime Time Programs on a Television Network," Am. Stat. 21, 21-23.

[34] ***Razooq, N. N. (1994),*** Analysis of Type II Censoring Data Related to the Weibull Distribution, M. Sc. Thesis, Collage of Science, Saddam University.

[35] ***Rubinstein, R. Y.*** **(1981),** Simulation and Monte Carlo Method, John Wiley, New York.

[36] ***Sigeo Aki and Katuomi Hirano (1995).*** Lifetime Distribution and Estimation Problems Consecutive-k-out-of-n: F system. Math. & Stat. Vol. 48, No. 1 / March 1996.

[37] ***Sinha, D., Tanner, M. A., and Hall, W. J. (1994),*** Maximization of the Marginal Likelihood of Grouped Survival Data, Biometrika, 81, 1, 53-60.

[38] ***Sparling, Y. H., Younes, N., and Lachin, J. M. (2006),*** Parametric Survival Models for Interval-Censored Data with Time-Dependent Covariates. Biostatistics, Vol. 7, No. 4, 599-614.

[39] ***Zippin, C., and Armitage, P. (1996),*** Use of Concomitant Variables and Incomplete Survival Information in the Estimation of an Exponential Survival Parameter, Biometrics, 21, 665-672.

Abstract

In this thesis, we consider a regression models for survival censored data of type II in which the underling distributions are exponential or gamma where the effect of the regressor variables on the means is multiplication given by the model

$$\mu_i = \exp(\underset{\sim}{x}_i' \underset{\sim}{\beta})$$

Three methods of estimation for the regression coefficients are considered, namely maximum likelihood (ML), weighted least squares (WLS), and suggest weighted least squares (SWLS). These methods are discussed theoretically and examined practically by Monte Carlo simulation for the case of a single explanatory variable.

Moments and higher moments properties of the estimators, such as, bias, variance, skewness, and kurtosis are examined, illustrated and compared.

Finally, a new bias reduction estimator to the ML estimator is proposed and shows a higher performance with respect to the other estimators.

Acknowledgments

Thanks to Allah for his kindness and blessing had given us to conduct this piece of work.

I am especially indebted and grateful to my supervisor Dr. Akram A. Al-Abood for valuable support, advice, suggestion, comments, and for the time and effort generously dedicated for supervising this work.

Also I'd like to express my thanks to the stuff of the mathematics and computer applications department: Dr. Alaudin N. Ahmed, Dr. Ahlam J. Khaleel, Dr. Fadhel S. Fadhel, and Dr. Radhi A. Al-Zboon.

I am grateful for the College of Sciences, Al-Nahrain University, for giving me the life opportunity to complete my study.

Last, but certainly not least, my great appreciation and sincere thanks to my family for their help and support.

Rasha A. Ali

2007

Contents

Examining committee certification

We certify that we read this thesis entitled "ORDER STATISTICS FOR TYPE II CENSORED EXPONENTIALLY DISTRIBUTED DATA IN ACCORDENCE OF EXPLANATORY VARIABLES" and as examining committee examined the student, Rasha Abdul Hussein Ali Al-Nea'amy in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

(Chairman)
Signature:
Name: Dr. Abdul Mejeed H. Al-Nasser Ahmead
Date: 27 / 9 /2007

(Member)
Signature:
Name: Dr. Alauldin N.
Date: 27 / 9 /2007

(Member)
Signature:
Name: Dr. Fadhel Subhi Fadhel
Date: 27 / 9 /2007

(Member and Supervisor)
Signature:
Name: Dr. Akram M. Al-Abood
Date: 27 / 9 /2007

Approved for the Dean of the College of Education

Signature:

Name: Assist. Prof. Dr. Laith Abdul Aziz Al-Ani

Dean of College of Science of Al-Nahrain University

Date: / /2007

List of Abbreviations

Eq.	Equation
Distn	Distribution
r.s.	Random Sample
r.v.	Random Variable
p.d.f.	Probability Density Function
c.d.f.	Cumulative Distribution Function
m.g.f.	Moment Generating Function
ML	Maximum Likelihood
WLS	Weighted Least Squares
SWLS	Suggest Weighted Least Squares
MSE	Mean Square Error
$X \sim U(a, b)$	The r.v. X has Uniform Distn with constant a and b.
$X \sim Exp(\lambda)$	The r.v. X has Exponential Distn with Failure Rate λ
$X \sim G(\alpha, \beta)$	The r.v. X has Gmma Distn with α and β
$X \sim \chi^2(r)$	The r.v. X has Chi-square Distn with r Degree of Freedom

Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science

ORDER STATISTICS FOR TYPE II CENSORED EXPONENTIALLY DISTRIBUTED DATA IN ACCORDENCE OF EXPLANATORY VARIABLES

A Thesis
Submitted to the Department of Mathematics,
College of Science, Al-Nahrain University, as a Partial
Fulfillment of the Requirements for the Degree of Master
of Science in Mathematics

By

Rasha Abdul Hussein Ali Al Nea'amy
(B.Sc. 2003)

Under Supervision of

Dr. Akram M. Al-Abood

June 2007 Rejab 1427

وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

الاحصائيات المرتبة لبيانات المراقبة من الصنف الثاني والموزعة أسياً والمتأثرة بوجود المتغيرات التفسيرية

رسالة

مقدمة إلى كلية العلوم في جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

رشا عبد الحسين علي النعيمي
(بكالوريوس علوم، ٢٠٠٣)

بإشراف

د. اكرم محمد العبود

حزيران ٢٠٠٧

المستخلص

في هذه الاطروحة، تطرقنا لنماذج الانحدار لبيانات البقاء المراقبة من الصنف الثاني في حالة كون التوزيع الاساسي هو التوزيع الآسي او غاما حيث ان تأثير المتغيرات التفسيرية على المعدلات معطاة بالنموذج:

$$\mu_i = \exp(\underset{\sim}{x}_i' \underset{\sim}{\beta})$$

طرق ثلاث من التخمين قد أعتمدت لتخمين معالم الانحدار وهي طريقة الامكان الاعظم (ML)، طريقة المربعات الصغرى الموزونة (WLS)، وطريقة المربعات الصغرى الموزونة المقترحة (SWLS). هذه الطرق قد اختبرت نظرياً وفحصت عملياً بأستخدام محاكاة مونت كارلو في حالة وجود متغير تفسيري واحد.

خواص العزوم والعزوم العليا مثل التحيز، التباين، الالتواء، والتفلطح قد جدولت وقورنت.

واخيراً اقترحنا مخمن متحيز مخفض جديد لمخمن الامكان الاعظم (ML) واظهر اداء اكثر كفاءة بالنسبة للمخمنات الاخرى.