Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science
Department of Mathematics
and Computer Applications

On the Homotopy Perturbation Method and its Applications

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Farah Lateef Joey
(B.Sc. Math, Al-Nahrain University, 2008)

Supervised by
Asst. Prof. Dr. Ahlam J. Khaleel *Lect. Dr.Shatha Ahmad Aziz*

Rabee Al-Thanee 1432
March 2011

بســـــمِ اللهِ الرحمنِ الرحِيـــــمِ

وسارِعوا إلى مغفِرةٍ مِن ربِكم وجنةٍ عرضها السموات والأرض اعدت لِلمتقيـــــــن

صدق الله العلي العظِيم

سورة ال عمران الاية (133)

الإهداء

إلى...

من أحيا القلوب بعد مماتها وأنارها بعد ظلمتها وآلف بينها بعد شتاتها الحبيب المصطفى محمد(صلى الله عليه وآله الغر الميامين).

الى مثلي الاعلى الذي كان يشجعني دوما
على الدراسة حتى وصلت الى هذه المرحلة والدي الشهيد

الى شمس اشرقت في وجهي وتدفعني
للامام دوما أمي وأسرتي

والى كل من يعتبر الرياضيات طريقاً في الحيـــاة

ـرح

Acknowledgements

Before anything …

Thanks to Allah for helping me to complete my thesis

I wish to express my deep appreciation and sincere thanks to my supervisors Dr. Ahlam Jameel and Dr. Shatha Ahmad for their appreciable advices, important comments support and encouragement during the research.

It is of my pleasure to record my respect to the staff members of Mathematics and Computer Applications, whom gave me all facilities during my work.

Thanks are extended to the College of Science of Al-Nahrain University for giving me the chance to complete my postgraduate study.

Also, I would like to express my deepest thanks to all my family and friends, and every one who helped me in my project.

Farah

March 2011

Abstract

The aim of this work is to use the homotopy perturbation method to solve special types of the local and nonlocal problems. This study including the following aspects:

(1) Give some basic concepts of the homotopy perturbation method.

(2) Use the homotopy perturbation method to solve some types of differential, integral and integro-differential equations.

(3) Describe some nonlocal problems and use the homotopy perturbation method to solve them.

(4) Use the homotopy perturbation method to solve some real life applications and these applications are advection-diffusion problems, gas dynamics problem and the ground -water level problem.

Contents

Chapter Three: The Homotopy Perturbation Method for Solving Some Nonlocal Problems

Chapter Four: Solutions of Some Real Life Applications Via the Homotopy Perturbation Method

Introduction

The homotopy perturbation method was first proposed by He J. Huan in 1999, where the solution of this method is considered as the sum of an infinite series which is very rapidly converge to the accurate solution, [13].

The homotopy perturbation method, presents some advantages: obtaining exact solutions with high accuracy, minimal calculations without loss of physical verification. This method has found application in different fields of nonlinear equations such as fluid mechanics and heat transfer, [10].

Many authors and researchers studied the homotopy perturbation method, say, He J. in 1999, used the homotopy perturbation method for solving nonlinear ordinary differential equations of the first and second orders, [13], He J. in 2003, solved the Nonlinear ordinary differential equations with n^{th} order, [14], He J. in 2004, solved the oscillators equation with discontinuities via the homotopy perturbation method, [17], He J. in 2005, studied the homotopy perturbation method for solving one dimensional nonlinear wave equation, [18], Li-Na Z. and He J. in 2006, solved the electrostatic potential differential equation, [19].

Yu-Xi Wang and et al. in 2008, used the homotopy perturbation method for solving reaction-diffusion equation, [38], Fatemeh S. and Mehdi D. in 2008, solved the deley differential equations via the homotopy perturbation method, [9], Jafari H. and et al. in 2008, used the homotopy perturbation method for solving Gas Dynamics Equation, [22], Jazbi B.

and Moini M. in 2008, used the homotopy perturbation method for solving Schrodinger equation, [23].

Lin Jin in 2008, studied the homotopy perturbation method to solve the three dimensions parabolic and hyperbolic partial differential equations with variable coefficients, [29], Behrouz R., in 2009, solved nonlinear Volttera partial integro-differential equations of the second kind, [4], Ghorbali A. R., and et al. in 2009, used the homotopy perturbation method for solving the three dimensional heat equation with variable coefficients, [11].

Mashallah M. and Mohammad S. in 2010, studied the homotopy perturbation method for solving linear fuzzy Fedholem integral equations, [33], Allahviranloo T. and et al. in 2010, solved the linear fuzzy Volttera integral equations of the second kind, [1], Roozi A. and et al. in 2011, studied the homotopy perturbation method for solving nonlinear parabolic and hyperbolic partial differential equations of one and two dimensions, [35].

The nonlocal problems plays an important role in real life applications and they used in various field of mathematical physics and in other fields, [3].

Many authors and researchers studied the nonlocal problem, say, Karakostas G. L. and Tsamatas P. Ch. In 2000, studied a nonlocal boundary value problem for a second order ordinary differential equations, [25], Beilin S. in 2002, studied the existence of Solutions for One-Dimensional Wave equations with Nonlocal conditions, [6], Ruyun M. in 2007, presented a servay of recent results on the existence and multiplicity of solutions of nonlocal boundary value problem involving second order ordinary differential equations, [36].

The purpose of this thesis is to give a full information for the homotopy perturbation method and its applications for solving the non-linear Fredholm integral and integro-differential equations. Also, this method is used to solve special types of nonlocal problems. Moreover, the solutions of some real life applications are obtained via the homotopy perturbation method.

This thesis consists of four chapters:

In chapter one, some basic concepts of the homotopy perturbation method are described and used to solve the nonlinear ordinary differential equations with and without initial conditions.

In chapter two, the solutions of the linear integral equation of the second kind are obtained via the homotopy perturbation method and its convergence is presented. Also, the solution of the non-linear integral and integro-differential equations of the second kind are obtained by means of the homotopy perturbation method.

In chapter three, the homotopy perturbation method is used to solve the one dimensional wave and hyperbolic integro-differential equations with non-homogeneous Neumann and nonlocal conditions, respectively.

In chapter four, the solutions of the advection-diffusion problem with initial boundary conditions, gas dynamics problem with initial condition and the ground water level problem with non-homogeneous Dirichlet and nonlocal conditions are obtained via the homotopy perturbation method.

Chapter One
The Homotopy Perturbation Method for Solving the Ordinary Differential Equations

Introduction:

The homotopy perturbation method proposed first by He J. Huan in 1999 for solving differential and integral equations, linear and nonlinear has been the subject of extensive analytic and numerical studies. This method has a significant advantage in that it provides an approximated solution to a wide range of nonlinear problems in applied sciences. In this method, the solution is considered as the summation of an infinite series which usually converges rapidly to the solutions, [13].

In this chapter, some basic ideas of this method has been explained.

This chapter consists of two sections:

In section one, some basic concepts of the homotopy perturbation method are described.

In section two, we use this method for solving the non-linear ordinary differential equations.

1.1 Some Basic Concepts of the Homotopy Perturbation Method:

In this section, we give some basic concepts of the homotopy perturbation method. To do this, we recall the following definition:

Definition (1.1.1), [32]:

Let X and Y be two topological spaces. Two continuous functions $f : X \longrightarrow Y$ and

$g: X \longrightarrow Y$ are said to be homotopic, denoted by $f \cong g$, if there exists a continuous function $H: X \times [0,1] \longrightarrow Y$, such that:

$$H(x,0) = f(x), \ \forall x \in X$$

$$H(x,1) = g(x), \ \forall x \in X$$

In this case, H is said to be a homotopy.

Now, to illustrate this definition, consider the following examples:

Example (1.1.2):

Let X and Y be any topological spaces, f be the identity function and g be the zero function, then define $H : X \times [0,1] \longrightarrow Y$ by:

$$H(x, p) = x(1 - p), \ \forall \mathrm{x} \in X, \ \forall \mathrm{p} \in [0,1]$$

Then H is a continuous function and

$$H(x,0) = x = f(x), \ \forall \mathrm{x} \in X$$

$$H(x,1) = 0 = g(x), \ \forall \mathrm{x} \in X$$

Therefore $f \cong g$.

Next, the following proposition appeared in [32] without proof, here we give its proof.

Proposition (1.1.3):

On the continuous functions $\cong$ is an equivalence relation.

Proof:

Let $f: X \longrightarrow Y$ be a continuous function, then define $H: X \times [0,1] \longrightarrow Y$ by:

$$H(x,p) = f(x),\ \forall x \in X,\ \forall p \in [0,1]$$

Therefore

$$H(x,0) = f(x),\ \forall x \in X$$

$$H(x,1) = f(x),\ \forall x \in X$$

and this implies that $f \cong f$. Therefore $\cong$ is a reflexive relation.

To prove $\cong$ is a symmetric relation, let $f \cong g$, then there exists a continuous function $H: X \times [0,1] \longrightarrow Y$ such that:

$$H(x,0) = f(x),\ \forall x \in X$$

$$H(x,1) = g(x),\ \forall x \in X$$

Define $K: X \times [0,1] \longrightarrow Y$ by:

$$K(x,p) = H(x,1-p),\ \forall x \in X,\ \forall p \in [0,1]$$

Then

$$K(x,0) = H(x,1) = g(x),\ \forall x \in X$$

$$K(x,1) = H(x,0) = f(x),\ \forall x \in X$$

Hence $g \cong f$.

To prove $\cong$ is a transitive relation, let $f \cong g$ and $g \cong w$, then there exist continuous functions $H: X \times [0,1] \longrightarrow Y$ and $K: X \times [0,1] \longrightarrow Y$

such that:

$$H(x,0) = f(x), \ \forall x \in X$$

$$H(x,1) = g(x), \ \forall x \in X$$

$$K(x,0) = g(x), \ \forall x \in X$$

$$K(x,1) = w(x), \ \forall x \in X$$

Define $L : X \times [0,1] \longrightarrow Y$ by:

$$L(x,p) = \begin{cases} f(x), & \mathrm{p} = 0 \\ H(x,p) + K(x,p) - g(x), & 0 < \mathrm{p} < 1 \\ w(x), & \mathrm{p} = 1 \end{cases}$$

Therefore

$$L(x,0) = f(x), \ \forall x \in X \ ,$$

$$L(x,1) = w(x), \ \forall x \in X$$

$$\begin{aligned} \lim_{p \to 0^+} L(x,p) &= \lim_{p \to 0^+} \left[H(x,p) + K(x,p) - g(x) \right] \\ &= H(x,0) + K(x,0) - g(x) \\ &= f(x) + g(x) - g(x) \\ &= f(x) \\ &= L(x,0), \quad \forall \mathrm{x} \in \mathrm{X}, \end{aligned}$$

and

$$\begin{aligned} \lim_{p \to 1^-} L(x,p) &= \lim_{p \to 1^-} \left[H(x,p) + K(x,p) - g(x) \right] \\ &= H(x,1) + K(x,1) - g(x) \\ &= g(x) + w(x) - g(x) \\ &= w(x) \\ &= L(x,1), \quad \forall \mathrm{x} \in \mathrm{X}, \end{aligned}$$

Hence L is a continuous function. Therefore $f \cong w$. Hence $\cong$ is an equivalence relation on the set of all continuous functions.

Remark (1.1.4):

Let X and Y be two topological spaces, let $f:\Re \longrightarrow \Re$ and $g:\Re \longrightarrow \Re$ be continuous functions. Define $H:\Re \times [0,1] \longrightarrow \Re$ by:

$$H(x,p)=(1-p)f(x)+pg(x), \quad \forall x \in X, \ \forall p \in [0,1]$$

Then

$$H(x,0)=f(x), \quad \forall x \in X$$

and

$$H(x,1)=g(x), \quad \forall x \in X$$

Therefore $f \cong g$.

Definition (1.1.5), [32]:

Let X and Y be two topological spaces and $f:X \longrightarrow Y$ be a continuous function. The equivalence class of f, denoted by $[f]$ is defined by:

$$[f]=\{g \mid g:X \longrightarrow Y \text{ be a continuous function and } f \cong g\}$$

and it is said to be a homotopy class of functions of f.

Remark (1.1.6):

(1) It is clear that $f \in [f]$, for every continuing function f defined from a topological space X into a topological space Y. Therefore $[f] \neq \phi$.

(2) By using theorem (2.4.4) in [31], the set of equivalence classes of $\cong$ form a partition of continuous functions.

Definition (1.1.7), [32]:

Let X and Y be two topological spaces. Two continuous functions f and g are said homotopic relative to $A \subseteq X$ if there exists a continuous function $H : X \times [0,1] \longrightarrow Y$ such that:

$$H(x,0) = f(x), \ \forall x \in X$$

$$H(x,1) = g(x), \ \forall x \in X$$

$$H(a,p) = f(a) = g(a), \ \ \forall \mathrm{p} \in [0,1], \ \ \forall a \in A$$

Now, to illustrate the basic idea of the homotopy perturbation method, we consider the following non-linear equation:

$$A(u) = f(x), \ \ \mathrm{x} \in \Omega \tag{1.1}$$

where A is any operator, f is a known function of x. The operator A can generally speaking be divided into two parts L and N, where L is a linear operator, and N is a non-linear operator. Therefore equation (1.1) can be rewritten as follows:

$$L(u) + N(u) - f(x) = 0$$

According to [13], we can construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies the homotopy equation:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0$$

or

$$H(v,p) = L(v(x)) - L(u_0(x)) + pL(u_0(x)) + p[N(v(x)) - f(x)] = 0 \qquad (1.2)$$

where p $\in [0,1]$, $\Re$ represents the set of all real numbers and u_0 is an initial approximation of the solution of equation (1.1).

Obviously, from equation (1.2) we have:

$$H(v,0) = L(v) - L(u_0) = 0$$

$$H(v,1) = A(v) - f(x) = 0$$

The changing process of p from zero to unity is just that of v(x,p) from u_0(x) to u(x).

Therefore

$$L(v) - L(u_0) \cong A(v) - f(x), \quad x \in \Omega$$

and

$$u_0(x) \cong u(x), \quad x \in \Omega.$$

Assume that the solution of equation (1.1) can be written as a power series in p as follows:

$$v(x,p) = \sum_{i=0}^{\infty} p^i v_i(x) \qquad (1.3)$$

By setting p=1 in equation (1.3), one can get:

$$u(x) = \lim_{p \to 1} v(x,p) = \sum_{i=0}^{\infty} v_i(x) \qquad (1.4)$$

which is the solution of equation (1.1)

The series (1.4) is convergent for some cases, [13].

1.2 The Homotopy Perturbation Method for Solving The Non-Linear Ordinary Differential Equations, [13]:

In this section, we use the homotopy perturbation method to solve the non-linear ordinary differential equations with or without (initial or boundary) conditions. To do this, consider the following non-linear ordinary differential equation:

$$A(y) = f(x), \quad x \in \Omega \tag{1.5}$$

where A is any differential operator, f is a known function of x. The operator A can be divided into two parts L and N where L is a linear operator while N is a non-linear operator. Therefore equation (1.5) can be rewritten as:

$$L(y) + N(y) - f(x) = 0, \quad x \in \Omega$$

So, by the homotopy perturbation method, we can construct a homotopy $u : \Omega \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(u, p) = L(u(x, p)) - L(y_0(x)) + pL(y_0(x)) + p[N(u(x, p)) - f(x)] = 0 \tag{1.6}$$

where $p \in [0,1]$, $\Re$ represents the set of all real numbers and y_0 is the initial approximation for the solution of equation (1.5) which satisfies the initial or boundary conditions if they exist.

Obviously, from equation (1.6) one can have:

$$H(u,0) = L(u(x,0)) - L(y_0(x)) = 0$$

$$H(u,1) = L(u(x,1)) + N(u(x,1)) - f(x) = 0$$

The changing process of p from zero to unity is just that of u(x,p) from $y_0(x)$ to $y(x)$.

Therefore

$$L(u) - L(y_0) \cong A(u) - f(x), \quad \mathrm{x} \in \Omega$$

and

$$y_0(x) \cong y(x), \qquad \mathrm{x} \in \Omega.$$

Assume that, the solution of equation (1.5) can be written as power series in p:

$$u(x,p) = \sum_{i=0}^{\infty} p^i u_i(x) \tag{1.7}$$

where u_0, u_1, ... are the unknown functions that must be determined. By setting p=1 in the above equation one can obtain:

$$y(x) = \lim_{p \to 1} u(x,p) = \sum_{i=0}^{\infty} u_i(x) \tag{1.8}$$

which is the solution of the differential equation (1.5).

The infinite series given by equation (1.8) is convergent for some cases. However, the convergence rate depends on the non-linear operator A. The following opinions are suggested by He J. to ensure that the convergence of the infinite series given by equation (1.8):

1. The second derivative of N(u) with respect to u must be small.

2. The norm of $L^{-1}\dfrac{\partial N}{\partial u}$ must be smaller than one.

To illustrate this method, consider the following examples.

Example (1.2.1):

Consider the first order nonlinear ordinary differential equation:

$$y'(x) + y^2(x) = 0, \quad |\mathrm{x}| < 1 \tag{1.9}$$

Here $A(y) = y' + y^2$ and $f(x) = 0$. The operator A can be divided into two parts L and N, where $L(y) = y'$ and $N(y) = y^2$.

In this case, equation (1.6) becomes:

$$u'(x) - y_0'(x) + py_0'(x) + p[u^2(x)] = 0, \quad \text{p} \in [0,1]$$

Assume the solution of the above equation can be written as given in equation (1.7). By substituting this solution into the above equation one can have:

$$\sum_{i=0}^{\infty} p^i u_i'(x) - y_0'(x) + py_0'(x) + p\left[\sum_{i=0}^{\infty} p^i u_i(x)\right]^2 = 0$$

By equating the terms with identical powers of p one can have:

$$p^0 : u_0'(x) - y_0'(x) = 0 \qquad (1.10.a)$$

$$p^1 : u_1'(x) + y_0'(x) + u^2{}_0(x) = 0 \qquad (1.10.b)$$

$$p^2 : u_2'(x) + 2u_0(x)u_1(x) = 0 \qquad (1.10.c)$$

$$p^3 : u_3'(x) + 2u_2(x)u_0(x) + u^2{}_1(x) = 0 \qquad (1.10.e)$$

$$\vdots$$

For simplicity, let $u_0(x) = y_0(x)$, then equation (1.10.a) is automatically satisfied.

Let $y_0(x) = 1$ be the initial approximation of the differential equation (1.9), then

$$u_0(x) = 1$$

By substituting u_0 and y_0 into equation (1.10.b) one can have:

$$u_1'(x) = -1$$

and this implies that:

$$u_1(x) = -x$$

By substituting u_0 and u_1 into equation (1.10.c) one can have:

$$u_2'(x) = 2x$$

and this implies that:

$$u_2(x) = x^2$$

By substituting u_1, u_2 and u_3 into equation (1.10.e) one can have:

$$u_3'(x) = -3x^3$$

and this implies that:

$$u_3(x) = -x^3$$

By continuing in this manner one can have:

$$u_i(x) = (-1)^i x^i, \text{ i = 0, 1, } \dots$$

By substituting these functions into equation (1.8) one can obtain:

$$y(x) = \sum_{i=0}^{\infty} (-1)^i x^i$$

$$= \frac{1}{1+x}.$$

which is the exact solution of the ordinary differential equation (1.9).

Example (1.2.2):

Consider the first order nonlinear ordinary differential equation:

$$y(x) + [y(x) + x]y'(x) = 0 \qquad (1.11)$$

Here $A(y) = y + (y + x)y'$ and $f(x) = 0$. The operator can be divided into two parts L and N, where $L(y) = y$ and $N(y) = (y + x)y'$

In this case, equation (1.6) becomes:

$$u(x) - y_0(x) + py_0(x) + p[u(x) + x]u'(x) = 0, \quad p \in [0, 1]$$

By substituting equation (1.7) into the above equation one can have:

$$\sum_{i=0}^{\infty} p^i u_i(x) - y_0'(x) + py_0(x) + p\left[\sum_{i=0}^{\infty} p^i u_i(x) + x\right]\left[\sum_{i=0}^{\infty} p^i u_i'(x)\right] = 0$$

By equating the terms with identical powers of p one can have:

$$p^0 : u_0(x) - y_0(x) = 0 \qquad (1.12.a)$$

$$p^1 : u_1(x) + y_0(x) + (u_0(x) + x)u_0'(x) = 0 \qquad (1.12.b)$$

$$p^2 : u_2(x) + [u_0(x) + x]u_1'(x) + u_1(x)u_0'(x) = 0 \qquad (1.12.c)$$

$$\vdots$$

Let $y_0(x) = x$, then from equation (1.12.a) one can have:

$$u_0(x) = x.$$

From equation (1.12.b) one can have:

$$u_1(x) = -3x$$

Therefore, the first approximation of equation (1.11) is:

$$y(x) = u_0(x) + u_1(x) = -2x$$

which is the exact solution of the ordinary differential equation (1.11).

That is, the first approximation in this example is sufficient to give the exact solution. Next, if we choose $y_0(x) = 1$, then $u_0(x) = 1$, from equation (1.12.b) one can have:

$$u_1(x) = -1$$

and from equation (1.12.c) one can get:

$$u_2(x) = 0$$

and in a similar manner one can obtain:

$$u_i(x) = 0, \quad \text{i} = 3,4,...$$

so

$$y(x) = \sum_{i=0}^{\infty} u_i(x) = 0$$

which is the exact solution of the ordinary differential equation (1.11).

Example (1.2.3), [14]:

Consider the first order nonlinear ordinary differential equation:

$$y'(x) + y^2(x) = 0, \quad 0 < \text{x} < 1 \tag{1.13.a}$$

together with the initial condition

$$y(0) = 1 \tag{1.13.b}$$

Here $A(y(x)) = y'(x) + y^2(x)$ and $f(x) = 0.$

To solve this example by the homotopy perturbation method, consider equations (1.10):

For simplicity, let $u_0(x) = y_0(x) = y(0) = 1,$ then equation (1.10.a) is automaticly satisfied.

Since

$$y(x)=\sum_{i=0}^{\infty}u_i(x)$$

Then

$$y(0)=\sum_{i=0}^{\infty}u_i(0)$$

But $u_0(0)=1,$ therefore $u_i(0)=0, \mathrm{i}=1,2,....$ Thus, equation (1.10.b) becomes:

$$\begin{aligned}u_1'(x)&=-y_0'(x)-u^2{}_0(x)\\&=-1.\end{aligned}$$

By integrating both sides of the above ordinary differential equation from 0 to x and using the initial condition $u_1(0)=0$ one can get:

$$u_1(x)=-x\,.$$

From equation (1.10.c) one can have:

$$\begin{aligned}u_2'(x)&=-2u_0(x)u_1(x)\\&=2\mathrm{x}.\end{aligned}$$

Then by integrating both sides of the above differential equation from 0 to x and by using the initial condition $u_2(0)=0$, one can get:

$$u_2(x)=x^2.$$

By continuing in this manner one can obtain:

$$u_i(x)=(-x)^i,\quad \mathrm{i}=0,1,....$$

Therefore

$$y(x)=\sum_{i=0}^{\infty}u_i(x)=1-x+x^2+\cdots$$

$$=\frac{1}{1+x}.$$

which is the exact solution of the above initial value problem.

Example (1.2.4):

Consider the first order nonlinear ordinary differential equation:

$$y(x)+y(x)y'(x)=1 \qquad (1.14.a)$$

together with the initial condition

$$y(0)=1 \qquad (1.14.b)$$

Here $A(y)=y+yy'$ and $f(x)=1$. The operator A can be divided into two parts L and N, where $L(y)=y$ and $N(y)=yy'$

In this case, equation (1.6) becomes:

$$u(x)-y_0(x)+py_0(x)+p[u(x)u'(x)-1]=0, \quad \text{p}\in[0,1]$$

By substituting equation (1.7) into the above equation one can have:

$$\sum_{i=0}^{\infty}p^iu_i(x)-y_0(x)+py_0(x)+p\left[\sum_{i=0}^{\infty}p^iu_i(x)\sum_{i=0}^{\infty}p^iu_i'(x)-1\right]=0$$

By equating the terms with identical powers of p one can have:

$$p^0:u_0(x)-y_0(x)=0 \qquad (1.15.a)$$

$$p^1:u_1(x)+y_0(x)+u_0(x)u_0'(x)-1=0 \qquad (1.15.b)$$

$$\vdots$$

Let $y_0(x) = 1$, then from equation (1.15.a) one can have:

$$u_0(x) = 1$$

From equation (1.15.b) one can have:

$$u_1(x) = 0$$

By continuing in this manner one can obtain:

$$u_i(x) = 0, \quad \mathrm{i} = 2,3,....$$

Therefore

$$y(x) = \sum_{i=0}^{\infty} u_i(x) = 1.$$

which is the exact solution of the above initial value problem.

Chapter Two
The Homotopy Perturbation Method for Solving the Integral and Integro-Differential Equations

Introduction:

Many researchers used analytical methods and numerical methods for solving linear and nonlinear integral and integro-differential equations [24], [26], [28].

The aim of this chapter is to use the homotopy perturbation method for solving special types of linear and nonlinear integral and integro-differential equations of the second kind.

This chapter consists of three sections:

In section one, we describe the homotopy perturbation method for sovling the linear Fredholm and Voltera integral equations of the second kind with its convergence.

In section two and three, we use the homotopy perturbation method to solve special types of non-linear Fredholm integral and integro-differential equations of the second kind.

2.1 The Homotopy Perturbation Method for Solving Linear Integral Equations, [2]:

In this section, we use the homotopy perturbation method to solve the linear integral equations of the second kind. To do this, first, consider the linear Fredholm integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad \mathrm{x} \in [\mathrm{a},\mathrm{b}] \tag{2.1}$$

where f and k are known functions. The function f is said to be the driving term and k is said to be the kernel function that depends on x, t and λ is a scalar parameter, a and b are known constants and u is the unknown function that must be determined.

We rewrite equation (2.1) as:

$$A(u) - f(x) = 0 \qquad (2.2)$$

where $A(u) = u(x) - \lambda\int_a^b k(x,t)u(t)dt$.

Then the integral operator A can be divided into two parts L and N such that equation (2.2) becomes:

$$L(u) + N(u) - f(x) = 0 \qquad (2.3)$$

where $Lu = u$ and $N = -\lambda\int_a^b k(x,t)udt.$

Accoding to [13], we construct a homotopy $v:[a,b]\times[0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = (1-p)[v(x,p) - u_0(x)] + p\left[v(x,p) - \lambda\int_a^b k(x,t)v(t,p)dt - f(x)\right] = 0 \qquad (2.4)$$

where p$\in$ [0,1], $\Re$ represents the set of all real numbers and u_0 is the initial approximation to the solution of equation (2.1).

By using equation (2.4) it follows that:

$$H(v,0) = v(x,0) - u_0(x) = 0$$

$$H(v,1) = v(x,1) - \lambda\int_a^b k(x,t)v(t,1)dt - f(x) = 0$$

and the changing process of p from zero to unity is just that of $v(x,p)$ from $u_0(x)$ to $u(x)$. Therefore

$$v(x,0) - u_0(x) \cong v(x,1) - \lambda\int_a^b k(x,t)v(t,1)dt - f(x), \qquad x \in [a,b]$$

and

$$u_0(x) \cong u(x), \qquad x \in [a,b].$$

Next, we assume that the solution of equation (2.4) can be expressed as

$$v(x,p) = \sum_{i=0}^{\infty} p^i v_i(x) \tag{2.5}$$

Therefore the approximated solution of the integral equation (2.1) can be obtained as follows:

$$\begin{aligned} u(x) &= \lim_{p \to 1} v(x,p) \\ &= \sum_{i=0}^{\infty} v_i(x) \end{aligned} \tag{2.6}$$

By substituting the approximated solution given by equation (2.5) into equation (2.4) one can get:

$$\sum_{i=0}^{\infty} p^i v_i(x) - u_0(x) + p u_0(x) + p\left[-\lambda \int_a^b k(x,t) \sum_{i=0}^{\infty} p^i v_i(t) dt - f(x)\right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : v_0(x) - u_0(x) = 0 \tag{2.7.a}$$

$$p^1 : v_1(x) + u_0(x) - f(x) - \lambda \int_a^b k(x,t) v_0(t) dt = 0 \tag{2.7.b}$$

$$p^j : v_j(x) - \lambda \int_a^b k(x,t) v_{j-1}(t) dt = 0, \quad j = 2,3,\ldots \tag{2.7.c}$$

For simplicity we set $v_0(x) = u_0(x) = f(x)$, then equation (2.7.a) is automatically satisfied. By substituting $u_0(x) = v_0(x) = f(x)$ into equation (2.7.b) one can have:

$$v_1(x) = \lambda\int_a^b k(x,t)\, f(t)dt$$

By substituting v_1 into equation (2.7.c) one can have:

$$v_2(x) = \lambda\int_a^b k(x,t)v_1(t)dt$$

In a similar manner one can get $v_i(x),\ \mathrm{i}=3,4,\ldots$. By substituting $v_i(x),\ \mathrm{i}=0,1,\ldots$ into equation (2.6) one can get the approximated solution of the integral equation (2.1).

Next, we study the convergence of the homotopy perturbation method for solving the integral equation (2.1). To do this, consider the iteration formula that is obtained by applying the homotopy perturbation method to solve the integral equation (2.1):

$$v_i(x) = \lambda\int_a^b k(x,t)v_{i-1}(t)dt, \quad \mathrm{i}=1,2,\ldots \qquad (2.8)$$

with the initial approximation $v_0(x) = f(x)$

According to the previous equation, we define the partial sum as follow:

$$s_n(x) = \sum_{i=0}^{n} v_i(x), \quad \mathrm{n}=0,1,\ldots \qquad (2.9)$$

where $s_0(x) = f(x)$.

In view of equations (2.8) and (2.9), one can have:

$$s_0(x) = f(x)$$

and

$$s_{n+1}(x)=\sum_{i=0}^{n+1}v_i(x)=v_0(x)+\sum_{i=0}^{n}v_i(x)$$

$$=f(x)+\lambda\int_a^b k(x,t)v_0(t)dt+\lambda\int_a^b k(x,t)v_1(t)dt+\cdots+\lambda\int_a^b k(x,t)v_n(t)dt$$

$$=f(x)+\lambda\int_a^b k(x,t)\left[v_0(t)+v_1(t)+\cdots+v_n(t)\right]dt$$

$$=f(x)+\lambda\int_a^b k(x,t)s_n(t)dt.$$

From [21], it is known that if $v_1,v_2,...$ be a sequence of functions, then the series $\sum_{i=1}^{\infty}v_i(x)$ is said to be convergence to u if the sequence $\{s_n\}$ of partial sums defined by:

$$s_n(x)=\sum_{i=0}^{n}v_i(x)$$

converges to u.

Now, we are in the position that we can give the following theorem.

Theorem (2.1), [21]:

Consider the iteration scheme:

$$s_0(x)=f(x)$$

and

$$s_{n+1}(x)=f(x)+\lambda\int_a^b k(x,t)s_n(t)dt,\quad \text{n}=0,1,...$$

to construct a sequence of successive iterations $s_n(x)$ to the solution of equation (2.1). Let $f \in L^2(a,b)$ and

$$\int_a^b\int_a^b |k(x,t)|^2\,dxdt = B^2 < \infty.$$

If $|\lambda| < \frac{1}{B}$, then the above iteration scheme convergence to the solution of equation (2.1).

To illustrate this method, consider the following example.

Example (2.2):

Consider the following linear Fredholm integral equation of the second kind:

$$u(x) = e^{3x} - \frac{1}{9}(2e^3+1)x + \int_0^1 xtu(t)dt, \quad 0 \le \mathrm{x} \le 1$$

Here $a=0$, $\mathrm{b} = \lambda = 1$, $\mathrm{f(x)} = \mathrm{e}^{3\mathrm{x}} - \frac{x}{9}\left[2e^3+1\right]$ and $k(x,t) = xt.$

Therefore

$$\int_a^b\int_a^b |k(x,t)|^2\,dxdt = \int_0^1\int_0^1 x^2t^2\,dxdt = \frac{1}{9} = B^2 < \infty.$$

and

$$|\lambda| = 1 < \frac{1}{B} = 3.$$

So, we can use the homotopy perturbation method to solve this example.

To do this, let

$$u_0(x) = v_0(x) = f(x) = e^{3x} - \frac{x}{9}\left[2e^3 + 1\right].$$

Hence

$$v_1(x,t) = \lambda \int_a^b k(x,t) v_0(t) dt$$
$$= \int_0^1 xt\left[e^{3t} - \frac{\mathrm{t}}{9}\left(2e^3 + 1\right)\right]dt$$
$$= \left[\frac{2 + 4e^3}{27}\right]x.$$

and

$$v_2(x,t) = \lambda \int_a^b k(x,t) v_1(t) dt$$
$$= \int_0^1 xt^2\left[\frac{2 + 4e^3}{27}\right]dt$$
$$= \left[\frac{2 + 4e^3}{81}\right]x.$$

By continuing in this manner, one can have:

$$v_i(x,t) = \lambda \int_a^b k(x,t) v_{i-1}(t) dt$$
$$= \left[\frac{2 + 4e^3}{27(3)^{i-1}}\right]x, \ \mathrm{i} = 1,2,....$$

Thus

$$u(x)=\sum_{i=0}^{\infty}v_i(x)$$

$$=\mathrm{e}^{3\mathrm{x}}-\frac{x}{9}\left[2e^3+1\right]+\sum_{i=1}^{\infty}\left[\frac{2+4e^3}{27(3)^{i-1}}\right]x$$

$$=\mathrm{e}^{3\mathrm{x}}-\frac{x}{9}\left[2e^3+1\right]+\left[\frac{2+4e^3}{27}\right]x\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i-1}$$

$$=\mathrm{e}^{3\mathrm{x}}-\frac{x}{9}\left[2e^3+1\right]+\frac{3}{2}\left[\frac{2+4e^3}{27}\right]x$$

$$=\mathrm{e}^{3\mathrm{x}}.$$

which is the exact solution of the above integral equation.

Example (2.3):

Consider the following linear Fredholm integral equation of the second kind:

$$u(x)=e^{3x}-\frac{8}{9}e^3x-\frac{4}{9}x+4\int_0^1 xtu(t)dt, \quad 0\le \mathrm{x}\le 1$$

Here $a=0$, $\mathrm{b}=1$, $\lambda=4$, $\mathrm{f(x)}=\mathrm{e}^{3\mathrm{x}}-\frac{8}{9}e^3x-\frac{4}{9}x$ and $k(x,t)=xt$.

Therefore

$$\int_a^b\int_a^b\left|k(x,t)\right|^2dxdt=\int_0^1\int_0^1 x^2t^2dxdt=\frac{1}{9}=B^2<\infty.$$

and

$$|\lambda|=4>\frac{1}{B}=3.$$

So, if we use the homotopy perturbation method to solve this example, then the solution that is obtained by applying this method may be convergent to the exact solution or may not. To do this, let

$$u_0(x) = v_0(x) = f(x) = e^{3x} - \left[\frac{8}{9}e^3 + \frac{4}{9}\right]x$$

Hence

$$\begin{aligned} v_1(x,t) &= 4\int_a^b xtf(t)dt \\ &= 4\int_0^1 xt\left[e^{3t} - \frac{8}{9}e^3 t - \frac{4}{9}t\right]dt \\ &= \left[\frac{-8}{27}e^3 - \frac{4}{27}\right]x \end{aligned}$$

and

$$\begin{aligned} v_2(x,t) &= 4\int_a^b xtv_1(t)dt \\ &= 4\int_0^1 xt^2\left[\frac{-8}{27}e^3 - \frac{4}{27}\right]dt \\ &= \left[\frac{-32}{81}e^3 - \frac{16}{81}\right]x. \end{aligned}$$

By continuing in this manner, one can have:

$$\begin{aligned} v_i(x,t) &= 4\int_0^1 xtv_{i-1}(t)dt \\ &= \left[\frac{-8(4)^{i-1}}{27(3)^{i-1}}e^3 - \frac{4(4)^{i-1}}{27(3)^{i-1}}\right]x, \ \mathrm{i} = 1,2,.... \end{aligned}$$

Thus

$$u(x)=\sum_{i=0}^{\infty}v_i(x)$$

$$=\mathrm{e}^{3\mathrm{x}}-\left[\frac{8}{9}e^3+\frac{4}{9}\right]x+\sum_{i=1}^{\infty}\left[\frac{-8(4)^{i-1}}{27(3)^{i-1}}e^3-\frac{4(4)^{i-1}}{27(3)^{i-1}}\right]x.$$

Since

$$\sum_{i=1}^{\infty}\left[\frac{-8}{27}\left(\frac{4}{3}\right)^{i-1}e^3-\frac{4}{27}\left(\frac{4}{3}\right)^{i-1}\right]x=\left[\frac{-8}{27}e^3-\frac{4}{27}\right]x\sum_{i=1}^{\infty}\left(\frac{4}{3}\right)^{i-1}.$$

But $\sum_{i=1}^{\infty}\left(\frac{4}{3}\right)^{i-1}$ is a geometric series, that is divergent since $|r|=\frac{4}{3}>1$. Therefore $\sum_{i=0}^{\infty}v_i(x)$ is divergent.

Next, consider the linear Volterra integral equation of the second kind:

$$u(x)=f(x)+\lambda\int_a^x k(x,t)u(t)dt \qquad (2.10)$$

where k is the kernel of the integral equation, f is the driving term, λ is a scalar parameter, a is a known constant and u is the unknown function that must be determined.

To solve this integral equation via the homotopy perturbation method, we rewrite equation (2.10) as:

$$A(u)-f(x)=0 \qquad (2.11)$$

where $A(u)=u(x)-\lambda\int_a^x k(x,t)u(t)dt$.

Then A can be divided into two parts L and N such that equation (2.11) becomes:

$L(u) + N(u) - f(x) = 0$

where $Lu = u$ and $Nu = -\lambda \int_a^x k(x,t)u(t)dt$.

Accoding to [13], we construct a homotopy $v:[a,\infty)\times[0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = (1-p)[v(x,p) - u_0(x)] + p\left[v(x,p) - \lambda \int_a^x k(x,t)v(t,p)dt\right] = 0 \qquad (2.12)$$

where p$\in$ [0,1] $\Re$ represents the set of all real numbers and u_0 is the initial approximation to the solution of equation (2.10).

By using equation (2.12) it follows that:

$$H(v,0) = v(x,0) - u_0(x) = 0$$

$$H(v,1) = v(x,1) - \lambda \int_a^x k(x,t)v(t,1)dt - f(x) = 0$$

and the changing process of p from zero to unity is just that of $v(x,p)$ from $v(x,0) - u(x)$ to

$v(x,1) - \lambda \int_a^x k(x,t)v(t,1)dt - f(x).$

Therefore

$$v(x,0) - u_0(x) \cong v(x,1) - \lambda \int_a^x k(x,t)v(t,1)dt - f(x), \qquad x \geq a$$

and

$$u_0(x) \cong u(x), \qquad x \geq a.$$

Next, we assume that the solution of equation (2.12) can be expressed as in equation (2.5). By substituting this approximated solution into equation (2.12) one can get:

$$\sum_{i=0}^{\infty} p^i v_i(x) - u_0(x) + pu_0(x) + p\left[-\lambda\int_a^x k(x,t)\sum_{i=0}^{\infty} p^i v_i(t)dt - f(x)\right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : v_0(x) - u_0(x) = 0$$

$$p^1 : v_1(x) + u_0(x) - f(x) - \lambda\int_a^x k(x,t)v_0(t)dt = 0$$

$$p^j : v_j(x) - \lambda\int_a^x k(x,t)v_{j-1}(t)dt = 0, \quad \text{j} = 2,3,...$$

So, if we choose $v_0(x) = u_0(x) = f(x)$, then one can get the iteration formula.

$$v_0(x) = f(x)$$

$$v_i(x) = \lambda\int_a^x k(x,t)v_{i-1}(t)dt, \quad \text{i} = 1,2,...$$

Next, we study the convergence of the homotopy perturbation method for solving the integral equation (2.10). To do this, consider the following theorem.

Theorem (2.4), [21]:

Consider the iteration scheme:

$$s_0(x) = f(x)$$

and

$$s_{n+1}(x) = f(x) + \lambda \int_a^x k(x,t) s_n(t)dt, \quad \mathrm{n} = 0,1,...$$

to construct a sequence of successive iterations $s_n(x)$ to the solution of the integral equation (2.10).

If f and k are real-valued continuous functions, then the above iteration scheme convergence to the solution of the integral equation (2.10) for all values of λ.

To illustrate this method consider the following example.

Example (2.5):

Consider the following linear Volterra integral equation of the second kind:

$$u(x) = x + \lambda \int_0^x (x-t)u(t)dt$$

To solve this example via the homotopy perturbation method, consider the iteration formula:

$$v_0(x) = u_0(x) = x.$$

$$v_{i+1}(x) = \lambda \int_0^x (x-t) v_i(t)dt, \quad \mathrm{i} = 0,1,....$$

Therefore

$$v_1(x) = \lambda \int_0^x t(x-t)dt = \frac{\lambda}{3!} x^3$$

and

$$v_2(x) = \frac{\lambda^2}{3!}\int_0^x t^3 (x-t)dt$$
$$= \frac{\lambda^2}{5!}x^5.$$

By continuing in this manner, one can have:

$$v_i(x,t) = \frac{\lambda^i}{(2i+1)!}x^{2i+1}, \text{ i} = 1,2,....$$

It is known that the sequence

$$s_n(x) = \sum_{i=0}^{n} \lambda^i \frac{x^{2i+1}}{(2i+1)!}$$

which is convergent for all values of λ and x.

Therefore

$$u(x) = \sum_{i=0}^{\infty} \lambda^i \frac{x^{2i+1}}{(2i+1)!}.$$

which is the exact solution of the above integral equation.

Note that, if $\lambda = -1,$ then $u(x) = \sin(x).$

2.2 The Homotopy Perturbation Method For Solving Non-Linear Integral Equations:

In this section, the homotopy perturbation method is used to solve special types of non-linear Fredholm integral equations with some illustrative examples. To do this, consider the following non-linear Fredholm integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)[u(t)]^q \, dt, \qquad a \le x \le b \tag{2.13}$$

where $q \in \mathbb{N}$, f is the driving term and k is the kernel of the integral equation that depends on x and t, λ is a scalar parameter, a and b are known constants and u is the unknown function that must be determined.

We rewrite equation (2.13) as:

$$A(u) - f(x) = 0 \tag{2.14}$$

where $A(u) = u(x) - \lambda \int_a^b k(x,t)[u(t)]^q \, dt.$

Then the integral operator A can be divided into two parts L and N equation (2.14) becomes:

$$L(u) + N(u) - f(x) = 0$$

where $L(u) = u$ and $N(u) = -\lambda \int_a^b k(x,t)[u(t)]^q \, tdt$.

Accoding to [13], we can construct a homotopy $v : [a,b] \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = (1-p)[v(x,p) - u_0(x)] + p\left[v(x,p) - \lambda \int_a^b k(x,t)[v(t,p)]^q dt - f(x)\right] = 0 \tag{2.15}$$

where $p \in [0,1]$, $\Re$ represents the set of all real numbers and u_0 is the initial approximation to the solution of equation (2.13).

By using equation (2.15) it follows that:

$$H(v,0) = v(x,0) - u_0(x) = 0$$

$$H(v,1) = v(x,1) - \lambda\int_a^b k(x,t)[v(t,1)]^p\,dt - f(x) = 0$$

and the changing process of p from zero to unity is just that of $v(v,p)$ from $u_0(x)$ to $u(x)$. Therefore

$$v(x,0) - u_0(x) \cong v(x,1) - \lambda\int_a^b k(x,t)[v(t,1)]^q dt - f(x), \qquad \mathrm{x} \in [a,b]$$

and

$$u_0(x) \cong u(x), \qquad \mathrm{x} \in [a,b].$$

Next, we assume that the solution of equation (2.15) can be expressed as in equation (2.5). By substituting the approximated solution given by equation (2.5) into equation (2.15) one can get:

$$\sum_{i=0}^{\infty} p^i v_i(x) - u_0(x) + pu_0(x) + p\left[-\lambda\int_a^b k(x,t)\left[\sum_{i=0}^{\infty} p^i v_i(t)\right]^q dt - f(x)\right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : v_0(x) - u\ \ (x) = 0 \qquad (2.16.a)$$

$$p^1 : v_1(x) + u_0(x) - f(x) - \lambda\int_a^b k(x,t)[v_0(t)]^q\,dt = 0 \qquad (2.16.b)$$

$$p^2:\begin{cases} v_2(x)-\lambda\int_a^b k(x,t)\left[2v_0(t)v_1(t)\right]dt=0, & \text{if } q=2\\ v_2(x)-\lambda\int_a^b k(x,t)\left[3(v_0(t))^2v_1(t)\right]dt=0, & \text{if } q=3\\ v_2(x)-\lambda\int_a^b k(x,t)\left[4(v_0(t))^3v_1(t)\right]dt=0, & \text{if } q=4\\ \vdots \end{cases} \quad (2.16.c)$$

$$p^j:\begin{cases} v_j(x)-\lambda\int_a^b k(x,t)\sum_{k=0}^{j-1}\left[v_k(t)v_{j-k-1}(t)\right]dt=0, & \text{if } q=2\\ v_j(x)-\lambda\int_a^b k(x,t)\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\left[v_i(t)v_k(t)v_{j-k-i-1}(t)\right]dt=0, & \text{if } q=3\\ v_j(x)-\lambda\int_a^b k(x,t)\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\sum_{l=0}^{j-i-k-1}\left[v_i(t)v_k(t)v_l(t)v_{j-l-k-i-1}(t)\right]dt=0, & \text{if } q=4\\ \vdots \end{cases} \quad (2.16.e)$$

where j=3,4,….

For simplicity, we set $v_0(x)=u_0(x)=f(x)$, then equation (2.16.a) is automatically satisfied. By substituting $v_0(x)=u_0(x)=f(x)$ into equation (2.16.b) one can have:

$$v_1(x)-\lambda\int_a^b k(x,t)[v_0(t)]^q dt=0$$

By substituting v_0, v_1 into equation (2.16.c) one can get $v_2(x)$.

In a similar manner, one can get $v_i(x), \text{i}=3,4,...$ By substituting $v_i(x), \text{i}=0,1,...$ into equation (2.6) one can get the approximated solution of the integral equation (2.13).

To illustrate this method consider the following examples.

Example (2.4):

Consider the following nonlinear Fredholm integral equation of the second kind:

$$u(x)=\frac{4}{5}x^2+\frac{1}{6}+\int_0^1 (x^2-t)[u(t)]^2\,dt, \quad 0\le \mathrm{x} \le 1 \tag{2.17}$$

Here $a=0,\ \ \mathrm{b}=\lambda=1,\ \ \mathrm{q}=2,\ \ \mathrm{f(x)}=\frac{4}{5}x^2+\frac{1}{6}$ and $k(x,t)=x^2-t.$

To solve this example via the homotopy perturbation method, let

$$v_0(x)=u_0(x)=f(x)=\frac{4}{5}x^2+\frac{1}{6}$$
$$\cong 0.8\mathrm{x}^2+0.16667.$$

Then

$$v_1(x)=\int_a^b k(x,t)(v_0(t))^q\,dt$$
$$=\int_0^1 (x^2-t)\left[\frac{4}{5}t^2+\frac{1}{6}\right]^2 dt$$
$$=-0.18722+0.24467x^2.$$

In this case, let N=1, then

$$u(x)=\sum_{i=0}^{\mathrm{N}} v_i(x)=v_0(x)+v_1(x)\cong 1.04466x^2-2.05555\times 10^{-2}$$

Next, we must find $v_2(x)$:

$$v_2(x)=\int_a^b k(x,t)[2v_0(t)v_1(t)]dt$$
$$=2\int_0^1 (x^2-t)\left[\frac{4}{5}t^2+\frac{1}{6}\right]\left[\frac{-337}{1800}+\frac{367}{1500}t^2\right]dt$$
$$=0.02046-0.05678x^2.$$

In this case, let N=2, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 0.98789x^2 - 2.62963 \times 10^{-5}$$

The following table gives the approximated solution of example (2.4) for different values of N.

Table (1.1) represents the approximated solutions of example (2.4) for different values of N.

N	u (x)	N	u(x)	N	u(x)
0	$0.8x^2+0.166667$	4	$1.00222x^2–1.26717\times10^{-3}$	8	$0.99993x^2+1.31635\times10^{-5}$
1	$1.04466x^2–2.05555\times10^{-2}$	5	$0.99924x^2+6.40980\times10^{-5}$	9	$1.00001x^2+1.52175\times10^{-5}$
2	$0.98789x^2–9.62963\times10^{-5}$	6	$0.99994x^2+2.05376\times10^{-4}$	10	$1.00001x^2–1.03366\times10^{-5}$
3	$0.99762x^2+3.58386\times10^{-3}$	7	$1.00016x^2–1.06638\times10^{-4}$	11	$1.00001x^2–1.00453\times10^{-5}$

Note that from the above table one can deduce that as N increases, the approximated solution of the integral equation (2.17) converges to the exact solution $u(x) = x^2$.

2.3 The Homotopy Perturbation Method for Solving Non-Linear Fredholm Integro-Differential Equations, [26]:

In this section, the homotopy perturbation method is employed for solving the initial value problems of special types of the first order non-linear Fredholm integro-differential equations with some illustrative examples.

To do this, consider the following first order nonlinear Fredholm integro-differential equation of the second kind:

$$u'(x) = f(x) + \lambda \int_a^b k(x,t)[u(t)]^q \, dt, \qquad a \le x \le b \tag{2.18.a}$$

together with the initial condition:

$$u(a) = \alpha \tag{2.18.b}$$

where $q \in \mathrm{N}$, f is the driving term and k is the kernel of the integro-differential equation that depends on x and t, λ is a scalar parameter, a and b are known constants and u is the unknown function that must be determined.

It is clear that, if q=1, then the integro-differential equation is linear, otherwise it is nonlinear.

We rewrite equation (2.18.a) as:

$$A(u) - f(x) = 0 \tag{2.19}$$

where $A(u) = \dfrac{du}{dx} - \lambda \int_a^b k(x,t)[u(t)]^q \, dt$.

Then the operator A can be divided into two parts L and N such that equation (2.19) becomes:

$$L(u) + N(u) - f(x) = 0$$

where $L(u) = \dfrac{du}{dx}$ and $N(u) = -\lambda \int_a^b k(x,t)[u(t)]^q \, dt$.

According to [13], we construct a homotopy $v : [a,b] \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p)=(1-p)\left[\frac{dv(x,p)}{dx}-\frac{du_0(x)}{dx}\right]+p\left[\frac{dv(x,p)}{dx}-\lambda\int_a^b k(x,t)[v(t,p)]^q\,dt-f(x)\right]=0 \quad (2.20)$$

where p$\in$ [0,1] $\Re$ represents the set of all real numbers and u_0 is the initial approximation to the solution of equation (2.18.a) which satisfies the initial condition given by equation (2.18.b).

By using equation (2.20) it follows that:

$$H(v,0)=\frac{dv(x,0)}{dx}-\frac{du_0(x)}{dx}=0$$

$$H(v,1)=\frac{dv(x,1)}{dx}-\lambda\int_a^b k(x,t)[v(t,1)]^q\,dt-f(x)=0$$

and the changing process of p from zero to unity is just that of $v(x,p)$ from $u_0(x)$ to $u(x)$. Therefore

$$\frac{\partial v(x,0)}{\partial x}-\frac{\partial v_0(x)}{\partial x}\cong\frac{\partial v(x,1)}{\partial x}-\lambda\int_a^b k(x,t)[v(t,1)]^q dt-f(x),\qquad x\in[a,b]$$

and

$$u_0(x)\cong u(x),\qquad x\in[a,b].$$

Next, we assume that the solution of equation (2.18.a) can be expressed as in equation (2.5).

By substituting the approximated solution given by equation (2.5) into equation (2.20) one can get:

$$\sum_{i=0}^{\infty}p^i\frac{dv_i(x)}{dx}-\frac{du_0(x)}{dx}+p\frac{du_0(x)}{dx}+p\left[-\lambda\int_a^b k(x,t)\left[\sum_{i=0}^{\infty}p^i v_i(t)\right]^q dt-f(x)\right]=0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : \frac{dv_0(x)}{dx} - \frac{du_0(x)}{dx} = 0 \qquad (2.21.a)$$

$$p^1 : \frac{dv_1(x)}{dx} + \frac{du_0(x)}{dx} - f(x) - \lambda \int_a^b k(x,t)[v_0(t)]^q dt = 0 \qquad (2.21.b)$$

$$p^2 : \begin{cases} \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x,t)[2v_0(t)v_1(t)]dt = 0, & \text{if } q = 2 \\ \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x,t)[3(v_0(t))^2 v_1(t)]dt = 0, & \text{if } q = 3 \\ \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x,t)[4(v_0(t))^3 v_1(t)]dt = 0, & \text{if } q = 4 \\ \vdots \end{cases} \qquad (2.21.c)$$

$$p^j : \begin{cases} \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,t) \sum_{i=0}^{j-1} [v_k(t) v_{j-k-1}(t)]dt = 0, & \text{if } q = 2 \\ \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,t) \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} [v_i(t) v_k(t) v_{j-k-i-1}(t)]dt = 0, & \text{if } q = 3 \\ \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,t) \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \sum_{l=0}^{j-i-k-1} [v_i(t) v_k(t) v_l(t) v_{j-l-k-i-1}(t)]dt = 0, & \text{if } q = 4 \\ \vdots \end{cases} \qquad (2.21.e)$$

where j=3,4,….

Since $u(a) = \alpha$ then we choose $u_0(x) = \alpha + \int_a^x f(t)dt$ and this implies that $u_0(a) = \alpha$. Also, for simplicity we set $v_0(x) = u_0(x) = \alpha + \int_a^x f(t)dt$. So equation (2.21.a) is automatically satisfied.

Therefore by substituting $x = a$ in equation (2.6) one can have:

$$u(a) = \sum_{i=0}^{\infty} v_i(a)$$

But $v_0(a) = u(a) = \alpha,$ hence $v_i(a) = 0, \mathrm{i} = 1,2,....$ By substituting $v_0(x) = u_0(x) = \alpha + \int_a^x f(t)dt$ into equation (2.21.b) one can get:

$$\frac{dv_1(x)}{dx} - \lambda \int_a^b k(x,t) \left[\alpha + \int_a^t f(z)dz \right]^q dt = 0$$

By integrating both sides of the above differential equation and by using the initial condition $u_1(a) = 0$ one can obtain:

$$v_1(x) = \lambda \int_a^x \int_a^b k(s,t) \left[\alpha + \int_a^t f(z)dz \right]^q dtds = 0 \qquad (2.22)$$

By substituting v_0 and v_1 into equation (2.21.c) and by solving the resulting first order linear ordinary differential equation together with the initial condition $v_2(a) = 0$ one can get $v_2(x)$. Then by substituting j= 3 v_0, v_1 and v_2 into equation (2.21.e) and by using initial condition $v_3(a) = 0$ one can solve the resulting first order linear ordinary differential equation to get $v_3(x)$. In a similar manner one can get $v_i(x), \mathrm{i} = 4,5,....$ By substituting $v_i(x), \mathrm{i} = 0,1,...$ into equation (2.6) one can get the approximated solution of the initial value problem given by equations (2.18).

To illustrate this method consider the following examples.

Example (2.5), [28]:

Consider the initial value problem that consists of the first order linear Fredholm integro-differential equation of the second kind:

$$u'(x) = \frac{-1}{2} - \frac{5}{6}x + \int_0^1 (xt+1)u(t)dt, \qquad 0 \le x \le 1 \qquad (2.23.a)$$

together with the initial condition:

$$u(0) = 1 \qquad (2.23.b)$$

Here $a = 0$, $\mathrm{b} = \lambda = 1, \mathrm{f(x)} = \frac{-1}{2} - \frac{5}{6}x$ and $k(x,t) = xt + 1$.

We use the homotopy perturbation method to solve this example. To do this, let

$$v_0(x) = u_0(x) = \alpha + \int_a^b f(t)dt = 1 + \int_0^x \left[\frac{-1}{2} - \frac{5}{6}t\right]dt$$

$$= 1 - \frac{1}{2}x - \frac{5}{12}x^2$$

$$\cong 1 - 0.5x - 0.41667x^2.$$

Then

$$v_1(x) = \int_0^x \int_0^1 (st+1)\left[1 - \frac{1}{2}t - \frac{5}{12}t^2\right]dtds$$

$$\cong 0.11458x^2 + 1.375x.$$

In this case, let N=1, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x) \cong 1 + 0.11111x - 0.30208x^2$$

Next, we must find $v_2(x)$

$$v_2(x) = \int_0^x \int_0^1 (st+1)\left[0.11458x^2 + 1.375x\right] dtds$$

$$= 0.11617x^2 + 0.34375x.$$

Therefore, for N=2,

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 1 + 0.45486\text{x} - 0.18591x^2$$

In a similar manner one can get $v_i(x), i = 3,4,...$ .The following table gives the approximated solutions for different values of N.

Table (1.2) represents the approximated solutions of example (2.5) for different values of N.

N	u(x)	N	u(x)	N	U(X)
0	$1-0.5x-0.41667x^2$	9	$1+0.98903x-3.7401\times10^{-3}x^2$	18	$1+0.99986x-4.6173\times10^{-5}x^2$
1	$1+0.11111x-0.30208x^2$	10	$1+0.99327x-2.2952\times10^{-3}x^2$	19	$1+0.99992x-2.8335\times10^{-5}x^2$
2	$1+0.45486x-0.18591x^2$	11	$1+0.99587x-1.4085\times10^{-3}x^2$	20	$1+0.99995x-1.7389\times10^{-5}x^2$
3	$1+0.79470x-0.070018\times10^{-2}x^2$	12	$1+0.99747x-8.6440\times10^{-4}x^2$	21	$1+0.99997x-1.0671\times10^{-5}x^2$
4	$1+0.87401x-4.2969\times10^{-2}x^2$	13	$1+0.99844x-5.3047\times10^{-4}x^2$	22	$1+0.99998x-6.5488\times10^{-6}x^2$
5	$1+0.92268x-2.6369\times10^{-2}x^2$	14	$1+0.99905x-3.2554\times10^{-4}x^2$	23	$1+0.99999x-4.0189\times10^{-6}x^2$
6	$1+0.95255x-1.6183\times10^{-2}x^2$	15	$1+0.99941x-1.9978\times10^{-4}x^2$	24	$1+0.99999x-2.4663\times10^{-6}x^2$
7	$1+0.97088x-9.9310\times10^{-3}x^2$	16	$1+0.99964x-1.2260\times10^{-4}x^2$	25	$1+x-1.5136\times10^{-6}x^2$
8	$1+0.98213x-6.0945\times10^{-3}x^2$	17	$1+0.99978x-7.5238\times10^{-5}x^2$	26	$1+x-1.00336\times10^{-6}x^2$

Note that from the above table one can deduce that as N increases the approximated solution of the initial value problem given by equations (2.23) converges to the exact solution u(x)=1+x.

Example (2.6), [26]:

Consider the initial value problem that consider of the first order nonlinear Fredholm integro-differential equation:

$$u'(x) = \frac{159}{160} - \frac{1}{64}x^2 + \int_0^{\frac{1}{2}} (x^2 + t)[u(t)]^3 dt \qquad (2.24.a)$$

together with the initial condition:

$$u(0) = 0 \qquad (2.24.b)$$

Here a=0, b= $\frac{1}{2}$, q=3, $\lambda = 1$, $f(x) = \frac{159}{160} - \frac{1}{64}x^2$ and $k(x,t) = x^2 + t$.

We use the homotopy perturbation method to solve this example. To do this, let

$$v_0(x) = u_0(x) = \int_0^x \left[\frac{156}{160} - \frac{1}{64}t^2\right] dt$$

$$= \frac{159}{160}x - \frac{1}{192}x^3 \quad .$$

Then

$$v_1(x) = \int_0^x \int_0^{\frac{1}{2}} (s^2 + t)\left[\frac{159}{160}t - \frac{1}{192}t^3\right]^3 dtds \, .$$

$$\cong 6.11634 \times 10^{-3} x + 5.09791 \times 10^{-3} x^3 .$$

In this case, let N=1, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x)$$

$$\cong 0.99987x - 1.10428 \times 10^{-4} x^3.$$

Next, we must find $v_2(x)$

$$v_2(x) = \int_0^x \int_0^{\frac{1}{2}} (s^2 + t)[3v_0(t)v_1(t)]dtds$$

$$\cong 8.54066 \times 10^{-4} x + 8.38354 \times 10^{-4} x^3.$$

Therefore, for N=2,

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 1.00072x - 7.27926 \times 10^{-4} x^3 .$$

In a similar manner one can get $v_i(x), i = 3,4,...$. The following table gives the approximated solutions of example (2.6) for different values of N.

Table (1.3) represents the approximated solutions of example (2.6) for different values of N.

N	u(x)	N	u(x)	N	u(x)
0	$0.99375x-5.20833\times10^{-3}x^3$	4	$1.00086x+8.63897\times10^{-4}x^3$	8	$1.00086x+8.66701\times10^{-4}x^3$
1	$0.99987x-1.10428\times10^{-4}x^3$	5	$1.00086x+8.66308\times10^{-4}x^3$	9	$1.00086x+8.66702\times10^{-4}x^3$
2	$1.00072x-7.27926\times10^{-4}x^3$	6	$1.00086x+8.66651\times10^{-4}x^3$	10	$1.00086x+8.66703\times10^{-4}x^3$
3	$1.00084x+8.46953\times10^{-4}x^3$	7	$1.00086x+8.66700\times10^{-4}x^3$	11	$1.00086x+8.66704\times10^{-4}x^3$

Note that from the above table one can deduce that as N increases the approximated solution of the initial value problem given by equations (2.24) converges to the exact solution u(x)=x.

Example (2.7), [26]:

Consider the initial value problem that consists of the nonlinear first order Fredholm integral differential equation:

$$u'(x) = \frac{11}{6}x - \frac{1}{5} + \int_0^1 (xt+1)[u(t)]^2\,dt \qquad (2.25.a)$$

together with the initial condition:

$$u(0) = 0 \qquad (2.25.b)$$

Here a=0, b=λ=1, q=2, $f(x) = \frac{11}{6}x - \frac{1}{5}$ and $k(x,t) = xt + 1$.

We use the homotopy perturbation method to solve this example. To do this, let

$$v_0(x) = u_0(x) = \int_0^x (\frac{11}{6}t - \frac{1}{5})dt$$

$$= \frac{11}{12}x^2 - \frac{1}{5}x\,.$$

Then

$$v_1(x) = \int_0^x\int_0^1 (s^2t+1)\left[\frac{11}{12}t^2 - \frac{1}{5}t\right]^2 dtds$$

$$\cong 0.03836x^2 + 0.8975x..$$

In this case, let N=1, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x)$$

$$\cong 0.95502x^2 - 0.11028x.$$

Next, we must find $u_2(x)$

$$v_2(x) = \int_0^x \int_0^1 k(s,t)[2v_0(t)v_1(t)]dtds$$

$$\cong 0.01629x^2 + 0.03939x..$$

Therefore, for N=2,

$$u(x) \cong \sum_{i=0}^{N} v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 0.97131\mathrm{x}^2 - 7.09997 \times 10^{-2}\ \mathrm{x}.$$

In a similar manner one can get $v_i(x), i = 3,4,...$. The following table gives the approximated solutions for different values of N.

Table (2.4) represents the approximated solutions of example (2.7) for different values of N

N	U(x)	N	u(x)	N	u(x)
0	$0.91667x^2–0.2x$	13	$0.998330x^2–4.16697\times10^{-3}x$	26	$0.999838x^2–4.04670\times10^{-4}x$
1	$0.95502x^2–0.11028x$	14	$0.998631x^2–3.41862\times10^{-3}x$	27	$0.999863x^2–3.43486\times10^{-4}x$
2	$0.97131x^2–7.09997\times10^{-2}x$	15	$0.998872x^2–2.81678\times10^{-3}x$	28	$0.999883x^2–2.91965\times10^{-4}x$
3	$0.98022x^2–4.90465\times10^{-2}x$	16	$0.999067x^2–2.32982\times10^{-3}x$	29	$0.999901x^2–2.48517\times10^{-4}x$
4	$0.98572x^2–3.54806\times10^{-2}x$	17	$0.999226x^2–1.93369\times10^{-3}x$	30	$0.999915x^2–2.11818\times10^{-4}x$
5	$0.98936x^2–2.64527\times10^{-2}x$	18	$0.999355x^2–1.60990\times10^{-3}x$	31	$0.999928x^2–1.80774\times10^{-4}x$
6	$0.99190x^2–2.01584\times10^{-2}x$	19	$0.999462x^2–1.34412\times10^{-3}x$	32	$0.999938x^2–1.54475\times10^{-4}x$
7	$0.99373x^2–1.56202\times10^{-2}x$	20	$0.999550x^2–1.12510\times10^{-3}x$	33	$0.999947x^2–1.32165\times10^{-4}x$
8	$0.99508x^2–1.22643\times10^{-2}x$	21	$0.999622x^2–9.43993\times10^{-4}x$	34	$0.999954x^2–1.13214\times10^{-4}x$
9	$0.996096x^2–9.73292\times10^{-3}x$	22	$0.999682x^2–7.93753\times10^{-4}x$	35	$0.999961x^2–9.70962\times10^{-5}x$
10	$0.996875x^2–7.79290\times10^{-3}x$	23	$0.999732x^2–6.68758\times10^{-4}x$	36	$0.999966x^2–8.33727\times10^{-5}x$
11	$0.997480x^2–6.28651\times10^{-3}x$	24	$0.999774x^2–5.64490\times10^{-4}x$	37	$0.999971x^2–7.16747\times10^{-5}x$
12	$0.997955x^2–5.10395\times10^{-3}x$	25	$0.999809x^2–4.77297\times10^{-4}x$	38	$0.999975x^2–6.16927\times10^{-5}x$

Note that from the above table one can deduce that as N increases, the approximated solution of the initial value problem given by equations (2.25) converges to the exact solution $u(x) = x^2$.

Chapter Three

The Homotopy Perturbation Method for Solving Some Nonlocal Problems

Introduction:

It is seen that in the modeling of many real life applications systems in various fields of physics, ecology, biology, etc, an integral term over the spatial domain is appeared in some part or in the whole boundary, [8]. Such boundary value problems are known as nonlocal problems. The integral term may appear in the boundary conditions. Nonlocal conditions appear when values of the function on the boundary are connected to values inside the domain, [3]

Many researchers studied the nonlocal problems, say, [7] used Galerkin method for solving the nonlocal problem for the diffusion equation, [6] discussed the existence of the solutions for the nonlocal problem of the one-dimensional wave equations, [30] used Fourier method to establish the existence of the solution for a class of linear hyperbolic equations with nonlocal conditions, [27] used the homotopy perturbation method for solving the one-dimensional parabolic integro-differential equations with some real life applications.

In this chapter, we use the homotopy perturbation method to solve some types of the nonlocal problems.

This chapter consists of two sections:

In section one, we use the homotopy perturbation method for solving the one-dimensional wave equation with non-homogeneous Neumann and nonlocal conditions.

In section two, we give the solution of hyperbolic integro-differential equations with non-homogeneous Neumann and nonlocal conditions via the homotopy perturbation method.

3.1 Solutions of One-Dimensional Wave Equation with Non-Homogeneous Neumann and Nonlocal Conditions:

Consider the one-dimensional non-homogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (\text{x}, \text{t}) \in \Omega \qquad (3.1.a)$$

together with the initial conditions:

$$u(x,0) = r_1(x), \quad 0 \le \text{x} \le \ell, \qquad (3.1.b)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = r_2(x), \quad 0 \le x \le \ell, \qquad (3.1.c)$$

the non-homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \alpha(t), \quad 0 \le t \le T \qquad (3.1.d)$$

and the non-homogeneous nonlocal condition:

$$\int_0^\ell u(x,t)dx = \beta(t), \quad 0 \le t \le T \qquad (3.1.e)$$

where f is a known function of x and t, $\Omega = \{(x,t) | \, 0 < x < \ell, 0 \le t \le T\}$, r_1, r_2, α and β are given functions that must satisfy the compatibility conditions:

$$r_1'(0) = \alpha(0), \; r_2'(0) = \alpha'(0), \; \int_0^\ell r_1(x)dx = \beta(0), \text{ and } \int_0^\ell r_2(x)dx = \beta'(0).$$

To solve this nonlocal problem by the homotopy perturbation method, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous Neumann condition and homogeneous nonlocal conditions. To do this we use the transformation that appeared in [1]:

$$w(x,t) = u(x,t) - z(x,t), \quad (\mathrm{x,t}) \in \Omega \tag{3.2}$$

where $z(x,t) = \alpha(t)\left[x - \frac{\ell}{2}\right] + \frac{\beta(t)}{\ell}$.

Then

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2 z(x,t)}{\partial t^2}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial x^2}.$$

Therefore the nonlocal problem given by equations (3.1) is transformed to the one-dimensional non-homogeneous wave equation:

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} + g(x,t), \quad (\mathrm{x,t}) \in \Omega \tag{3.3.a}$$

together with the initial conditions:

$$w(x,0) = q_1(x), 0 \le \mathrm{x} \le \ell \tag{3.3.b}$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = q_2(x), 0 \le \mathrm{x} \le \ell \tag{3.3.c}$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0, \quad \mathrm{t} \ge 0 \tag{3.3.c}$$

and the homogeneous nonlocal condition:

$$\int_0^\ell w(x,t)dx = 0, \quad \mathrm{t} \ge 0 \tag{3.3.d}$$

where $g(x,t) = f(x,t) - \frac{\partial^2 z(x,t)}{\partial t^2}$, $q_1(x) = r_1(x) - z(x,0)$ and $q_2(x) = r_2(x) - \left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0}$.

To solve this nonlocal problem by the homotopy perturbation method, we construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = \frac{\partial^2 v(x,t,p)}{\partial t^2} - \frac{\partial^2 w_0(x,t)}{\partial t^2} + p\frac{\partial^2 w_0(x,t)}{\partial t^2} + p\left[-\frac{\partial^2 v(x,t,p)}{\partial x^2} - g(x,t)\right] = 0 \qquad (3.4)$$

where $p \in [0,1]$, $\Re$ represents the set of all real numbers and w_0 is the initial approximation to the solution of equation (3.3.a) which satisfies the initial conditions, the Neumann condition and the nonlocal condition given by equations (3.3.b)-(3.3.d).

By using equation (3.4) it follows that:

$$H(v,0) = \frac{\partial^2 v(x,t,0)}{\partial t^2} - \frac{\partial^2 w_0(x,t)}{\partial t^2} = 0$$

$$H(v,1) = \frac{\partial^2 v(x,t,1)}{\partial t^2} - \frac{\partial^2 v(x,t,1)}{\partial x^2} - g(x,t) = 0$$

Next, we assume that the solution of equation (3.4) can be expressed as:

$$w(x,t,p) = \sum_{i=0}^{\infty} p^i v_i(x,t) \qquad (3.5)$$

Therefore the approximated solution of the nonlocal problem given by equations (3.3) can be obtained as follows:

$$w(x,t) = \lim_{p\to 1} v(x,t,p) = \sum_{i=0}^{\infty} v_i(x,t) \qquad (3.6)$$

By substituting the approximated solution given by equation (3.5) into equation (3.4) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\left[-\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial x^2}-g(x,t)\right]=0.$$

Then by equating the terms with identical powers of p one can have:

$$p^0:\frac{\partial^2 v_0(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}=0 \qquad (3.7.a)$$

$$p^1:\frac{\partial^2 v_1(x,t)}{\partial t^2}+\frac{\partial^2 w_0(x,t)}{\partial t^2}-\frac{\partial^2 v_0(x,t)}{\partial x^2}-g(x,t)=0 \qquad (3.7.b)$$

$$p^j:\frac{\partial^2 v_j(x,t)}{\partial t^2}-\frac{\partial^2 v_{j-1}(x,t)}{\partial x^2}=0,\ j=2,3,... \qquad (3.7.c)$$

For simplicity, we take $v_0(x,t)=w_0(x,t)$. In this case equation (3.7.a) is automatically satisfied.

Let $w_0(x,t)=q_1(x)+q_2(x)t$ then

$$w_0(x,0)=q_1(x), \quad 0\le x\le \ell,$$

$$\left.\frac{\partial w_0(x,t)}{\partial t}\right|_{t=0}=q_2(x), \quad 0\le x\le \ell,$$

$$\left.\frac{\partial w_0(x,t)}{\partial x}\right|_{x=0}=q_1'(0)+q_2'(0)t=r_1'(0)-\left.\frac{\partial z(x,t)}{\partial x}\right|_{\substack{t=0\\x=0}}+\left[r_2'(0)-\left.\frac{\partial^2 z(x,t)}{\partial t\partial x}\right|_{\substack{t=0\\x=0}}\right]t$$
$$=r_1'(0)-\alpha(0)+r_2'(0)t-\alpha'(0)t=0, \quad 0\le t\le T$$

and

$$\int_0^{\ell}w_0(x,t)dx=\int_0^{\ell}q_1(x)dx+t\int_0^{\ell}q_2(x)dx=\int_0^{\ell}(r_1(x)-z(x,0))dx+t\int_0^{\ell}\left[r_2(x)-\left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0}\right]dx$$
$$=\beta(0)-\beta(0)+\beta'(0)t-\beta'(0)t=0, \qquad t\ge 0.$$

Therefore w_o satisfies the initial conditions, the Neumann condition and the nonlocal condition given by equations (3.3.b)-(3.3.d). Therefore by substituting t=0 in equation (3.6) one can have:

$$w(x,0)=\sum_{i=0}^{\infty}v_i(x,0).$$

But $v_0(x,0)=q_1(x)$ and $w(x,0)=q_1(x),$ hence $v_i(x,0)=0, \mathrm{i}=1,2,....$ By substituting $v_0(x,t)=$ $w_0(x,t)=q_1(x)+q_2(x)t$ into equation (3.7.b) one can get:

$$\frac{\partial^2 v_1(x,t)}{\partial t^2}=q_1''(x)+tq_2''(x)+g(x,t)$$

By integrating twice for both sides of the above differential equation with respect to t and by using the initial conditions $\mathrm{v}_1(\mathrm{x},0)=0$ and $\left.\frac{\partial v_1(x,t)}{\partial t}\right|_{t=0}=0$ one can obtain:

$$v_1(x,t)=\frac{t^2}{2}q_1''(x)+\frac{t^3}{6}q_2''(x)+\int_0^t\int_0^s g(x,\tau)d\tau ds.$$

By substituting v_1 into equation (3.7.c) and by solving the resulting second order linear partial differential equation together with the initial conditions $v_2(x,0)=0$ and $\left.\frac{\partial v_2(x,t)}{\partial t}\right|_{t=0}=0$ one can get $v_2(x,t).$ In a similar manner one can get $v_i(x,t), \mathrm{i}=3,4,....$ By substituting $v_i(x,t), \mathrm{i}=0,1,...$ into equation (3.6) one can get the approximated solution w of the nonlocal problem given by equations (3.3). Therefore from equation (3.2):

$$u(x,t)=w(x,t)+z(x,t)=\sum_{i=0}^{\infty}v_i(x,t)+z(x,t), \quad (\mathrm{x},\mathrm{t})\in\Omega$$

which is the solution of the original nonlocal problem given by equations (3.1).

To illustrate this method consider the following example

Example (3.1):

Consider the homogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}=0, \quad 0\le x\le \pi,\ 0\le t\le 1 \qquad (3.8.a)$$

together with the initial conditions:

$$u(x,0)=\cos(x), \qquad 0\le \mathrm{x}\le \pi \qquad (3.8.b)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=-\cos(x), \quad 0\le \mathrm{x}\le \pi \qquad (3.8.c)$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=0, \quad 0\le \mathrm{t}\le 1 \qquad (3.8.d)$$

and the homogeneous nonlocal condition:

$$\int_0^{\pi} u(x,t)dx=0, \quad 0\le \mathrm{t}\le 1 \qquad (3.8.d)$$

It is easy to check that the compatibility conditions are satisfied for this nonlocal problem. We use the homotopy perturbation method to solve this example. To do this, let $v_0(x,t)=u_0(x,t)=u(x,0)+\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} t=\cos(x)-t\cos(x)$. From equation (3.7.b) and by using the initial conditions:

$\left.\frac{\partial v_1(x,t)}{\partial t}\right|_{t=0}=v_1(x,0)=0$ one can have:

$$v_1(x,t)=\frac{1}{2!}t^2\cos(x)-\frac{1}{3!}t^3\cos(x).$$

Hence

$$v_0(x,t)+v_1(x,t)=\left[1-t+\frac{1}{2}t^2-\frac{1}{6}t^3\right]\cos(x).$$

From equation (3.7.c) one can get:

$$\begin{aligned} v_2(x,t) &= \int_0^t\int_0^\tau \frac{\partial^2 v_1(x,s)}{\partial x^2}\,ds\,d\tau \\ &= \frac{1}{4!}t^4\cos(x)-\frac{1}{5!}t^5\cos(x). \end{aligned}$$

and this implies:

$$u(x,t)=\sum_{i=0}^{2}v_i(x,t)=\left[1-t+\frac{1}{2}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5\right]\cos(x).$$

and by continuing in this manner one can have:

$$u(x,t)=\sum_{i=0}^{\infty}v_i(x,t)=e^{-t}\cos(x).$$

which is the exact solution of the nonlocal problem given by equations (3.8).

Example (3.2):

Consider the one-dimensional non-homogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}=-x\sin(t)-4e^{-2x},\quad 0\le x\le\frac{\pi}{2},\ 0\le t\le 1 \qquad (3.9.a)$$

together with the initial conditions:

$$u(x,0)=e^{-2x},\qquad 0\le x\le\frac{\pi}{2} \qquad (3.9.b)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = x, \qquad 0 \le x \le \frac{\pi}{2} \tag{3.9.c}$$

the non-homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \sin(t) - 2, \qquad 0 \le t \le 1 \tag{3.9.d}$$

and the non-homogeneous nonlocal condition:

$$\int_0^{\frac{\pi}{2}} u(x,t)dx = \frac{1}{8}\pi^2 \sin(t) - \frac{1}{2}e^{-\pi} + \frac{1}{2}, \qquad 0 \le t \le 1 \tag{3.9.d}$$

It is easy to check that the compatibility conditions are satisfied for this nonlocal problem. We use the homotopy perturbation method to solve this example. To do this, consider the transformation given by equation (3.2). In this case:

$$z(x,t) = (\sin(t) - 2)(x - \frac{1}{4}\pi) + \frac{1}{\pi}\left[2(\frac{1}{8}\pi^2 \sin(t)e^{\pi} - \frac{1}{2})e^{-\pi} + 1\right]$$

Therefore the nonlocal problem given by equations (3.9) is transformed to the one-dimensional non-homogeneous wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} - 4e^{-2x}, \quad 0 \le x \le \frac{\pi}{2}, \quad 0 \le t \le 1,$$

together with the initial conditions:

$$w(x,0) = e^{-2x} + 2x - \frac{\pi}{2} - \frac{1}{\pi}\left[1 - e^{-\pi}\right], \qquad 0 \le x \le \frac{\pi}{2},$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0, \qquad 0 \le x \le \frac{\pi}{2},$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le \mathrm{t} \le 1$$

and the homogeneous nonlocal condition:

$$\int_0^{\frac{\pi}{2}} w(x,t)dx = 0, \qquad 0 \le \mathrm{t} \le 1.$$

To solve this nonlocal problem by using the homotopy perturbation method, let

$$v_0(x,t) = w_0(x,t) = q_1(x) + q_2(x)t$$
$$= \frac{1}{2\pi}\left[2\pi e^{-2x} + 4\pi x - \pi^2 + 2e^{-\pi} - 2\right]$$

From equation (3.24.b) one can have:

$$v_1(x,t) = \frac{t^2}{2}q_1''(x) + \frac{t^3}{6}q_2''(x) + \int_0^t\int_0^s g(x,\tau)d\tau ds$$
$$= 0$$

Thus

$$v_i(x,t) = 0, \qquad \mathrm{i} = 2,3,....$$

Therefore

$$w(x,t) = w_0(x,t)$$
$$= \frac{1}{2\pi}\left[2\pi e^{-2x} + 4\pi x - \pi^2 + 2e^{-\pi} - 2\right].$$

which is the exact solution of the above nonlocal problem.

Hence

$$u(x,t) = w_0(x,t) + z(x,t)$$
$$= \mathrm{e}^{-2x} + x\sin(x).$$

which is the exact solution of the original nonlocal problem.

3.2 Solutions of the Hyperbolic Integro-Differential Equations with Non-Homogeneous Neumann and Nonlocal Conditions:

Consider the hyperbolic integro-differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + cu(x,t) = \int_0^t k(t,s)u(x,s)ds + f(x,t), \quad (\mathrm{x,t}) \in \Omega \tag{3.10.a}$$

together with the initial conditions:

$$u(x,0) = r_1(x), \qquad 0 \le \mathrm{x} \le \ell \tag{3.10.b}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = r_2(x), \qquad 0 \le x \le \ell \tag{3.10.c}$$

the non-homogeneous Neumann condition:

$$\frac{\partial u(x,t)}{\partial x} = \alpha(t), \qquad 0 \le t \le T \tag{3.10.d}$$

and the non-homogeneous nonlocal condition:

$$\int_0^\ell u(x,t)dx = \beta(t), \qquad 0 \le \mathrm{t} \le \mathrm{T} \tag{3.10.e}$$

Where c is a known constant, f is a known function of x and t, $\Omega = \{(x,t) | 0 < x < \ell, 0 \le t \le T\}$, r_1, r_2, α and β are given functions that must satisfy the previous compatibility conditions.

To solve this nonlocal problem by the homotopy perturbation method, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous Neumann condition and homogeneous nonlocal condition. To do this we use the transformation given by equation (3.2). Therefore the nonlocal problem given by equations (3.10) is transformed to the hyperbolic integro-differential equation:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} + cw(x,t) = \int_0^t k(t,s)w(x,s)ds + g(x,t), \quad (x,t) \in \Omega \qquad (3.11.a)$$

together with the initial conditions:

$$w(x,0) = q_1(x), \quad 0 \le x \le \ell, \qquad (3.11.b)$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = q_2(x), \quad 0 \le x \le \ell, \qquad (3.11.c)$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0, \quad t \ge 0 \qquad (3.11.d)$$

and the homogeneous nonlocal condition:

$$\int_0^\ell w(x,t)dx = 0, \quad t \ge 0 \qquad (3.11.e)$$

where

$$g(x,t) = f(x,t) - \frac{\partial^2 z(x,t)}{\partial t^2} - cz(x,t) + \int_0^t k(t,s)z(x,s)ds,$$

$$q_1(x) = r_1(x) - z(x,0)$$

and

$$q_2(x) = r_2(x) - \left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0}.$$

To solve this nonlocal problem by the homotopy perturbation method, we construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p)=\frac{\partial^2 v(x,t,p)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\frac{\partial^2 w_0(x,t)}{\partial t^2}+ p\left[-\frac{\partial^2 v(x,t,p)}{\partial x^2}+cv(x,t,p)-\int_0^t k(t,s)v(x,s,p)ds-g(x,t)\right]=0 \tag{3.12}$$

where $p\in[0,1]$, $\Re$ represents the set of all real numbers and w_0 is the initial approximation to the solution of equation (3.11.a) which satisfies the initial condition, the Neumann condition and the nonlocal condition given by equations (3.11.b)-(3.11.d).

By using equation (3.12) it follows that:

$$H(v,0)=\frac{\partial^2 v(x,t,0)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}$$

$$H(v,1)=\frac{\partial^2 v(x,t,1)}{\partial t^2}-\frac{\partial^2 v(x,t,1)}{\partial x^2}+cv(x,t,1)-\int_0^t k(t,s)v(x,s,1)ds-g(x,t)=0$$

Next, we assume that the solution of equation (3.11) can be expressed as in equation (3.5). Therefore the approximated solution of the nonlocal problem given by equations (3.11) is given by equation (3.6).

By substituting the approximated solution given by equation (3.5) into equation (3.12) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\frac{\partial^2 w_0(x,t)}{\partial t^2}+ p\left[-\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial x^2}+c\sum_{i=0}^{\infty}p^i v(x,t)-\int_0^t k(t,s)\sum_{i=0}^{\infty}p^i v_i(x,s)ds-g(x,t)\right]=0$$

Then by equating the terms with identical powers of p one can have:

$$p^0:\frac{\partial^2 v_0(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}=0 \tag{3.13.a}$$

$$p^1 : \frac{\partial^2 v_1(x,t)}{\partial t^2} + \frac{\partial^2 w_0(x,t)}{\partial t^2} - \frac{\partial^2 v_0(x,t)}{\partial x^2} + cv_0(x,t) - \int_0^t k(t,s) v_0(x,s) - g(x,t) = 0 \qquad (3.13.b)$$

$$p^j : \frac{\partial^2 v_j(x,t)}{\partial t^2} - \frac{\partial^2 v_{j-1}(x,t)}{\partial x^2} + cv_{j-1}(x,t) - \int_0^t k(t,s) v_{j-1}(x,s) ds = 0,\; j = 2,3,... \qquad (3.13.c)$$

Similar to the previous, we take $v_0(x,t) = w_0(x,t) = q_1(x) + q_2(x)t$. In this case equation (3.13.a) is automatically satisfied. By substituting it into equation (3.13.b) one can get:

$$\frac{\partial^2 v_1(x,t)}{\partial t^2} = q_1''(x) + q_2''(x)t - c\left[q_1(x) + q_2(x)t\right] + \int_0^t k(t,s)\left[q_1(x) + q_2(x)s\right] ds + g(x,t)$$

By integrating twice for both sides of the above differential equation with respect to t and by using the initial conditions $v_1(x,0) = 0$ and $\left.\frac{\partial v_1(x,t)}{\partial t}\right|_{t=0} = 0$ one can obtain:

$$v_1(x,t) = \frac{t^2}{2} q_1''(x) + \frac{t^3}{6} q_2''(x) - cq_1(x)\frac{1}{2}t^2 - cq_2(x)\frac{t^3}{6} + \int_0^{t_2}\int_0^{\tau} k(\tau,s)\left[q_1(x) + q_2(x)s\right] ds\, d\tau\, dt_2 + \int_0^t\int_0^s g(x,\tau)\, d\tau\, ds$$

In a similar manner one can get $v_i(x,t),\ i = 2,3,...$. By substituting $v_i(x,t),\ i = 0,1,...$ into equation (3.6) one can get the approximated solution w of the nonlocal problem given by equations (3.11). Therefore from equation (3.2):

$$u(x,t) = w(x,t) + z(x,t) = \sum_{i=0}^{\infty} v_i(x,t) + z(x,t), \quad (x,t) \in \Omega$$

which is the solution of the original nonlocal problem given by equations (3.10).

To illustrate this method consider the following example.

Example (3.3):

Consider the hyperbolic integro-differential equation:

$$\frac{\partial u^2(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) = \int_0^t (t+s^2)u(x,s)ds - 6xt + 3tx^3 - \frac{1}{4}x^3t^4 - \frac{1}{2}x^3t^3,$$

$$0 \le x \le 1, 0 \le t \le 1. \qquad (3.14.a)$$

together with the initial conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1, \qquad (3.14.b)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = x^3, \quad 0 \le x \le 1, \qquad (3.14.c)$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \quad 0 \le t \le 1 \qquad (3.14.d)$$

and the non-homogenous nonlocal condition:

$$\int_0^1 u(x,t)dx = \frac{1}{4}t, \quad 0 \le t \le 1 \qquad (3.14.e)$$

We use the homotopy perturbation method to solve this example. To do this, we transform this nonlocal problem into one but with homogeneous nonlocal condition. To do this, consider the transformation given by equation (3.2). In this case:

$$z(x,t) = \frac{1}{4}t$$

and the nonlocal problem given by equations (3.14) consisting of the hyperbolic integro-differential equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} + 3w(x,t) = \int_0^1 (t+s^2)w(x,s)ds - 6xt + 3tx^3 - \frac{1}{4}x^3t^4 - \frac{1}{2}x^3t^3 - \frac{3}{4}t + \frac{1}{16}t^4 + \frac{1}{8}t^3, \quad 0 \le x \le 1, \quad 0 \le t \le 1,$$

together with the initial conditions:

$$w(x,0) = 0 \qquad 0 \le x \le 1,$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = x^3 - \frac{1}{4}, \qquad 0 \le x \le 1,$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le 1$$

and the homogeneous nonlocal condition:

$$\int_0^1 w(x,t)dx = 0, \qquad 0 \le t \le 1.$$

To solve this nonlocal problem by using the ho motopy perturbation method, let

$$v_0(x,t) = w_0(x,t) = (x^3 - \frac{1}{4})t$$

From equation (3.13.b) one can have:

$$\frac{\partial^2 v_1(x,t)}{\partial t^2} = 0$$

therefore

$$v_i(x,t) = 0, \qquad i = 1,2,...$$

and this implies that

$$w(x,t) = w_0(x,t) = v_0(x,t) = (x^3 - \frac{1}{4})t.$$

which is the exact solution of the above nonlocal problem.

Hence

$$u(x,t) = w(x,t) + z(x,t)$$
$$= tx^3.$$

which is the exact solution of the original nonlocal problem.

Chapter Four

Solution of Some Real Life Applications Via the Homotopy Perturbation Method

Introduction:

In this chapter we use the homotopy perturbation method for solving some real life applications, namely advection-diffusion problems, gas dynamics equation and the ground-water level equation.

This chapter consists of four sections:

In section one and two, we solved advection-diffusion problems and gas dynamics via the homotopy perturbation method.

In section three, we present the homotopy perturbation method for solving the ground-water level problem.

4.1 Advection-Diffusion Problems:

Problems involving diffusion-advection equations arise in many domains of science. There are several methods for solving these quations, like the differential transform method, [34]. In this section, we use the homotopy perturbation method to solve the advection-diffusion problem that consists of the advection-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \beta \frac{\partial u(x,t)}{\partial x} + s(x), \quad (\mathrm{x,t}) \in \Omega \qquad (4.1.a)$$

together with initial condition:

$$u(x,0) = r(x), \quad 0 \le \mathrm{x} \le \ell, \qquad (4.1.b)$$

and the boundary conditions:

$$u(0,t) = f(t), \quad \mathrm{t} \ge 0, \qquad (4.1.c)$$

$$u(\ell,t) = g(t), \quad \mathrm{t} \ge 0, \qquad (4.1.d)$$

where the first two terms on the right hand side represent different physical processes:

$\alpha \frac{\partial^2 u(x,t)}{\partial x^2}$ corresponds to normal diffusion while $\beta \frac{\partial u(x,t)}{\partial x}$ describes advection which is why the equation is also known as the advection-diffusion equation. Further u is the variable of interest (species concentration for mass transfer). And α, β are non-negative real numbers where α is the diffusivity for species or heat transfer and β is the velocity, u is a known function of x only and $\Omega = \{(x,t) | 0 < x < \ell,\ t>0\}$, r, f and g are given functions that must satisfy the compatibility conditions:

$$r(0) = f(0)$$

and

$$r(\ell) = g(0).$$

To solve this problem by the homotopy perturbation method, we first transform this problem into another problem, but with homogeneous boundary conditions. To do this we use the transformation:

$$w(x,t) = u(x,t) - z(x,t),\ (x,t) \in \Omega \tag{4.2}$$

where

$$z(x,t) = f(t) - \frac{1}{\ell}\left[f(t) - g(t)\right]x$$

Then

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial w(x,t)}{\partial t} + \frac{\partial z(x,t)}{\partial t}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial x^2}.$$

Therefore the problem given by equations (4.1) is transformed to the partial differential equation:

$$\frac{\partial w(x,t)}{\partial t} = \alpha \frac{\partial^2 w(x,t)}{\partial x^2} - \beta \frac{\partial w(x,t)}{\partial x} + g(x,t), \quad (x,t) \in \Omega \tag{4.3.a}$$

together with the initial condition:

$$w(x,0) = q(x),\ 0 \le x \le \ell, \tag{4.3.b}$$

and the homogeneous boundary conditions:

$$w(0,t) = 0,\ t \ge 0 \tag{4.3.c}$$

$$w(\ell,t) = 0,\ t \ge 0 \tag{4.3.d}$$

where $g(x,t) = s(x) - \beta \frac{\partial z(x,t)}{\partial x} - \frac{\partial z(x,t)}{\partial t}$ and $q(x) = r(x) - z(x,0)$.

To solve this problem by the homotopy perturbation method, we rewrite equation (4.3.a) as

$$A(w) - g(x,t) = 0$$

where $A(w) = \frac{\partial w}{\partial t} - \alpha \frac{\partial^2 w}{\partial x^2} + \beta \frac{\partial w}{\partial x}$. Then the operator A can be divided into two parts L and N such that equation (4.3.a) becomes:

$$L(w) + N(w) - g(x,t) = 0$$

where $L = \frac{\partial}{\partial t}$ and $N = -\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial x}$.

According to [13], we can construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies

$$H(v,p) = \frac{\partial v(x,t,p)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} + p \frac{\partial w_0(x,t)}{\partial t} + p\left[-\alpha \frac{\partial^2 v(x,t,p)}{\partial x^2} + \beta \frac{\partial v(x,t,p)}{\partial x} - g(x,t) \right] = 0$$

(4.4)

where $p \in [0,1]$ is an embedding parameter and w_o is the initial approximation to the solution of equation (4.3.a) which satisfies the initial condition and the boundary conditions given by equations (4.3.b)-(4.3.d).

Next, we assume that the solution of equation (4.4) can be expressed as

$$v(x,t,p) = \sum_{i=0}^{\infty} p^i v_i(x,t) \tag{4.5}$$

Therefore the approximated solution of the problem given by equations (4.3) can be obtained as follows:

$$w(x,t) = \lim_{p\to 1} v(x,t,p) = \sum_{i=0}^{\infty} v_i(x,t) \tag{4.6}$$

By substituting the approximated solution given by equation (4.5) into equation (4.4) one can get:

$$H(v,p) = \sum_{i=0}^{\infty} p^i \frac{\partial v_i(x,t)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} + p\frac{\partial w_0(x,t)}{\partial x} + p\left[-\alpha \sum_{i=0}^{\infty} p^i \frac{\partial^2 v_i(x,t)}{\partial x^2} + \beta \sum_{i=0}^{\infty} p^i \frac{\partial v_i(x,t)}{\partial x} - g(x,t)\right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : \frac{\partial v_0(x,t)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} = 0 \tag{4.7.a}$$

$$p^1 : \frac{\partial v_1(x,t)}{\partial t} + \frac{\partial w_0(x,t)}{\partial t} - \alpha\frac{\partial^2 v_0(x,t)}{\partial x^2} + \beta\frac{\partial v_0(x,t)}{\partial x} - g(x,t) = 0 \tag{4.7.b}$$

and in general

$$p^j : \frac{\partial v_j(x,t)}{\partial t} - \alpha\frac{\partial^2 v_{j-1}(x,t)}{\partial x^2} + \beta\frac{\partial v_j(x,t)}{\partial x} = 0, \quad j=2,3,... \tag{4.7.c}$$

For simplicity, we take $v_0(x,t)=w_0(x,t)$. In this case equation (4.7.a) is automatically satisfied. Let $w_0(x,t)=q(x)$ then

$$w_0(x,0)=q(x),\ 0\le x\le \ell,\ \ w_o(0,t)=q(0)=r(0)-z(0,0)=r(0)-f(0)=0,\ t\ge 0$$

and

$$w_o(\ell,t)=q(\ell)=r(\ell)-z(\ell,0)=g(0)-g(0)=0,\ t\ge 0$$

Therefore w_o satisfies the initial and the boundary conditions given by equations (4.3.b)-(4.3.d). Therefore by substituting t=0 in equation (4,6) one can have:

$$w(x,0)=\sum_{i=0}^{\infty} v_i(x,0)$$

But $w_0(x,0)=q(x)$ and $w(x,0)=q(x)$, hence $v_i(x,0)=0$, i=1,2,.... By substituting $v_0(x,0)=w_0(x,t)=q(x)$ into equation (4.7.b) one can get:

$$\frac{\partial v_1(x,t)}{\partial t}=\alpha q''(x)-\beta q'(x)+g(x,t)$$

By integrating both sides of the above differential equation and by using the initial condition $v_1(x,0)=0$ one can obtain:

$$v_1(x,t)=\left[\alpha q''(x)-\beta q'(x)\right]t+\int_0^t g(x,\tau)d\tau$$

By substituting v_1 into equation (4.7.c) and by solving the resulting first order linear partial differential equation together with the initial condition $v_2(x,0)=0$ one can get $v_2(x,t)$. In a similar manner one can get $v_i(x,t),\ i=3,4,....$ By substituting $v_i(x,t),\ i=0,\ 1,...$ into equation (4.6) one can get the approximated solution of the problem given by equations (4.3). Therefore from equation (4.2):

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = \mathrm{w}(\mathrm{x},\mathrm{t}) + \mathrm{z}(\mathrm{x},\mathrm{t}) = \sum_{\mathrm{i}=1}^{\infty} \mathrm{v}_{\mathrm{i}}(\mathrm{x},\mathrm{t}) + \mathrm{z}(\mathrm{x},\mathrm{t}),\ (\mathrm{x},\mathrm{t}) \in \Omega$$

which is the solution of the advection-diffusion problem given by equations (4.1).

4.2 Gas Dynamics problem:

Consider the nonlinear non-homogeneous gas dynamic equation:

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - u(x,t)[1 - u(x,t)] = f(x,t), \quad (\mathrm{x},\mathrm{t}) \in \Omega \qquad (4.8.a)$$

together with the initial condition:

$$u(x,0) = r(x), \quad 0 \le \mathrm{x} \le 1 \qquad (4.8.b)$$

where f is a known function of x and t $\Omega = \{(x,t) \mid 0 \le x \le 1, \mathrm{t} > 0\}$

In [22], they use the homotopy perturbation method for solving the homogeneous gas dynamic equation in case $r(x) = e^{-x}, 0 \le \mathrm{x} \le 1$. Here we use the same method to solve the non-homogeneous gas dynamic equation for any choice of the initial condition. To do this, we construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = \frac{\partial v(x,t,p)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} + p\frac{\partial u_0(x,t)}{\partial t} +$$

$$p\left[- v(x,t,p)\frac{\partial v(x,t,p)}{\partial x} - v(x,t,p)[1 - v(x,t,p)]\right] = 0 \qquad (4.9)$$

where $p \in [0,1]$ and u_{o} is the initial approximation to the solution of equation (4.8.a) which satisfies the initial condition given by equation (4.8.b).

By using equation (4.9) it follows that:

$$H(x,0) = \frac{\partial v(x,t,0)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} = 0$$

$$H(v,1)=\frac{\partial v(x,t,1)}{\partial t}+v(x,t,1)\frac{\partial v(x,t,1)}{\partial t}-v(x,t,1)[1-v(x,t,1)]=0.$$

Next, we assume that the solution of equation (4.9) can be expressed as in equation (3.3). By substituting this approximated solution into equation (4.9) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial v_i(x,t)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}+p\frac{\partial u_0(x,t)}{\partial t}+$$
$$p\left[-\sum_{i=0}^{\infty}p^iv_i(x,t)\sum_{i=0}^{\infty}p^i\frac{\partial v_i(x,t)}{\partial x}-\sum_{i=0}^{\infty}p^iv_i(x,t)\left(1-\sum_{i=0}^{\infty}p^iv_i(x,t)\right)\right]=0`$$

Then by equating the terms with identical powers of p one can have:

$$p^0:\frac{\partial v_0(x,t)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}=0 \qquad (4.10.a)$$

$$p^1:\frac{\partial v_1(x,t)}{\partial t}+\frac{\partial u_0(x,t)}{\partial t}-v_0(x,t)\frac{\partial v_0(x,t)}{\partial x}-v_0(x,t)[1-v_0(x,t)]=0 \qquad (4.10.b)$$

$$p^2:\frac{\partial v_2(x,t)}{\partial t}-v_0(x,t)\frac{\partial v_1(x,t)}{\partial t}-v_1(x,t)\frac{\partial v_0(x,t)}{\partial x}+2v_0(x,t)v_1(x,t)-v_1(x,t)=0 \qquad (4.10.c)$$

$\vdots$

For simplicity, we take $v_0(x,t)=u_0(x,t)$. In this case equation (4.10.a) is automatically satisfied. Let $u_0(x,t)=r(x)$ then $u_0(x,0)=r(x), 0\le x\le 1$. Therefore u_0 satisfies the the initial condition given by equation (4.8.a). Thus:

$$u(x,0)=\sum_{i=0}^{\infty}v_i(x,0)$$

But $v_0(x,0)=u(x,0)=r(x)$ hence $v_i(x,0)=0$, $i=1,2,\ldots$. By substituting $v_0(x,0)=u_0(x,t)$ $=r(x)$ into equation (4.10.b) one can get:

$$v_1(x,t)=\left(-r(x)r'(x)+r(x)-[r(x)]^2\right)t$$

By substituting v_1 into equation (4.10.c) and by solving the resulting first order linear partial differential equation together with the initial condition $v_2(x,t)=0$ one can get $v_2(x,t)$. In a similar manner one can get $v_i(x,t), \mathrm{i}=3,4,...$. Thus $u(x,t)=\sum_{i=0}^{\infty} v_i(x,t)$ is the approximated solution of the initial value problem given by equations (4.8).

4.3 The Ground Water Level problem:

Consider the linear partial differential equation:

$$\frac{\partial^2\theta(x,t)}{\partial t}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)=h(x,t), \quad (\mathrm{x,t})\in(0,\mathrm{b})\times(0,\mathrm{T}] \qquad (4.11.a)$$

together with the initial conditions:

$$\theta(x,0)=r(x),\ 0\le \mathrm{x}\le \mathrm{b} \qquad (4.11.b)$$

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=0}=p(x),\ \ 0\le \mathrm{x}\le \mathrm{b} \qquad (4.11.c)$$

the non-homogeneous Dirichlet condition:

$$\theta(b,t)=\alpha(t),\ 0\le t\le \mathrm{T} \qquad (4.11.d)$$

and the non-homogeneous nonlocal condition:

$$\frac{1}{b}\int_0^b \theta(x,t)dx=\beta(t), \quad 0\le \mathrm{t}\le \mathrm{T} \qquad (4.11.e)$$

where θ is the ground water level, $\beta(t)$ is the mean value of θ at time t and h is a known function of x and t and r,p,α and β are given functions that must satisfy the compatibility conditions:

$$r(b)=\alpha(0),$$

$$p(b) = \alpha'(0),$$

$$\frac{1}{b}\int_0^b r(x)dx = \beta(0),$$

and

$$\frac{1}{b}\int_0^b p(x)dx = \beta'(0).$$

To solve this nonlocal problem by the homotopy perturbation method, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous Dirichlet and nonlocal conditions. To do this we use the transformation that appeared in [5]:

$$w(x,t) = \theta(x,t) - z(x,t), \quad (\mathrm{x,t}) \in (0,b) \times (0,T] \tag{4.12}$$

where $z(x,t) = [2\beta(t) - \alpha(t)] - \frac{2x}{b}[\beta(t) - \alpha(t)]$. Then the nonlocal problem given by equations (4.11) is transformed to the one-dimensional non-homogeneous linear partial differential equation:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w(x,t)}{\partial x}\right) = g(x,t), \qquad (\mathrm{x,t}) \in (0,1) \times (0,\mathrm{T}] \tag{4.13.a}$$

together with the initial conditions:

$$w(x,0) = q_1(x), \quad 0 \le \mathrm{x} \le 1 \tag{4.13.b}$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = q_2(x), \quad 0 \le \mathrm{x} \le 1 \tag{4.13.c}$$

and the homogeneous Dirichlet conditions:

$$w(b,t) = 0, \quad 0 \le \mathrm{t} \le \mathrm{T} \tag{4.13.d}$$

and

$$\frac{1}{b}\int_0^b w(x,t)dx = 0, \quad 0 \le \mathrm{t} \le \mathrm{T} \tag{4.13.e}$$

where $g(x,t) = h(x,t) - \dfrac{\partial^2 z(x,t)}{\partial t^2} + \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial z(x,t)}{\partial x}\right)$ and $q_1(x) = r(x) - z(x,0)$ and

$q_2(x) = p(x) - \left.\dfrac{\partial z(x,t)}{\partial t}\right|_{t=0}$.

To solve this nonlocal problem by the homotopy perturbation method, we can construct a homotopy $v:(0,1)\times(0,T]\times[0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = \frac{\partial^2 v(x,t,p)}{\partial t^2} - \frac{\partial^2 w_0(x,t)}{\partial t^2} + p\frac{\partial^2 w_0(x,t)}{\partial t^2} + p\left[-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w(x,t)}{\partial x}\right) - g(x,t)\right] = 0$$

(4.14)

where $p \in [0,1]$ and w_0 is the initial approximation to the solution of equation (4.13.a) which satisfies the initial condition and the nonlocal conditions given by equations (4.13.b)-(4.13.d).

By using equation (4.21) it follows that:

$$H(v,0) = \frac{\partial^2 v(x,t,0)}{\partial t^2} - \frac{\partial^2 w_0(x,t,0)}{\partial t^2} = 0$$

$$H(v,1) = \frac{\partial^2 v(x,t,1)}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v(x,t,1)}{\partial x}\right) - g(x,t) = 0.$$

Next, we assume that the solution of equation (4.14) can be expressed as:

$$v(x,t) = \sum_{i=0}^{\infty} p^i v_i(x,t) \tag{4.15}$$

Therefore the approximated solution of the nonlocal problem given by equations (4.13) is given by:

$$w(x,t)=\sum_{i=0}^{\infty} v_i(x,t) \tag{4.16}$$

By substituting the approximated solution given by equation (4.15) into equation (4.14) one can get:

$$H(v,p)=\sum_{i=0}^{\infty} p^i \frac{\partial^2 v_i(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\left[-\frac{1}{x}\frac{\partial}{\partial x}\left(x\sum_{i=0}^{\infty} p^i \frac{\partial v_i(x,t)}{\partial x}\right)-g(x,t)\right]=0.$$

Then by equating the terms with identical powers of p one can obtain:

$$p^0:\frac{\partial^2 v_0(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}=0 \tag{4.17.a}$$

$$p^1:\frac{\partial^2 v_1(x,t)}{\partial t^2}+\frac{\partial^2 w_0(x,t)}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0(x,t)}{\partial x}\right)-g(x,t)=0 \tag{4.17.b}$$

$$p^j:\frac{\partial^2 v_j(x,t)}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_{j-1}(x,t)}{\partial x}\right)=0,\quad \mathrm{j}=2,3,\ldots \tag{4.17.c}$$

For simplicity, we take $v_0(x,t)=w_0(x,t)$. In this case equation (4.17.a) is automatically satisfied. Let $w_0(x,t)=q_1(x)+q_2(x)t$ then

$$w_0(x,0)=q_1(x),\ 0\le x\le b,$$

$$\left.\frac{\partial w_0(x,t)}{\partial t}\right|_{t=0}=q_2(x),\qquad 0\le \mathrm{x}\le \mathrm{b},$$

$$\begin{aligned}w_0(b,t)&=q_1(b)+q_2(b)t\\ &=r(b)-z(b,0)+p(b)t-t\left.\frac{\partial z(x,t)}{\partial t}\right|_{\substack{x=b\\t=0}}\\ &=\alpha(0)-\alpha(0)+\alpha'(0)\mathrm{t}-\alpha'(0)\mathrm{t}\\ &=0,\qquad\qquad 0\le \mathrm{t}\le \mathrm{T}\end{aligned}$$

and

$$\frac{1}{b}\int_0^b w_0(x,t)dx = \frac{1}{b}\int_0^b [q_1(x)+q_2(x)t]dx$$

$$= \frac{1}{b}\int_0^b r(x)dx - \frac{1}{b}\int_0^b z(x,0)dx + \frac{1}{b}t\int_0^b p(x)dx - \frac{t}{b}\int_0^b \left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0} dx$$

$$= \beta(0) - \frac{1}{b}\int_0^b [2\beta(0)-\alpha(0)]dx + \frac{1}{b}\int_0^b \frac{2x}{b}[\beta(0)-\alpha(0)]dx + t\beta'(0)$$

$$-\frac{1}{b}t\int_0^b [2\beta'(0)-\alpha'(0)]dx + \frac{1}{b}t\int_0^b \frac{2x}{b}[\beta'(0)-\alpha'(0)]dx$$

$$= \beta(0) - 2\beta(0) + \alpha(0) + \beta(0) - \alpha(0) + t\beta'(0) - t[2\beta'(0)-\alpha'(0)] + t[\beta'(0)-\alpha'(0)]$$

$$= 0, \qquad 0 \le \mathrm{t} \le \mathrm{T}.$$

Therefore w_o satisfies the initial condition and the nonlocal conditions given by equations (4.13.b)-(4.13.d). By substituting $v_0(x,0) = w_0(x,t) = q(x)$ into equation (4.13.b) and by using the initial condition $v_1(x,0) = 0$ one can get:

$$v_1(x,t) = q''(x)t + \int_0^t g(x,\tau)d\tau$$

From equation (4.13.c) and by using the initial condition $\mathrm{v_i}(\mathrm{x},0) = 0$, i=2,3,... one can have:

$$v_j(x,t) = \int_0^t \frac{\partial^2 v_{j-1}(x,\tau)}{\partial x^2} d\tau, \qquad j = 2,3,...$$

Therefore from equation (4.12):

$$u(x,t) = w(x,t) + z(x,t) = \sum_{i=1}^{\infty} v_i(x,t) + z(x,t), \quad (\mathrm{x},\mathrm{t}) \in (0,1)\times(0,T]$$

which is the solution of the original nonlocal problem given by equations (4.11).

Conclusions and Recommendation

From the present study, we can conclude the following:

(1)The homotopy perturbation method can also be used to solve the linear Volttera integral equation of the first kind by transferring it into an equivalent integral equation of the second kind.

(2) The homotopy perturbation method for solving linear integral equations of the second order is precisely the method of successive in case the initial approximation is $u_0(x) = f(x)$ instead of $u_0(x) = 0$.

(3) The homotopy perturbation method for solving any initial or boundary value problems requires the initial approximation to the solution of these problems must satisfy the initial or boundary conditions associated with these problems.

(4) The homotopy perturbation method can be also used to solve systems of differential, integral and integro-differential equations.

Also, we recommend the following for future work:-

(1) Discuss the convergence of the homotopy perturbation method for the prescribed non-local problems.

(2) Use the homotopy analysis method to solve the nonlocal problems.

(3) Solve the fuzzy integro-differential equations via the homotopy perturbation method.

References

[1] Allahviranloo T., Khezerloo M., Ghanbari M. and Khezerloo S., "The Homotopy Perturbation Method for Solving Fuzzy Volttera Integral Equations", Int. J. of Comp. Co., Vol.8, No.2, pp.31-37, 2010.

[2] Abbasbandy S., "Numerical Solutions of the Integral Equations", Appl. Math., Vol.173, pp.493-500, 2006.

[3] Abasbandy A. and Roohani H. G., "The He's Variational Iteration Method for Solving the Integro- Differential Parabolic Problem with Integral Conditions" Appl. and Appl. Math. In., Special Issue, No.1, pp.12-23, 2010.

[4] Behrouz R., "Application of He's Homotopy Perturbation Method and Variational Iteration Method for Nonlinear Partial Integro-Differential Equations ", World Appl.Sc. J., Vol.7, No.4, pp.399-404, 2009.

[5] Bouziani A., "On Initial Boundary Value Problem with Dirichlet Integral Conditions for a Hyperbolic Equation with the Bessel Operator", J. Appl. Math., Vol.10, pp.487-502, 2003.

[6] Beilin S., "Existence of Solutions for One-Dimensional Wave Equations with Nonlocal Conditions", Elect. J. Diff. Eq., Vol.76, pp.1-8, 2002.

[7] Cannon J. and Lin Y., "A Galerkin Procedure for Diffusion Equations with Boundary Integral Conditions", Inter. J. Engng. Sc., Vol.28, pp.579-587, 1990.

[8] Carlson D., "Linear Thermoelasticity", Encyclopedia of Physics, Springer, Berlin, 1972.

[9] Fatemeh Sh., Mehdi D., "Solution of Delay Differential Equations via a Homotopy Perturbation Method ", J. Appl. Sc., Vol.8, No.7, pp.1256-1261, 2008.

[10]El Qarnia H., "Application of Homotopy Perturbation Method to Non-Homogeneous Parabolic Partial and Nonlinear Differential Equations" Worl. J. Model. Simul., Vol.5, No.3, pp.225-231, 2009.

[11] Ghorbali A. R., Rezania A., Ganji D.D.and Bararnia H., "Application on Homotopy Perturbation and Variational Iteration Methods for Heat Equation", Aust. J. Bas. Appl. Sc., Vol.3, No.3, pp.1863-1874, 2009.

[12] Hwang C., Lin S., Shen L., "Effects of Wall Conduction and Interface Thermal Resistance on the Phase Change Problem", Int. J. Heat. Mass Trans., Vol.37, No.13, pp.1844-1855, 1994.

[13] He J. H., "Homotopy Perturbation Technique", Comput. Math. Appl. Mech. Engrg., Vol.178, pp.257-262, 1999.

[14] He J. H., "Homotopy Perturbation Method for Solving Nonlinear Problem", Appl. Math. Comput., Vol.135, pp.73-79, 2003.

[15] He J. H., "A coupling Method of A Homotopy Technique and A Perturbation Technique for Non-Linear Problems", Int. Non-Linear Mech, Vol.35, pp.37-43, 2000.

[16] He J. H., "A New Non-linear Analytical Technique", Appl. Math. Comp., Vol.135, pp.73-79, 2000.

[17] He J. H., "The Homotopy Perturbation Method for Solving Non-Linear Oscillators with Discontinuities", Appl. Math. Comp., Vol.151, pp.287-292, 2004.

[18] He J. H.,"The Homotopy Perturbation Method for Solving Nonlinear Wave Equation", Choas. Solit. and Fract., Vol.26, pp.695-700, 2005.

[19] He J. H. and Li-Naz., " Homotopy Perturbation Method for the Solution of The Electrostatic Potential Differential Equation", Math. Probl. Engin., Vol.10, pp.1-6, 2006.

[20] Han M., Huang P. "Perturbation Solutions of Planar Diffusion-Controlled Moving-Boundary Problems". Int, J. Heat. Mass Transf., Vol.18, pp.689-695, 1975.

[21] Jafari H., Alipour M. and Tajadodi H. , "Convergence of Homotopy Perturbation Method for Solving Integral Equations", Thai J. Math., Vol.8, No.3, pp.511-520, 2010.

[22] Jafari H., Zabihi v and Saidy M., "Application of Homotopy Perturbation Method For Solving Gas Dynamics Equation", Appl. Math. Sc., Vol.2, No.48, pp.2393-2396, 2008.

[23] Jazbi B. and Moini. M. "Application of He’s homotopy Perturbation Method for Solving Schrodinger Equation". Iranian J. Math. Sc. Inf., Vol.3, No.2, pp.13-19, 2008.

[24] Javidi, M. and Golbabai A., "A Numerical Solution for Solving System of Fredholm Integral Equations by Using Homotopy Perturbation Method", Appl. Math. Comput., Vol.189, pp.1921-1928, 2007.

[25] Karakostas G. L. and Tsamatos P. Ch. "Positive Solutions for a Nonlocal Boundary-Value Problem with Increasing Response", Elect. J. Diff. Eq.,Vol.2000, No.37, pp.1-8, 2000.

[26] Khaleel A., "Homotopy Perturbation Method for Solving Spaecial Type of Nonlinear Fredholm Integor-Differential Equations", J. Al-Nahrain University,Vol. 13, No.2, pp.219-224, 2010.

[27] Khaleel A., "Homotopy Perturbation Method for Solving Some Types of Nonlocal Problem With Some Real Life Applications" to be appear in J. AL-Nahrain University

[28] Khamas A., "Some Approximation Methods for Solving Linear System of Integr-Differential Equations", Msc. Thesis, College of Education, Ibn Al-Hatham, University of Baghdad, 2009.

[29] Lin J., "Homotopy Perturbation Technique for Solving Partial Differential Equations with Variable Coefficients", Int. J. Contemp. Math. Sc., Vol.3, No. 28, pp.1395-1407, 2008

[30] Mohammad A., "A Class of Hyperbolic Equations with Nonlocal Conditions", Int. J. of Math. Analy., Vol.2, No.10, pp.491-498, 2008.

[31] Max D. L., "Fundamental Concepts of Modern Mathematics", Addision-Wesley, 1970.

[32] Massey W. S., "Algebric Topology", Harcourt, Brace Worl. Inc., 1967.

[33] Mashallah M. and Mohammad S., "The Homotopy Perturbation Method for Solving Fuzzy Integral Equations", The J. of Math. Comput. Sc.,Vol.1, No.4, pp.377-385, 2010.

[34] Patrıcio M. F. and P. M. Rosa, "The Differential Trnasform Method for Solving Reaction-Diffusion Equations", Int. J. Comput. Math. Sc., Vol.1, No.4, pp.258-262, 2007.

[35] Roozi A., Alibeiki E., Hosseini S.S., Shafiof S.M. and Ebrahimi M." Homotopy Perturbation Method for Special Nonlinear Partial Differential Equations", J. King Saud University, Vol.23, pp.99-103, 2011.

[36] Ruyun M. "A Survey On Nonlocal Boundary Value Problems". Appl. Math., Vol.7, pp.257-279, 2007.

[37] Saeed R. K., "Homotopy Perturbation Method for Solving System of Non-Linear Fredholm Integral Equations of The Second Kind", J. Appl. Scien. Res., Vol.4, No.10, pp.1166-1173, 2008.

[38] Yu-Xi W., Hua-You S., and Lu-Feng M., "Homotopy Perturbation Method for Solving Reaction-Diffusion Equations", Math. Probl. Eng., Vol.10, pp.1-5, 2008.

المستخلص

الهدف الرئيسي من هذا العمل هو استعمال طريقة الهوموتوبي المتخلخل لحل انواع خاصة من المسائل المحلية واللامحلية .

هذه الدراسة شملت المحاور التالية :

(1) اعطاء بعض المفاهيم الاساسية لطريقة الهوموتوبي المتخلخل.

(2) استعمال طريقة الهوموتوبي المتخلخل لحل انواع من المعادلات التفاضلية و التكاملية و التكاملية-التفاضلية.

(3) وصف بعض المسائل اللامحلية و استعمال طريقة الهوموتوبي المتخلخل لحلها.

(4) استعمال طريقة الهوموتوبي المتخلخل لحل بعض المسائل الحياتية و هي مسالة تافق انتشار الحرارة في مستوى افقي، مسالة ديناميكية الغاز و مسألة مستوى المياه الجوفية .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم
قسم الرياضيات و تطبيقات الحاسوب

حول طريقة الهوموتوبي المتخلل وتطبيقاتها

رسالة

مقدمة إلى كلية العلوم - جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير
في علوم الرياضيات

من قبل

فرح لطيف جوي

(بكالوريوس علوم رياضيات،جامعة النهرين، 2008)

باشراف

ا. م. د. احـــلام جميل خليل	م. د. شذى احمد عزيز

ربيع الثاني 1432
آذار **2011**